Class Notes for Quantum Field Theory: Section III

Path Integrals in Quantum Mechanics, Path Integrals in Field Theory, Functional Methods for Fermions, Path Integrals for Fermions, Path Integrals for Abelian Gauge Theory, Path Integrals for Non-Abelian Gauge Theory — the Fadeev-Popov Procedure, Derivation of QCD Feynman Rules, Color Algebra Techniques, Spinor Algebra Techniques, Some Explicit QCD Process Calculations

Based on Abers and Lee (Physics Reports), Ryder, Cheng and Li, Peskin and Schroeder and Ramond.

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Introduction to Path Integrals in Quantum Mechanics

My notes may differ somewhat in some places from Ryder; they are based on the notation of the Abers and Lee, Physics Reports.

Introductory Ideas

- In the usual formulation of QM, the quantities q and p are replaced by operators which obey Heisenberg commutation relations. The mathematics is that of operators in Hilbert space.

- The *path integral formulation* is instead based directly on the notion of a propagator $K(q_f, t_f; q_i, t_i)$ which is defined such that

$$\psi(q_f, t_f) = \int K(q_f, t_f; q_i, t_i)\psi(q_i, t_i)dq_i \tag{1}$$

  i.e. the wave function at later time is given by a Huygen's principle in terms of the wave function at an earlier time, where we have to integrate over all the points q_i since all can, in principle, send out little wavelets that would influence the value of the wave function at q_f at the later time t_f.

This equation is very general and is simply an expression of causality.

- According to the usual interpretation of QM, $\psi(q_f, t_f)$ is the probability amplitude that the particle is at the point q_f and the time t_f, which means that $K(q_f, t_f; q_i, t_i)$ is the probability amplitude for a transition from q_i and t_i to q_f and t_f.

  The probability that the particle is observed at q_f at time t_f if it began at q_i at time t_i is

$$P(q_f, t_f; q_i, t_i) = |K(q_f, t_f; q_i, t_i)|^2 . \tag{2}$$

- Let us now divide the time interval between t_i and t_f into two, with t as the intermediate time, and q the intermediate point in space.

  Repeated application of Eq. (1) gives

$$\psi(q_f, t_f) = \int \int K(q_f, t_f; q, t) dq K(q, t; q_i, t_i) \psi(q_i, t_i) dq_i \tag{3}$$

from which it follows that

$$K(q_f, t_f; q_i, t_i) = \int dq K(q_f, t_f; q, t) K(q, t; q_i, t_i) \,. \tag{4}$$

In words, this equation says that the transition from (q_i, t_i) to (q_f, t_f) may be regarded as the result of the transition from (q_i, t_i) *to all available intermediate points* q followed by a transition from (q, t) to (q_f, t_f).

- The two-slit experiment provides a useful reference point for intuition. Moving from left to right, consider the light source as located at position 1, the two slits as located at positions 2A and 2B and the screen on which we see the interference pattern is at location 3.

  The result of Eq. (4) then simplifies to

$$K(3; 1) = K(3; 2A) K(2A; 1) + K(3; 2B) K(2B; 1) \tag{5}$$

and the intensity pattern of the screen 3 is given by the probability

$$P(3; 1) = |K(3; 1)|^2 \tag{6}$$

which will clearly contain interference terms characteristic of the quantum theory.

Note that we cannot say that the electron traveled either through slit 2A or 2B — we had to allow *at the amplitude level* **for it to travel over both paths (if not detected at the slits).**

This notion of all possible paths is crucial in the path integral formulation of QM.

- **We now show that**

$$K(q_f, t_f; q_i, t_i) = \langle q_f, t_f | q_i, t_i \rangle_H \,. \tag{7}$$

To see this, we need only remember that

$$\psi(q, t) = \langle q | \psi, t \rangle_S, \tag{8}$$

where the state vector $|\psi, t\rangle_S$ in the Schroedinger picture is related to that in the Heisenberg picture $|\psi\rangle_H$ by (recall that in the Heisenberg

H picture the states are time-independent and the operators are time-dependent, whereas in the Schroedinger S picture the states are time dependent and the operators time-independent)

$$|\psi, t\rangle_S = e^{-iHt}|\psi\rangle_H \,, \tag{9}$$

or, equivalently,

$$|\psi\rangle_H = e^{iHt}|\psi, t\rangle_S \,. \tag{10}$$

We also define the vector

$$|q, t\rangle_H = e^{iHt}|q\rangle_S \tag{11}$$

which is the Heisenberg version of the Schroedinger state $|q\rangle$. Then, we can equally well write

$$\psi(q, t) = \langle q, t|\psi\rangle_H \,. \tag{12}$$

By completeness of states we can write

$$\langle q_f, t_f|\psi\rangle_H = \int \langle q_f, t_f|q_i, t_i\rangle_H \langle q_i, t_i|\psi\rangle_H dq_i \,, \tag{13}$$

which with the definition of Eq. (12) becomes

$$\psi(q_f, t_f) = \int \langle q_f, t_f | q_i, t_i \rangle_H \psi(q_i, t_i) dq_i. \tag{14}$$

Upon comparing to where we started, Eq. (1), we see that

$$\langle q_f, t_f | q_i, t_i \rangle_H = K(q_f, t_f; q_i, t_i) \tag{15}$$

as claimed. (Note: Ryder implicitly drops the H subscript, but I have kept it explicitly up to this point. I will drop it in what follows.)

Setting up the Path Integral

- We begin with the Heisenberg state $|q, t\rangle_H$, where q is a coordinate, so that $Q_H(t)|q, t\rangle_H = q|q, t\rangle_H$ where $Q_H(t) = e^{iHt} Q_S e^{-iHt}$, where subscript S denotes the Schroedinger representation.

  Of course,

$$|q\rangle_S = e^{-iHt}|q, t\rangle_H\,, \quad |q, t\rangle_H = e^{iHt}|q\rangle_S \tag{16}$$

and we will denote $|q\rangle_S \equiv |q\rangle$.

- Let us now compute the NRQM propagator

$$\langle q', t'|q, t\rangle_H = \langle q'|e^{-iH(t'-t)}|q\rangle. \tag{17}$$

- We will rewrite this propagator using the path integral approach which will incorporate the quantization of the coordinates.

- The first step is to divide up the time interval into $n+1$ tiny pieces: $t_l = l\epsilon + t$ with $t' = (n+1)\epsilon + t$.

- By completeness, we can then write

$$\begin{aligned} \langle q', t'|q, t\rangle_H &= \int dq_1(t_1) \ldots \int dq_n(t_n) \langle q', t'|q_n, t_n\rangle_H \\ &\quad \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle_H \ldots \langle q_1, t_1|q, t\rangle_H \,. \end{aligned} \tag{18}$$

The integral $\int dq_1(t_1) \ldots dq_n(t_n)$ is an integral over all possible "trajectories". These are not trajectories in the normal sense, since there is no requirement of continuity. The paths are really what are called Markov chains.

- Now notice that for small ϵ we can write (dropping the H subscripts to avoid so much writing):

$$\langle q', \epsilon | q, 0\rangle = \langle q' | e^{-i\epsilon H} | q\rangle = \delta(q'-q) - i\epsilon \langle q' | H | q\rangle \,. \tag{19}$$

- Now, we need to recall that $H = H(P,Q)$ (where P, Q are the momentum and coordinate operators).

  For example, let us assume that $H = \frac{1}{2}P^2 + V(Q)$.

  Then,

$$\begin{aligned}
\langle q' | H(P,Q) | q\rangle &= \langle q' | \frac{1}{2} P^2 | q\rangle + V\left(\frac{q+q'}{2}\right)\delta(q'-q) \\
&= \int \frac{dp}{2\pi} \langle q' | p\rangle \langle p | \frac{1}{2} P^2 | q\rangle + V\left(\frac{q+q'}{2}\right) \int \frac{dp}{2\pi} e^{i(q'-q)p} \\
&= \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[\frac{1}{2} p^2 + V\left(\frac{q+q'}{2}\right)\right]
\end{aligned}$$

$$= \int \frac{dp}{2\pi} e^{ip(q'-q)} H\left(p, \frac{1}{2}(q'+q)\right) . \tag{20}$$

In the above, we inserted a complete set of momentum basis states in order to evaluate the operator P in terms of the number p.

- Putting this into our earlier form we obtain

$$\langle q', \epsilon | q, 0\rangle \simeq \int \frac{dp}{2\pi} \exp\left[i \left\{ p(q'-q) - \epsilon H\left(p, \frac{1}{2}(q'+q)\right)\right\}\right], \tag{21}$$

where the 0th order in $\epsilon \Rightarrow \delta(q'-q)$ and the 1st order in $\epsilon \Rightarrow -i\epsilon\langle q'|H(P,Q)|q\rangle$.

- We now need to substitute many such forms into Eq. (18). This will yield:

$$\langle q', t'|q, t\rangle_H = \lim_{n\to\infty} \int \prod_{i=1}^{n} dq_i \prod_{k=1}^{n+1} \frac{dp_k}{2\pi} \exp\Big\{ i \sum_{j=1}^{n+1} \big[p_j(q_j - q_{j-1}) - H\left(p_j, \frac{1}{2}(q_j + q_{j-1})\right)(t_j - t_{j-1}) \big] \Big\}, \tag{22}$$

with $q_0 = q$ and $q_{n+1} = q'$. It is important to keep track of what the above formula says.

- **For each little interval, we had a p_k over which we had to integrate.**
- **Each interval is defined by the q_i on either end, and we needed n $q_i(t_i)$'s and these were integrated over in our original form of Eq. (18).**
- **We multiplied exponentials associated with each of these little intervals together and got an exponential of the sum of all the arguments of the exponentials.**
- **Roughly, the above formula says to integrate over all possible momenta and coordinate values associated with a small interval, *weighted* by something that's going to turn into the exponential of the action in the limit where $\epsilon \to 0$.**
- **Again, it should be stressed that the different q_i and p_k integrals are independent. This implies that p_k for one interval can be completely different from the $p_{k'}$ for some other interval (including the neighboring intervals). See fig 5.4 of Ryder for a picture.**
- **You could question whether or not the complicated integral above can be defined mathematically. Below, we will claim that it should be defined by analytic continuation into the complex plane of, for example, the p_k integrals. This is something you should be familiar with from**

the 204 course. It is actually related to the $+i\epsilon$ prescription for the Feynman propagator. The appropriate analytic continuation is one of several different choices that could be made and the one made will correspond to the Feynman boundary conditions.

- In fact, what we do now is to go to the differential limit where we call $t_j - t_{j-1} \equiv d\tau$ and write $\frac{(q_j - q_{j-1})}{(t_j - t_{j-1})} \equiv \dot{q}$, in which case the above formula takes the form

$$\langle q', t' | q, t \rangle = \int \left[\frac{dpdq}{2\pi\hbar} \right] \exp \left\{ \frac{i}{\hbar} \int_t^{t'} [p\dot{q} - H(p,q)]\, d\tau \right\} \tag{23}$$

where we have used the shorthand notation

$$\int \prod_\tau \frac{dq(\tau)dp(\tau)}{2\pi\hbar} \equiv \int \left[\frac{dpdq}{2\pi\hbar} \right] \tag{24}$$

and we have temporarily put back in the $\hbar$'s so that you will recognize the quantum mechanical action exponential.

Note that the above integration is an integration over the p and q values *at every time* τ. This is what we call a **functional integral**.

We can think of a given set of choices for all the $p(\tau)$ and $q(\tau)$ as defining a path in the 6-dimensional phase space.

- The most important point of the above result is that we have obtained an expression for a quantum mechanical transition amplitude in terms of an integral involving only pure c-numbers (no operators).

- We can actually perform the above integral for Hamiltonians of the type $H = H(P, Q)$ (i.e. no products of P and Q). We use square completion in the exponential for this, defining the integral in the complex p plane (i.e. by writing $p \to p(1 - i\theta)$ and continuing to the physical situation by taking $\theta \to 0$)

  In particular, we have

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left\{ i\epsilon(p\dot{q} - \frac{1}{2}p^2) \right\} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left\{ -\frac{1}{2}i\epsilon(p - \dot{q})^2 + \frac{1}{2}i\epsilon\dot{q}^2 \right\}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{1}{2}i\epsilon}} \exp\left[\frac{1}{2}i\epsilon\dot{q}^2 \right]$$

$$= \frac{1}{\sqrt{2\pi i\epsilon}} \exp\left[\frac{1}{2} i\epsilon \dot{q}^2\right] . \quad (25)$$

- Substituting this result into Eq. (22), we obtain

$$\begin{aligned} \langle q', t' | q, t\rangle_H &= \frac{1}{\sqrt{2\pi i\epsilon}} \lim_{n\to\infty} \int \prod_{i=1}^{n} \frac{dq_i}{\sqrt{2\pi i\epsilon}} \exp\Big\{ i\epsilon \sum_{j=1}^{n+1} \Big[\frac{1}{2}\left(\frac{q_j - q_{j-1}}{\epsilon}\right)^2 \\ &\quad - V\left(\frac{q_j + q_{j-1}}{2}\right)\Big]\Big\} \\ &\equiv \int \left[\frac{dq}{\sqrt{2\pi i\epsilon}}\right] \exp\left\{ i \int_t^{t'} L(q, \dot{q}) d\tau \right\} , \end{aligned} \quad (26)$$

where there is actually the extra $\frac{1}{\sqrt{2\pi i\epsilon}}$ that gets lost in the nice (but too simplistic) general notation. In the above,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \quad (27)$$

is the Lagrangian, and

$$S = \int L(q, \dot{q}) d\tau \tag{28}$$

is the standard action.

- Let us check that this is the correct answer in the case being discussed. To do that, we must see how the Schroedinger equation emerges from our expression. For this, we separate out the very last dq_n integral and write

$$\langle q', t' | q, t \rangle_H = \int_{-\infty}^{\infty} \frac{dq_n}{\sqrt{2\pi i \epsilon}} \exp\left\{ i \frac{(q' - q_n)^2}{2\epsilon} - i\epsilon V\left(\frac{q' + q_n}{2}\right)\right\} \langle q_n, t' - \epsilon | q, t \rangle_H \,. \tag{29}$$

Now, it is clear that the rapid oscillation of the exponential guarantees that q_n must be very close to q'. This implies that we can expand the above and write

$$\begin{aligned} \langle q', t' | q, t \rangle_H &= \int_{-\infty}^{\infty} \frac{dq_n}{\sqrt{2\pi i \epsilon}} \exp\left\{ i \frac{(q' - q_n)^2}{2\epsilon} \right\} \left[1 - i\epsilon V(q') \right] \\ &\quad \times \left[1 + (q_n - q') \frac{\partial}{\partial q'} + \frac{1}{2} (q_n - q')^2 \frac{\partial^2}{\partial q'^2} + \ldots \right] \langle q', t' - \epsilon | q, t \rangle_H \end{aligned} \tag{30}$$

We now perform the dq_n integral using

$$\int dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int dx x e^{-ax^2} = 0, \quad \int dx x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{31}$$

to obtain

$$\begin{aligned} \langle q', t' | q, t \rangle_H &= \frac{\sqrt{2\pi i \epsilon}}{\sqrt{2\pi i \epsilon}} \left[1 - i\epsilon V(q') + \frac{i\epsilon}{2} \frac{\partial^2}{\partial q'^2} + \ldots \right] \langle q', t' - \epsilon | q, t \rangle_H \\ &= \left[1 - i\epsilon V(q') + \frac{i\epsilon}{2} \frac{\partial^2}{\partial q'^2} + \ldots \right] \left[\langle q', t' | q, t \rangle_H - \epsilon \frac{\partial}{\partial t'} \langle q', t' | q, t \rangle_H \right] \end{aligned} \tag{32}$$

Noting that the terms of order ϵ^0 cancel and comparing terms of order ϵ^1 we find

$$i \frac{\partial}{\partial t'} \langle q', t' | q, t \rangle_H = \left[-\frac{1}{2} \frac{\partial^2}{\partial q'^2} + V(q') \right] \langle q', t' | q, t \rangle_H \tag{33}$$

which says that our propagator (which in Eq. (1) we called $K(q', t'; q, t)$) obeys the Schroedinger equation (with $m = \hbar = c = 1$), as it must given that it propagates the wave function, see again Eq. (1).

- More complicated H forms can lead to an "$S_{\rm eff}$" which is a bit more complicated than the above. Typical cases in which this happens are when the Lagrangian L depends upon $\dot{q}$ and q in an entwined fashion: e.g. $L = \frac{\dot{q}^2}{2} f(q)$.

- As you see, we should not worry about the $\prod_{i=1}^{n} \frac{1}{\sqrt{\epsilon}}$ which seems to diverge as $\epsilon \to 0$. In fact, it is canceled because of the fact that neighboring q_i's must be very close to one another in order to escape rapid oscillations of the exponential phase. In field theory, such factors disappear since we deal only with normalized transition amplitudes.

- Always keep in mind that the pretty continuous expressions *are defined by the finite interval expressions*, and it is only at the end that we can take $\epsilon \to 0$.

Generalization to many degrees of freedom

$$\langle q_1' \ldots q_N', t' | q_1 \ldots q_N, t \rangle$$
$$= \int \prod_{n=1}^{N} \left[\frac{dq_n dp_n}{2\pi\hbar} \right] \exp\left\{ \frac{i}{\hbar} \int_t^{t'} \left[\sum_{n=1}^{N} p_n \dot{q}_n - H(p_1, \ldots, p_N; q_1, \ldots, q_N) \right] d\tau \right\} \tag{34}$$

where $q_n(t) = q_n$ and $q_n(t') = q'_n$ for all $n = 1, N$, and we are allowing for the full Hamiltonian of the system to depend upon all the N momenta and coordinates collectively. Of course, it could be that H is a sum of N independent H_n's in some cases, but this is not necessary to the derivation.

- For the moment, we will continue with some additional results keeping $N = 1$.

- We will shortly see how it is that time ordering enters into this path integral game.

- Consider first,

$$\langle q', t'|Q(t_0)|q, t\rangle$$
$$= \int \prod_i dq_i(t_i) \langle q', t'|q_n, t_n\rangle \ldots \langle q_{i0}, t_{i0}|Q(t_0)|q_{i-1}, t_{i-1}\rangle \ldots \langle q_1, t_1|q, t\rangle \,, \quad (35)$$

  where we choose one of the time interval ends to coincide with t_0, i.e. $t_{i0} = t_0$. If we operate $Q(t_0)$ to the left, then it is replaced by its eigenvalue $q_{i0} = q(t_0)$.

  Aside from this one addition, everything else is evaluated just as before and we will obviously obtain:

$$\langle q', t' | Q(t_0) | q, t \rangle = \int \left[\frac{dqdp}{2\pi} \right] q(t_0) \exp \left\{ i \int_t^{t'} [p\dot{q} - H(p,q)] \, d\tau \right\} . \tag{36}$$

- Next, suppose we want a path integral expression for

$$\langle q', t' | Q(t_1) Q(t_2) | q, t \rangle \tag{37}$$

in the case where $t_1 > t_2$. Well, we have to insert as intermediate states $|q_{i1}, t_{i1}\rangle\langle q_{i1}, t_{i1}|$ with $t_{i1} = t_1$ and $|q_{i2}, t_{i2}\rangle\langle q_{i2}, t_{i2}|$ with $t_{i2} = t_2$ *and since we have ordered the times at which we do the insertions we must have the first insertion to the left of the 2nd insertion when* $t_1 > t_2$.

Once these insertions are done, we evaluate $\langle q_{i1}, t_{i1} | Q(t_1) = \langle q_{i1}, t_{i1} | q(t_1)$ and $\langle q_{i2}, t_{i2} | Q(t_2) = \langle q_{i2}, t_{i2} | q(t_2)$ and then proceed as before with everything else and obtain

$$\langle q', t' | Q(t_1) Q(t_2) | q, t \rangle = I \tag{38}$$

with

$$I = \int \left[\frac{dqdp}{2\pi}\right] q(t_1)q(t_2) \exp\left\{i \int_t^{t'} [p\dot{q} - H(p,q)]\, d\tau\right\}. \qquad (39)$$

Now, let us ask what the above integral is equal to if $t_2 > t_1$? Well, I assume it is obvious that what you get is $I = \langle q', t'|Q(t_2)Q(t_1)|q, t\rangle$.

In short,

$$I = \langle q', t'|T\{Q(t_1)Q(t_2)\}|q, t\rangle . \qquad (40)$$

Clearly, this generalizes to an arbitrary number of Q operators.

- Of course, when we go to field theory, the Q's will be replaced by fields, since it is the fields that play the role of coordinates in the 2nd quantization conditions.

Functional Techniques and Scattering

- The type of boundary condition employed above, i.e. $q(t_f) = q_f$ and $q(t_i) = q_i$, while appropriate in the motion of classical particles, is not what we meet in field theory.

- **Its analogue there would be, for example, $\psi(t_i) = \psi_i$ and $\psi(t_f) = \psi_f$. But, in our applications in field theory, what happens is that particles are scattering from one another (colliding) and perhaps creating other particles, and then the final state particles are destroyed by the detector which observes them.**

- **The act of creation may be represented as a source, and that of destruction by a sink, which is, in a manner of speaking, also a source. The boundary conditions and sequence of events that define the problem may then be represented as:**

  1. **starting with a vacuum state at $t \to -\infty$,**
  2. **followed by creation of the scattering particles,**
  3. **followed by their interaction,**
  4. **followed by destruction of the final state particles emerging from the interaction,**
  5. **followed by a vacuum state at $t \to +\infty$.**

  What this means is that we want the *vacuum-to-vacuum transition amplitude in the presence of the required sources and sinks.*

- **The source is represented by modifying the Lagrangian:**

$$L \to L + \hbar J(t) q(t) \,. \tag{41}$$

- **Let us define $|0, t\rangle^J$ as the ground state (vacuum) vector (in the moving frame — i.e. with the e^{iHt} included, that is multiplying the Schroedinger state) in the presence of the source. The required transition amplitude is**

$$Z[J] \propto \langle 0, \infty | 0, -\infty \rangle^J \,, \tag{42}$$

where we have omitted a proportionality factor that will not matter in the end as we will have a ratio of two things with the same proportionality factor.

- **The source $J(t)$ plays a role analogous to that of an electromagnetic current, which acts as a "source" of the electromagnetic field. In other words, think $J_\mu A^\mu$, where J_μ is the current from a scalar or Dirac field acting as a source of A^μ.**

  In the same way, we can always define a current J that acts as the source for some arbitrary field ϕ.

- $Z[J]$ is a functional of J (indicated by the $[J]$ notation), and we now derive an expression for it, i.e. for the transition amplitude up to a constant factor.

  We will later return to what we mean by a functional in case you are not used to this language. Basically, a functional, as opposed to a function, is a quantity that depends upon another function (such as $J(t)$, which is itself a function, in this case a function of time).

- Let us consider $L \neq L(t)$ (no explicit time dependence).

- Let us define energy eigenstates *in the absence of the source term*: $|n\rangle$ with $H|n\rangle = E_n|n\rangle$.

- Let us define $\phi_n(q) = \langle q|n\rangle$ with $\phi_0(q)$ being the ground state wave function.

  Recall that $|q\rangle$ is the Schroedinger representation state. (We use the notation $|q,t\rangle = e^{iHt}|q\rangle$ for the Heisenberg state.)

- We wish to calculate the amplitude for $\phi_0(q)$ at $T \to -\infty$ to transition to $\phi_0(q)$ at $T' \to +\infty$ in the presence of an external source term $J(t)q(t)$ added to L during the interval $[T, T']$.

We will see that this is useful later in the field theory context.

- Consider first,

$$
\begin{aligned}
&\langle Q', T' | Q, T \rangle^J \\
= &\int \left[\frac{dpdq}{2\pi}\right] \exp\left\{ i \int_T^{T'} \left[p(\tau)\dot{q}(\tau) - H(p,q) + J(\tau)q(\tau) \right] d\tau \right\} \\
= &\int dq' \int dq \langle Q', T' | q', t' \rangle \langle q', t' | q, t \rangle^J \langle q, t | Q, T \rangle \,, \qquad (43)
\end{aligned}
$$

where from t' to T' and from T to t, $J = 0$; i.e. $J(\tau) \neq 0$ only in the interval $\tau \in [t, t']$.

- Now, we write

$$
\begin{aligned}
\langle q, t | Q, T \rangle &= \langle q | \exp\{-iH(t-T)\} | Q \rangle \\
&= \sum_n \langle q | n \rangle \langle n | \exp\{-iH(t-T)\} | Q \rangle \\
&= \sum_n \phi_n(q) \phi_n^*(Q) \exp\{-iE_n(t-T)\} \qquad (44)
\end{aligned}
$$

as allowed in the time interval being considered where $\boldsymbol{J} = \mathbf{0}$.

Similarly,

$$\begin{aligned} &\langle Q', T' | q', t' \rangle \\ = &\sum_m \phi_m^*(q') \phi_m(Q') \exp\{iE_m(t' - T')\} \,. \end{aligned} \tag{45}$$

- **We now substitute these expressions into Eq. (43) and consider the limit $T' \to \infty e^{-i\delta}$ and $T \to -\infty e^{-i\delta}$, where δ is an arbitrary angle $0 < \delta \leq \pi/2$. This is equivalent to rotating the overall time axis so that it has a downward slant as time runs from $-\infty$ to $+\infty$.**

  Since the imaginary part of T is $i|T|\sin\delta$ and is getting very large, the term iE_nT in the exponential of Eq. (44) is being damped, with the damping being larger for larger E_n. The least damped term will be that with the lowest energy, namely $n = 0$.

  A similar argument implies that in the T' case only the $m = 0$ term survives in the limit.

- **We then end up with**

$$\begin{aligned}
& \lim_{T' \to \infty e^{-i\delta}, T \to -\infty e^{-i\delta}} \left[\frac{\langle Q', T' | Q, T \rangle^J}{\phi_0^*(Q)\phi_0(Q') \exp\left[-iE_0(T'-T)\right]} \right] \\
= & \int dq' dq \phi_0^*(q', t') \langle q', t' | q, t \rangle^J \phi_0(q, t) \\
= & \int dq' dq \langle 0 | q', t' \rangle \langle q', t' | q, t \rangle^J \langle q, t | 0 \rangle \\
\equiv & \langle 0 | 0 \rangle^J \\
\equiv & Z[J]
\end{aligned} \tag{46}$$

i.e. we have an expression for the ground state to ground state transition amplitude. In getting the above expression, we have used $|q, t\rangle_H = e^{iHt}|q\rangle_S$ (but with H and S subscripts implicit) in the following manner:

$$\begin{aligned}
\phi_0(q, t) &= \langle q, t | 0 \rangle = \langle q | e^{-iHt} | 0 \rangle = e^{-iE_0 t} \langle q | 0 \rangle = e^{-iE_0 t} \phi_0(q) \\
\phi_0^*(q', t') &= \langle 0 | q', t' \rangle = \langle 0 | e^{iHt'} | q' \rangle = e^{iE_0 t'} \langle 0 | q' \rangle = e^{iE_0 t'} \phi_0^*(q')
\end{aligned}$$

(47)

- By recognizing that (just insert $\sum_n |n\rangle\langle n|$ and take limit)

$$
\begin{aligned}
& \lim_{T'\to\infty e^{-i\delta}, T\to -\infty e^{-i\delta}} \langle Q', T'|Q, T\rangle \\
= & \lim_{T'\to\infty e^{-i\delta}, T\to -\infty e^{-i\delta}} \sum_n \phi_n(Q')\phi_n^*(Q) \exp\left[-iE_n(T'-T)\right] \\
= & \phi_0(Q')\phi_0^*(Q) \exp\left[-iE_0(T'-T)\right]
\end{aligned} \tag{48}
$$

we can write

$$
Z[J] = \langle 0|0\rangle^J = \lim_{T'\to\infty e^{-i\delta}, T\to -\infty e^{-i\delta}} \frac{\langle Q', T'|Q, T\rangle^J}{\langle Q', T'|Q, T\rangle}. \tag{49}
$$

Note that $Z[J=0]=1$! That is, we are always implicitly dealing with a normalized ratio. In what follows, it is most convenient to simply drop the denominator in the above expression and write:

$$
\begin{aligned}
Z[J] &= \lim_{T' \to \infty e^{-i\delta}, T \to -\infty e^{-i\delta}} \langle Q', T' | Q, T \rangle^J \\
&= \int \left[\frac{dpdq}{2\pi} \right] \exp\left\{ i \int_T^{T'} \left[p(\tau)\dot{q}(\tau) - H(p,q) + J(\tau)q(\tau) \right] d\tau \right\}
\end{aligned}
\tag{50}
$$

keeping in mind that at the very end we should normalize so that $Z[J=0] = 1$.

- Instead of rotating the time axis to have an imaginary component, the ground state contribution may also be isolated by adding a small negative imaginary component to the Hamiltonian:

$$
H \to H - \frac{1}{2} i\epsilon q^2 , \quad \text{or} \quad L \to L + \frac{1}{2} i\epsilon q^2 . \tag{51}
$$

With this addition $E_n \to E_n - i\epsilon'_n$ where ϵ'_n is larger for larger n by virtue of $\langle q^2 \rangle_n$ being larger at larger n. As a result, for example,

$$\begin{aligned} & \sum_n \phi_n(q)\phi_n^*(q) \exp\left[-iE_n(t-T)\right] \\ \rightarrow \quad & \sum_n \phi_n(q)\phi_n^*(q) \exp\left[-i(E_n - i\epsilon_n')(t-T)\right] \end{aligned} \tag{52}$$

implying a damping factor as $T \to -\infty \propto e^{\epsilon_n' T}$ which isolates the $n=0$ state since ϵ_n' is smallest for $n=0$.

Functional Techniques

- We now need to have a brief excursion into the mathematics of functional techniques.

- As we said earlier, a functional is an object that maps a function into a number.

  For example, in

$$\int \left[\frac{dpdq}{2\pi}\right] \exp\left\{ i \int_T^{T'} \left[p(\tau)\dot{q}(\tau) - H(p,q) + J(\tau)q(\tau)\right] d\tau \right\} \tag{53}$$

the argument of the exponential depends upon the functions $q(\tau)$ and $p(\tau)$ and we then integrate over all possible forms of these two functions. So the exponential is a *functional* that maps a choice for these two functions into a number.

The functions are defined on a manifold M (for example, the time τ above, or in another case the coordinate space $\vec{x}$ set of numbers, or maybe the 4-d space consisting of $\tau, \vec{x}$). If the function is n-fold differentiable, then a function on that manifold is denoted $C^n(M)$.

The functional then defines a mapping

$$\text{functional}: \quad C^n(M) \to \mathcal{R} \,. \tag{54}$$

- This differs from a normal *function* that maps a number or set of numbers into a number or set of numbers.

  For example, $\phi(\vec{x})$ maps the set of numbers $\vec{x}$ into a single number, or $\vec{E}(\vec{x})$ maps the set of numbers $\vec{x}$ into another set of numbers, namely the electric field 3-vector at point $\vec{x}$.

$$\text{function}: \mathcal{R}^n \to \mathcal{R}^m \,, \tag{55}$$

- **Anyway, the thing that we need is the concept of a** *functional derivative*. **We define the derivative of a functional $F[f]$ with respect to the function $f(y)$ as**

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} . \tag{56}$$

A specific example. Consider

$$F[f] = \int f(x) dx . \tag{57}$$

Then,

$$\frac{\delta F[f]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} = \int \delta(x-y) dx = 1 . \tag{58}$$

As a 2nd example, consider

$$F_x[f] = \int G(x,y) f(y) dy . \tag{59}$$

Then,

$$\frac{\delta F_x[f]}{\delta f(z)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int \{G(x,y)[f(y) + \epsilon\delta(y-z)]\} \, dy - \int G(x,y) f(y) dy \right)$$
$$= \int G(x,y)\delta(y-z) dy$$
$$= G(x,z) \, . \tag{60}$$

It is this last type of case that will be crucial in what follows.

Back to $Z[J]$

- We had

$$Z[J] = \int dq' dq \phi_0^*(q',t') \langle q',t'|q,t\rangle^J \phi_0(q,t) \tag{61}$$

inside of which

$$\langle q',t'|q,t\rangle^J = \int \left[\frac{dpdq}{2\pi}\right] \exp\left\{ i\int_t^{t'} [p(\tau)\dot q(\tau) - H(p(\tau),q(\tau)) + J(\tau)q(\tau)] d\tau \right\} \, . \tag{62}$$

Thus, we have

$$
\begin{aligned}
& \lim_{J\to 0}\left[\frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(t_1)\ldots\delta J(t_n)}\right] \\
= \quad & i^n \frac{\int dq dq' \phi_0^*(q',t')\phi_0(q,t)\int\left[\frac{dpdq}{2\pi}\right]\exp\left\{i\int_t^{t'}[p\dot{q}-H(p,q)]d\tau\right\}q(t_1)\ldots q(t_n)}{\int dq dq' \phi_0^*(q',t')\phi_0(q,t)\int\left[\frac{dpdq}{2\pi}\right]\exp\left\{i\int_t^{t'}[p\dot{q}-H(p,q)]d\tau\right\}} \\
= \quad & i^n \frac{\langle 0|T\left\{Q(t_1)\ldots Q(t_n)\right\}|0\rangle}{\langle 0|0\rangle}\,, \qquad (63)
\end{aligned}
$$

where we always need to remember that $H(p,q)$ contains the $-\frac{1}{2}i\epsilon q^2$ term that isolates the ground state to ground state transition denoted in Ryder as $|0,-\infty\rangle \to |0,+\infty\rangle$.

- For a quadratically completable $H(p,q)$, the p integral can be performed as done earlier (the $-\frac{1}{2}i\epsilon q^2$ addition to H was chosen so as to not interfere with this) yielding

$$
Z[J] = \langle 0,+\infty|0,-\infty\rangle^J \propto \int [dq]\exp\left\{i\int_{-\infty}^{+\infty}d\tau(L+Jq+\frac{1}{2}i\epsilon q^2)\right\}\,, \qquad (64)
$$

where all quantities under the $d\tau$ integral are functions of τ.

On to Field Theory

- Let us now treat, just as we did in the earlier commutator technique for 2nd quantization, the abstract field $\phi(x)$ (for the moment we discuss the scalar field case) as a coordinate in the sense that we imagine dividing space up into many little cubes and the average value of the field $\phi(x)$ in that cube is treated as a coordinate for that little cube (just as we might use the multi-coordinate generalization discussed earlier).

  Then, we go through the multi-coordinate analogue of the procedure we just considered and take the continuum limit.

  The result would be

$$Z[J] \propto \int [d\phi] \exp\left\{ i \int d^4x \left[\mathcal{L}(\phi) + J(x)\phi(x) + \frac{1}{2} i\epsilon\phi^2 \right] \right\} \tag{65}$$

  where for $\mathcal{L}$ we would employ the Klein Gordon Lagrangian form.

  In the above, the dx_0 integral is the same as $d\tau$, while the $d^3\vec{x}$ integral is simply summing over the sub-Lagrangians of all the different little cubes of space and then taking the continuum limit. $\mathcal{L}$ is the Lagrangian density describing the Lagrangian for each little cube after taking the many-cube limit.

- Perhaps this is completely obvious. But let me present a direct derivation.

  Recall the generalization to many degrees of freedom given earlier. Normal commutator quantization of these many independent degrees of freedom is clearly equivalent to the path-integral computation represented by (H subscripts on the 'Kernel' or overlap bra and ket are implicit as always)

$$
\begin{aligned}
&\langle q_1' \ldots q_N', t' | q_1 \ldots q_N, t \rangle \\
&\quad = \int \prod_{m=1}^{N} \left[\frac{dq_m dp_m}{2\pi\hbar} \right] \exp\left\{ \frac{i}{\hbar} \int_t^{t'} \Big[\sum_{n=1}^{N} p_n \dot{q}_n - H(\{p_i\}, \{q_i\}) \Big] d\tau \right\},
\end{aligned}
\tag{66}
$$

where $\{p_i\}$ denotes the set of all the momenta $p_1, \ldots, p_N$, and similarly $\{q_i\}$ denotes the set $q_1, \ldots, q_N$. As usual, such an expression is actually defined by the discretized version where we go back and discretize the integral over all possible functional forms for the q_n and p_n functions by dividing up into many steps in time:

$$\begin{aligned}
&\langle q'_1 \ldots q'_N, t' | q_1 \ldots q_N, t\rangle = \\
&\int \prod_{\gamma=1}^{N} \lim_{n\to\infty, \epsilon\to 0} \left[\prod_{i=1}^{n} [dq_\gamma(t_i)] \prod_{i=1}^{n+1} \left[\frac{dp_\gamma(t_i)}{2\pi}\right]\right] \\
&\times \exp\left[i \sum_{j=1}^{n} \left\{ \sum_{\alpha=1}^{N} p_\alpha(t_j)[q_\alpha(t_j) - q_\alpha(t_{j-1})] - \epsilon H\left(p_\alpha(t_j), \frac{q_\alpha(t_j)+q_\alpha(t_{j-1})}{2}\right)\right\}\right].
\end{aligned} \tag{67}$$

- This can be directly applied to field theory as suggested earlier by dividing space into cubes of size ϵ and defining an average value ϕ_α for each little cube:

$$\phi_\alpha(t) = \frac{1}{\epsilon^3}\int_{V_\alpha} d^3\vec{x}\,\phi(\vec{x}, t) \to q_\alpha(t)\,. \tag{68}$$

We can then approximate the Lagrangian by

$$L = \int d^3\vec{x}\,\mathcal{L} = \sum_\alpha \epsilon^3 \mathcal{L}_\alpha\left(\dot{\phi}_\alpha(t), \phi_\alpha(t), \phi_{\alpha\pm s}(t)\right)\,, \tag{69}$$

where $\mathcal{L}_\alpha$ is obviously a Lagrangian density, and the conjugate momentum for cell α will be

$$p_\alpha(t) = \frac{\partial L}{\partial \dot{\phi}_\alpha(t)} = \epsilon^3 \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\phi}_\alpha(t)} \equiv \epsilon^3 \pi_\alpha(t) \,. \tag{70}$$

Similarly, we will have

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L = \sum_\alpha \epsilon^3 \mathcal{H}_\alpha \,, \tag{71}$$

where

$$\mathcal{H}_\alpha = \pi_\alpha \dot{\phi}_\alpha - \mathcal{L}_\alpha \,. \tag{72}$$

We now imagine 2nd quantizing this system by assuming the usual $[q_\alpha, p_\beta] \to \epsilon^3[\phi_\alpha, \pi_\beta] = i\delta_{\alpha\beta}$, where α and β are labels for different cells of size ϵ^3. Making these replacements in our multi-coordinate path integral generalization that we have shown is equivalent to commutator quantization yields the form (where $\{\phi_\alpha\}$ denotes the set of average fields values in all the spatial cells labeled by the index α)

$$
\begin{aligned}
&\langle\{\phi_\alpha(t')\}|\{\phi_\alpha(t)\}\rangle \\
= \quad & \lim_{n\to\infty,\epsilon\to 0}\int\prod_{\gamma=1}^{N}\left[\prod_{i=1}^{n} d\phi_\gamma(t_i)\prod_{i=1}^{n+1}\frac{\epsilon^3 d\pi_\gamma(t_i)}{2\pi}\right] \\
& \exp\left[i\sum_{j=1}^{n+1}\epsilon\sum_\alpha\epsilon^3\left[\pi_\alpha(t_j)\frac{\phi_\alpha(t_j)-\phi_\alpha(t_{j-1})}{\epsilon}-\mathcal{H}_\alpha\left(\pi_\alpha(t_j),\frac{\phi_\alpha(t_j)+\phi_\alpha(t_{j-1})}{2},\ldots\right)\right]\right] \\
\equiv \quad & \int[d\phi]\left[\frac{\epsilon^3 d\pi}{2\pi}\right]\exp\left[i\int_t^{t'}d\tau\int d^3\vec{x}\left[\pi(\vec{x},\tau)\frac{\partial\phi(\vec{x},\tau)}{\partial\tau}-\mathcal{H}(\vec{x},\tau)\right]\right], \qquad (73)
\end{aligned}
$$

where $\pi(\vec{x},t)=\frac{\partial\mathcal{L}}{\partial\dot{\phi}(\vec{x},t)}$ the cell average of which is just the $\pi_\alpha(t)$ defined earlier.

- We now move from the above to $Z[J]$ by adding $-\frac{1}{2}i\epsilon\phi_\alpha^2$ to each $\mathcal{H}_\alpha$ and by adding a $J_\alpha\phi_\alpha$ for each cell yielding the result:

$$
Z[J]\propto\int[d\phi]\left[\frac{\epsilon^3 d\pi}{2\pi}\right]\exp\left[i\int d^4x\left[\pi(x)\dot{\phi}(x)-\mathcal{H}(x)+\frac{1}{2}i\epsilon\phi^2(x)+J(x)\phi(x)\right]\right]. \qquad (74)
$$

Further, from the QM example, we know that

$$\left[\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right]_{J=0} \propto i^n \langle 0|T\left\{\phi(x_1)\ldots\phi(x_n)\right\}|0\rangle\,. \tag{75}$$

- As in the QM case, if $\mathcal{H}$ is simple enough, we can perform the $d\pi$ integral(s) explicitly. In particular, if

$$\mathcal{H} = \frac{1}{2}\pi^2(x) + f[\phi(x), \vec{\nabla}\phi(x)] \tag{76}$$

we have $d\pi$ integrals of the form

$$\begin{aligned} & \int\left[\frac{d\pi}{2\pi}\right]\exp\left[i\left(\int d^4x[\pi(x)\dot\phi(x) - \frac{1}{2}\pi^2(x)]\right)\right] \\ = & \int\left[\frac{d\pi}{2\pi}\right]\exp\left[\int d^4x\left[-\frac{1}{2}i(\pi-\dot\phi)^2 + \frac{1}{2}i\dot\phi^2\right]\right] \\ \propto & \exp\left[i\int d^4x\frac{1}{2}\dot\phi^2\right], \end{aligned} \tag{77}$$

where we have really done these integrals cell-by-cell and then converted to the continuum limit. In the above, cell-by-cell means the following:

$$\begin{aligned} & \int \left[\frac{d\pi}{2\pi}\right] \exp\left[\int d^4x \left[-\frac{1}{2}i(\pi-\dot{\phi})^2+\frac{1}{2}i\dot{\phi}^2\right]\right] \\ = & \int \prod_{\gamma=1}^{N}\prod_i \frac{d\pi_\gamma(t_i)}{2\pi} \exp\left[\sum_j \sum_{\alpha=1}^{N} \epsilon^4 \left[-\frac{1}{2}i(\pi_\alpha(t_j)-\dot{\phi}_\alpha(t_j))^2+\frac{1}{2}i\dot{\phi}^2_\alpha(t_j)\right]\right] \\ = & \prod_{\gamma=1}^{N}\prod_i \left(\frac{1}{\sqrt{2\pi i\epsilon^4}}\right) \exp\left[\sum_j \sum_{\alpha=1}^{N} \epsilon^4 \frac{1}{2}i\dot{\phi}^2_\alpha(t_j)\right] \propto \exp\left[i\int d^4x \frac{1}{2}\dot{\phi}^2\right]. \end{aligned} \quad (78)$$

The result is

$$Z[J] \propto \int [d\phi] \exp\left\{i\int d^4x[\mathcal{L}(x)+J(x)\phi(x)]\right\} \quad (79)$$

with

$$\mathcal{L}(x) = \frac{1}{2}(\partial_0\phi)^2 - f(\phi(x),\vec{\nabla}\phi(x)) + \frac{1}{2}i\epsilon\phi^2(x)\,, \quad (80)$$

where the $\frac{1}{2}(\partial_0\phi)^2$ term in $\mathcal{L}(x)$ simply comes from the residual

$$\exp\left[i\int d^4x\frac{1}{2}\dot{\phi}^2\right] \tag{81}$$

obtained above.

As always, the exact form of the proportionality constant will be irrelevant in our applications.

- For the example of Klein Gordon theory, we would use

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I\,, \quad \mathcal{L}_0 = \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2]\,, \quad \mathcal{L}_I = \mathcal{L}_I(\phi)\,. \tag{82}$$

- In order to define the above $Z[J]$, we have to include the standard convergence factor that we are using to isolate the vacuum to vacuum transition. The result is

$$\mathcal{L}_0 \to \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2 + i\epsilon\phi^2] \tag{83}$$

so that

$$Z_0[J] \propto \int [d\phi] \exp\left\{i \int d^4x \left(\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2 + i\epsilon\phi^2] + J\phi\right)\right\} \tag{84}$$

is the appropriate generating function in the free field theory case.

- To actually compute this object, we first of all perform a partial integration on the $\partial_\mu\phi\partial^\mu\phi$ term to obtain

$$\begin{aligned} Z_0[J] &\propto \int [d\phi] \exp\left\{i \int d^4x \left(\frac{1}{2}\phi[-\partial^2 - \mu^2 + i\epsilon]\phi + J\phi\right)\right\} \\ &= \int [d\phi] \exp\left\{i \int d^4x \int d^4y \left(\frac{1}{2}\phi(x)[-\partial^2 - \mu^2 + i\epsilon]\delta^4(x-y)\phi(y)\right) + i\int d^4x J(x)\phi(x)\right\} \end{aligned} \tag{85}$$

and then we have to go back to dividing space-time up into cells (where in the following equation λ, α and β are labels for space-time cells — e.g. $\lambda \equiv (\gamma, i)$ of earlier equations, where γ is a space cell index and i denotes a particular one of the time divisions)

$$Z_0[J] = \lim_{\epsilon\to 0} \int \prod_\lambda d\phi_\lambda \exp\left[i\left\{\sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \frac{1}{2}\phi_\alpha K_{\alpha\beta}\phi_\beta + \sum_\alpha \epsilon^4 J_\alpha \phi_\alpha\right\}\right], \tag{86}$$

where

$$\lim_{\epsilon\to 0} K_{\alpha\beta} = (-\partial^2 - \mu^2 + i\epsilon)\delta^4(x-y)\,. \tag{87}$$

We perform the $d\phi_\lambda$ integrations explicitly by completing the square (see later and Ryder 6.2) to obtain

$$Z_0[J] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\det K_{\alpha\beta}}} \prod_\alpha \sqrt{\frac{2\pi i}{\epsilon^8}} \exp\left\{-\frac{1}{2} i \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 J_\alpha \frac{(K^{-1})_{\alpha\beta}}{\epsilon^8} J_\beta\right\}. \tag{88}$$

Here, K^{-1} is defined by

$$\sum_\gamma \left[\epsilon^4 K_{\alpha\gamma}\right]\left[\frac{(K^{-1})_{\gamma\beta}}{\epsilon^8}\right] = \frac{\delta_{\alpha\beta}}{\epsilon^4}\,, \tag{89}$$

where I have inserted some ϵ powers for the appropriate identifications below. In particular, as $\epsilon\to 0$ we have

$$\frac{1}{\epsilon^4}\delta_{\alpha\beta} \to \delta^4(x-y)\,, \quad \sum_\alpha \epsilon^4 \to \int d^4x\,,\,, \tag{90}$$

as can be checked from the identity:

$$1 = \int d^4x \delta^4(x-y) \to \sum_\alpha \epsilon^4 \frac{1}{\epsilon^4} \delta_{\alpha\beta} = 1 \,. \tag{91}$$

So, let us define

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^8} (K^{-1})_{\alpha\beta} = \Delta_F(x-y) \tag{92}$$

so that Eq. (89) becomes in the $\epsilon \to 0$ limit

$$\int d^4z (-\partial_x^2 - \mu^2 + i\epsilon) \delta^4(x-z) \Delta_F(z-y) = \delta^4(x-y) \tag{93}$$

which is equivalent to

$$(-\partial_x^2 - \mu^2 + i\epsilon) \Delta_F(x-y) = \delta^4(x-y) \,, \tag{94}$$

the defining equation for the Feynman propagator function. Thus, in the $\epsilon \to 0$ limit we have

$$Z_0[J] \propto \exp\left\{ -\frac{1}{2} i \int d^4x \int d^4y J(x) \Delta_F(x-y) J(y) \right\} \,, \tag{95}$$

where (as before) the solution to Eq. (94) is the Feynman propagator form

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{k^2-\mu^2+i\epsilon} \,. \tag{96}$$

Finally, the appropriate normalization (as we have seen earlier and will review below) is to define the proportionality constant so that $Z_0[J=0]=1$, i.e.

$$Z_0[J] = \exp\left\{-\frac{1}{2}i\int d^4x\int d^4y J(x)\Delta_F(x-y)J(y)\right\} \,. \tag{97}$$

This is effectively equivalent to normal ordering. With this normalization,

$$\left[\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right]_{J=0} = i^n\langle 0|T\left\{\phi(x_1)\ldots\phi(x_n)\right\}|0\rangle \,. \tag{98}$$

- Let us check this by showing that $\langle 0|T\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle = i\Delta_F(x_1-x_2)$ when the l.h.s. is computed using the functional techniques. We have

$$\begin{aligned}\langle 0|T\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle &= i^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z_0[J]\Big|_{J=0}\\ &= \left(i^2\frac{\delta}{\delta J(x_1)}\left[\left(-i\int d^4x' J(x')\Delta_F(x'-x_2)\right)\exp\left\{-\frac{1}{2}i\int d^4x\int d^4y J(x)\Delta_F(x-y)J(y)\right\}\right]\right)_{J=0}\end{aligned}$$

where the factor of $\frac{1}{2}$ was canceled by virtue of the fact that we could differentiate either of the two $J's$ in the exponential.

Turning to the 2nd derivative, it can act either on the stuff multiplying the exponential or on the exponential itself.

– If it acts on the exponential, another factor just like the 1st multiplying $(\ldots)$ is brought down, and each of these $(\ldots)$ vanishes when $J = 0$.
– If it acts on the existing $(\ldots)$, then we get

$$
\begin{aligned}
& \langle 0| T \{\phi(x_1)\phi(x_2)\} |0\rangle \\
= \quad & \left(i^2 \left(-i\Delta_F(x_1 - x_2)\right) \exp\left\{ -\frac{1}{2} i \int d^4x \int d^4y J(x) \Delta_F(x-y) J(y) \right\} \right)_{J=0} \\
= \quad & i\Delta_F(x_1 - x_2)
\end{aligned}
\tag{99}
$$

which is the required result.

- Before going on to introduce interactions, let us return to the derivation that takes us from

$$
Z_0[J] = \lim_{\epsilon \to 0} \int \prod_\lambda d\phi_\lambda \exp\left[i \left\{ \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \frac{1}{2} \phi_\alpha K_{\alpha\beta} \phi_\beta + \sum_\alpha \epsilon^4 J_\alpha \phi_\alpha \right\} \right], \tag{100}
$$

to

$$Z_0[J] = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\det K_{\alpha\beta}}} \prod_\alpha \sqrt{\frac{2\pi i}{\epsilon^8}} \exp\left\{ -\frac{1}{2} i \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 J_\alpha \frac{(K^{-1})_{\alpha\beta}}{\epsilon^8} J_\beta \right\}, \tag{101}$$

by square completion.

We begin by using vector notation:

$$\epsilon^4 \phi_\alpha = \vec{x} \equiv x \,, \tag{102}$$

where the final object x is to be thought of as a column matrix with let us say n (the total number of cells) components. Then, we can write (^{T} means transpose)

$$\begin{aligned} & \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \frac{1}{2} \phi_\alpha K_{\alpha\beta} \phi_\beta + \sum_\alpha \epsilon^4 J_\alpha \phi_\alpha \\ = \;& \frac{1}{2}[x^T K x] + J^T x \equiv \frac{1}{2}(x, Kx) + (J, x) \\ = \;& \frac{1}{2}(x + K^{-1}J - K^{-1}J, K(x + K^{-1}J - K^{-1}J)) + (J, x) \end{aligned}$$

$$= \frac{1}{2}(x + K^{-1}J, K(x + K^{-1}J)) - \frac{1}{2}(J, K^{-1}J)\,. \tag{103}$$

In writing this, we must remember that we are in a real vector space and that K is a real (in the limit $\epsilon \to 0$) symmetric matrix. As a result, K^{-1} is also a real symmetric matrix, i.e. $K^{-1} = [K^{-1}]^T$ and, for example,

$$(K^{-1}J, Kx) \equiv (K^{-1}J)^T KX = J^T[K^{-1}]^T KX = J^T K^{-1} Kx = J^T x \equiv (J, x) \tag{104}$$

$$(K^{-1}J, KK^{-1}J) \equiv (K^{-1}J)^T J = J^T[K^{-1}]^T J = J^T K^{-1} J \equiv (J, K^{-1}J)\,. \tag{105}$$

We shift variables to $\bar{x} = x + K^{-1}J$.

The original integral then reduces to

$$\exp\left\{-i\frac{1}{2}(J, K^{-1}J)\right\}\left(\frac{1}{\epsilon^4}\right)^n \int d^n\bar{x} e^{i\frac{1}{2}(\bar{x}, K\bar{x})} \tag{106}$$

To evaluate we recall that

$$\int e^{-ay^2/2} dy = \left(\frac{2\pi}{a}\right)^{1/2}, \tag{107}$$

which implies that

$$\int \exp\left(-\frac{1}{2}\sum_{i=1}^{n} a_i y_i^2\right) dy_1 \dots dy_n = \frac{(2\pi)^{n/2}}{\prod_{i=1}^{n} a_i^{1/2}}. \tag{108}$$

Now, let A be a diagonal matrix with elements $a_1, \dots a_n$, and let y be an n-vector $(y_1, \dots, y_n)$. Then, the exponent is the inner product:

$$\sum_{i=1}^{n} a_i y_i^2 = (y, Ay) \tag{109}$$

and the determinant of A is

$$\det A = \prod_{i=1}^{n} a_i. \tag{110}$$

With this identification, we can rewrite Eq. (108) as

$$\int e^{-(y,Ay)/2}(d^n y) = (2\pi)^{n/2} (\det A)^{-1/2} . \tag{111}$$

Since this holds for any diagonal matrix, it also holds for any real symmetric, positive, non-singular matrix, since one can always rotate to a diagonal basis without changing the inner product.

Finally, we must include the i factor. Assuming the presence of an appropriate convergence factor, Eq. (111) can be rewritten by defining $\vec{y} = \sqrt{-i}\vec{x}$ to obtain

$$\int e^{i(x,Ax)/2}(d^n\sqrt{-i}x) = (2\pi)^{n/2}(\det A)^{-1/2} , \tag{112}$$

or

$$\int e^{i(x,Ax)/2}(d^n x) = (2\pi i)^{n/2}(\det A)^{-1/2} . \tag{113}$$

At this point, in our earlier expression we identify A with K, x with $\bar{x}$, and n with the number of cells labeled by α to obtain

$$\begin{aligned} & \exp\left\{-i\frac{1}{2}(J,K^{-1}J)\right\}\left(\frac{1}{\epsilon^4}\right)^n\int d^n\bar{x}e^{i\frac{1}{2}(\bar{x},K\bar{x})} \\ = \; & \exp\left\{-i\frac{1}{2}(J,K^{-1}J)\right\}\left(\prod_\alpha\sqrt{\frac{2\pi i}{\epsilon^8}}\right)(\det K_{\alpha\beta})^{-1/2} . \end{aligned} \tag{114}$$

In this way, we obtain Eq. (101), i.e.

$$Z_0[J] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\det K_{\alpha\beta}}} \prod_\alpha \sqrt{\frac{2\pi i}{\epsilon^8}} \exp\left\{-\frac{1}{2} i \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 J_\alpha \frac{(K^{-1})_{\alpha\beta}}{\epsilon^8} J_\beta\right\}, \tag{115}$$

which we evaluated and found

$$Z_0[J] = \exp\left\{-\frac{1}{2} i \int d^4x \int d^4y J(x) \Delta_F(x-y) J(y)\right\}, \tag{116}$$

when the normalization is chosen so that $Z_0[J=0]=1$.

- We are now ready to introduce interactions. Our starting point is, assuming the simple form of the Hamiltonian,

$$Z[J] = \int [d\phi] \exp\left\{ i \int d^4x [\mathcal{L}_0(\phi(x)) + \mathcal{L}_I(\phi(x)) + J(x)\phi(x)]\right\}. \tag{117}$$

We evaluate this using a power expansion technique:

$$Z[J] = \int [d\phi] \exp\left\{ i \int d^4x [\mathcal{L}_0(\phi(x)) + \mathcal{L}_I(\phi(x)) + J(x)\phi(x)]\right\}$$

$$
\begin{aligned}
&= \int [d\phi] \sum_n \frac{1}{n!} \left[i \int \mathcal{L}_I(\phi(z)) d^4 z \right]^n \exp\left\{ i \int d^4x [\mathcal{L}_0(\phi(x)) + J(x)\phi(x)] \right\} \\
&= \int [d\phi] \sum_n \frac{1}{n!} \left[i \int \mathcal{L}_I \left(\frac{\delta}{i\delta J(z)} \right) d^4 z \right]^n \exp\left\{ i \int d^4x [\mathcal{L}_0(\phi(x)) + J(x)\phi(x)] \right\} \\
&= \exp\left[i \int \mathcal{L}_I \left(\frac{\delta}{i\delta J(z)} \right) d^4 z \right] \int [d\phi] \exp\left\{ i \int d^4x [\mathcal{L}_0(\phi(x)) + J(x)\phi(x)] \right\} \\
&= \frac{\exp\left[i \int \mathcal{L}_I \left(\frac{\delta}{i\delta J(z)} \right) d^4 z \right] \exp\left\{ -\frac{1}{2} i \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right\}}{\left(\exp\left[i \int \mathcal{L}_I \left(\frac{\delta}{i\delta J(z)} \right) d^4 z \right] \exp\left\{ -\frac{1}{2} i \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right\} \right)_{J=0}}, \qquad (118)
\end{aligned}
$$

where in the last line I have introduced the normalization factor required for $Z[J=0]=1$.

Obviously, the expansion in the second line above actually defines the perturbation theory expansion in powers of $\mathcal{L}_I$.

Some higher order expansion in the free-field case.

- Before setting $J=0$, we had (shifting $x_1 \to x_2$ and $x_2 \to x_3$, and shortening d^4z to dz etc.)

$$
\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_3)} Z_0[J] = i\Delta_F(x_2 - x_3) \exp\left(-\frac{i}{2} \int J \Delta_F J \right)
$$

$$+\int \Delta_F(x_2-x')J(x')dx' \int \Delta_F(x_3-y')J(y')dy' \exp\left(-\frac{i}{2}\int J\Delta_F J\right) . \tag{119}$$

Further differentiation then gives

$$\begin{aligned}
&\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_3)}Z_0[J] = \\
&-i\Delta_F(x_2-x_3)\int \Delta_F(x_1-x')J(x')dx' \exp\left(-\frac{i}{2}\int J\Delta_F J\right) \\
&-i\Delta_F(x_2-x_1)\int \Delta_F(x_3-y')J(y')dy' \exp\left(-\frac{i}{2}\int J\Delta_F J\right) \\
&-i\Delta_F(x_3-x_1)\int \Delta_F(x_2-x')J(x')dx' \exp\left(-\frac{i}{2}\int J\Delta_F J\right) \\
&-\int \Delta_F(x_2-x')J(x')dx' \int \Delta_F(x_3-y')J(y')dy' \int \Delta_F(x_1-z')J(z')dz' \exp\left(-\frac{i}{2}\int J\Delta_F J\right) .
\end{aligned} \tag{120}$$

In the $J=0$ limit, this gives 0, implying that

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle = 0 . \tag{121}$$

Going to the next level of differentiation, we get

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_3)}\frac{1}{i}\frac{\delta}{\delta J(x_4)}Z_0[J] =$$

$$
\begin{aligned}
&-\Delta_F(x_2-x_3)\Delta_F(x_1-x_4)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\\
&-\Delta_F(x_2-x_1)\Delta_F(x_3-x_4)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\\
&-\Delta_F(x_3-x_1)\Delta_F(x_2-x_4)\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\\
&+(\text{terms which vanish when } J=0)\,. \qquad (122)
\end{aligned}
$$

As a result, we have (using $-1 = i^2$)

$$
\begin{aligned}
&\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\
&=\Big[i\Delta_F(x_2-x_3)i\Delta_F(x_1-x_4)\\
&+i\Delta_F(x_2-x_1)i\Delta_F(x_3-x_4)\\
&+i\Delta_F(x_3-x_1)i\Delta_F(x_2-x_4)\Big]\,, \qquad (123)
\end{aligned}
$$

which is precisely the statement of Wick's theorem in the non-interacting case.

- Let us now consider the interacting case, using the example of $\mathcal{L}_I =$

$-\frac{\lambda}{4!}\phi^4$, **and keeping only the first order term in the expansion:**

$$\exp\left[i\int \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)dz\right] Z_0[J]$$
$$= \left[1 - \frac{i\lambda}{4!}\int\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 dz + \mathcal{O}(\lambda^2)\right]\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right) \quad (124)$$

To order λ^0, we just have the free-particle generating function and the free particle results just discussed. Let us look, therefore, at the order λ^1 term. We have:

$$\frac{1}{i}\frac{\delta}{\delta J(z)}\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right)$$
$$= -\int \Delta_F(z-x')J(x')dx' \exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right) \quad (125)$$

$$\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^2\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right)$$
$$= \left\{i\Delta_F(0) + \left[\int \Delta_F(z-x')J(x')dx'\right]^2\right\}\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right) \quad (126)$$

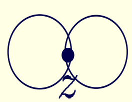
$$\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^3\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right)$$

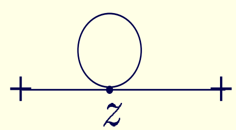
$$= \left\{ 3[-i\Delta_F(0)] \int \Delta_F(z-x')J(x')dx' - \left[\int \Delta_F(z-x')J(x')dx'\right]^3 \right\} \times$$
$$\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right) \tag{127}$$

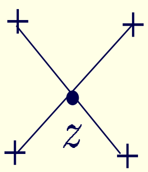
$$\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 \exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right)$$
$$= \left\{ -3[\Delta_F(0)]^2 + 6i\Delta_F(0)\left[\int \Delta_F(z-x')J(x')dx'\right]^2 + \left[\int \Delta_F(z-x')J(x')dx'\right]^4 \right\} \times$$
$$\exp\left(-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)dxdy\right) \tag{128}$$

We may represent this result diagrammatically as

$$\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 \exp\left\{-\frac{i}{2}\int J\Delta_F J\right\}$$

$= [-3$ 

$+6i$ 

$+$ 

$] \exp\left\{-\frac{i}{2}\int J\Delta_F J\right\}$

Figure 1: Graphical representation of the ϕ^4 interaction; $+$'s indicate J attachments and closed circle $= \Delta_F(0)$.

Altogether, $Z[J]$, including the λ^0 and λ^1 terms and normalizing to $Z[J=0]=1$ is $Z[J]=$

$$\frac{\left[1-\frac{i\lambda}{4!}\int dz\left\{-3[\Delta_F(0)]^2+6i\Delta_F(0)\left[\int\Delta_F(z-x)J(x')dx'\right]^2+\left[\int\Delta_F(z-x')J(x')dx'\right]^4\right\}\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)}{\left[1-\frac{i\lambda}{4!}\int dz\left\{-3[\Delta_F(0)]^2\right\}\right]}$$

$$\sim\left[1-\frac{i\lambda}{4!}\int dz\left\{6i\Delta_F(0)\left[\int\Delta_F(z-x')J(x')dx'\right]^2+\left[\int\Delta_F(z-x')J(x')dx'\right]^4\right\}\right]\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\,, \quad (12$$

where, in the last line, I have dropped extra $\mathcal{O}(\lambda^2)$ terms from the denominator expansion.

- We must now use this expression for $Z[J]$ to derive expressions for the time-ordered products of fields in the presence of the interactions. This means we must do further functional derivatives.

  Let us begin with the 2-point function

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle=\frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)}\bigg|_{J=0}\,. \qquad (130)$$

  From the above expression for $Z[J]$, we observe that the $[1\ldots]$ term simply gives the free particle propagator. The interesting piece is the correction coming from the λ^1 terms. Of the two terms of this order, the $\left[\int \Delta_F(z-x')J(x')dx'\right]^4$ term can give no contribution after setting $J=0$ since the J derivatives can remove at most two of the J's. Thus, the only term of interest in the $J=0$ limit will be the double derivative of

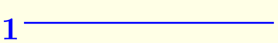
$$\frac{\lambda}{4}\Delta_F(0)\int dx'dy'dz\Delta_F(z-x')J(x')\Delta_F(z-y')J(y')\exp\left(-\frac{i}{2}\int J\Delta_F J\right) \tag{131}$$

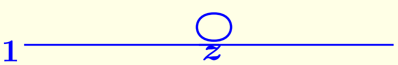
The only term that will survive for $J=0$ after the double differentiation is that in which the two J derivatives both act on the external multiplicative factor. This term takes the form:

$$\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}(\ldots)\Big|_{J=0} = -\frac{\lambda}{4}\Delta_F(0)2\int dz\Delta_F(z-x_1)\Delta_F(z-x_2)\,. \tag{132}$$

Combining with the λ^0 term, we obtain

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= i\Delta_F(x_1-x_2)-\frac{\lambda}{2}\Delta_F(0)\int dz\Delta_F(z-x_1)\Delta_F(z-x_2) \\ &= i\ x_1 \text{———} x_2\ -\frac{\lambda}{2}\ x_1 \text{———} z \text{———} x_2\ +\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{133}$$





As discussed in Ryder, this causes a shift in the mass-squared of the particle away from the tree-level value of μ^2. Naively, this shift is infinite, but in fact the integrals are cut off at some scale (perhaps the Planck scale or some lower new-physics scale). I will not dwell further on this since it is the subject of renormalization.

Instead, I want to turn to the tree-level expression for the 4-point interaction.

- The 4-point vertex function

  We want

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = \frac{1}{i^4}\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} \tag{134}$$

- Starting from the master form for $Z[J]$ of Eq. (129), the 1 term in $[1\ldots]$ simply gives us the free-particle 4-point function that we discussed earlier.

- The next term is

$$\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\frac{\lambda}{4}\left\{\Delta_F(0)\int dx' dy' dz \Delta_F(x'-z)\Delta_F(y'-z)J(y')J(x')\right\}\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\bigg|_{J=0}$$

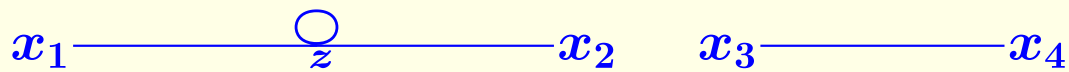
$$= -i\frac{\lambda}{2}\Delta_F(0)\int dz\, \left[\Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(x_3-x_4) + \text{5 more terms from permutations}\right] \, . \tag{135}$$

The terms in the last line above are obtained when two of the $\delta/\delta J$'s act on the external factor (which yields an extra factor of 2 since the two $\delta/\delta J$'s that act on the external factor can act on either of the J's appearing in the external factor yielding identical final result), the 3rd $\delta/\delta J$ acts on the exponential, and the final $\delta/\delta J$ acts on the term brought down from the exponential by the action of the 3rd one above.

The structure of the above is (referring to the 1st term written explicitly) a free propagator from x_3 to x_4 times a bubble correction to the propagator from x_1 to x_2.

Note that these terms have a "disconnected" form in which one particle just passes straight to the final state without interaction.

Diagrammatically, the first term can be represented as

$-i\frac{\lambda}{2}\Big[$ 

$\Big]$, (136)

and there are 5 more terms with the roles of the $x_{1,2,3,4}$ permuted to give distinct structures.

- **It is the last term that is most interesting as it gives the Feynman vertex in which 4 particles interact. We have (dropping all the prime notation which is getting cumbersome)**

$$\left[\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\times\right.$$

$$\left.\left[-i\frac{\lambda}{4!}\left\{\int dzdxdydsdt\Delta_F(x-z)J(x)\Delta_F(y-z)J(y)\Delta_F(s-z)J(s)\Delta_F(t-z)J(t)\right\}\exp\left(-\frac{i}{2}\int J\Delta_F J\right)\right]\right]_{J=0}$$

$$= -i\lambda\int dz\Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(z-x_3)\Delta_F(z-x_4)\,. \qquad (137)$$

Note that the $4!$ in the denominator was canceled by the $4!$ ways of doing the derivatives, all of which give the same answer.

This term clearly has a connected form in which all the external particles come together and interact at point z.

$$-i\lambda\ \left[\begin{matrix} x_1 & & x_3 \\ & \overset{z}{\times} & \\ x_2 & & x_4\end{matrix}\right] \qquad (138)$$

The above is the same result as we had in our other approach. It simply contains the external propagators that connect the external coordinate

locations to the central vertex point at z. The basic vertex Feynman rule is encoded in the $-i\lambda$ that multiplies these connecting free-particle propagators.

- It is worth noting that we can automatically avoid generating the disconnected diagrams if we define

$$W[J] = -i \ln Z[J] \tag{139}$$

and take functional derivatives of $W[J]$. This is discussed in Ryder, but I will not go into the details in these notes.

What you want to "take home" is the means we employed for getting the basic Feynman rule for the vertex of the perturbation theory.

Functional Methods for Fermions

- Clearly, we are going to have to somehow incorporate the idea of Fermi statistics when generalizing our functional techniques to fermions.

  The generalization is based upon Grassmann algebra, which is the algebra and mathematics of anti-commuting c-numbers.

- This algebra begins with introducing a set of generators C_i of an n-dimensional Grassmann algebra obeying

$$\{C_i, C_j\} \equiv C_i C_j + C_j C_i = 0\,, \tag{140}$$

  where $i, j = 1, \ldots, n$. In particular, $C_i^2 = 0$. The expansion of a function as a Taylor series in these new "coordinates" terminates; for example, for $n = 2$, we have

$$f(C_1, C_2) = a_0 + a_1 C_1 + a_2 C_2 + a_3 C_1 C_2 = a_0 + a_1 C_1 + a_2 C_2 - a_3 C_2 C_1 \tag{141}$$

where the a_i might or might not be ordinary c-numbers (possibly functions of the regular coordinates x). There are no terms such as $C_1^2 C_2$, etc. If the function f is to be a true c-number itself, i.e. not some kind of mixed part-Grassmann and part non-Grassmann, then a_1 and a_2 should also be Grassmann objects.

- We now need to define differentiation and integration in the Grassmann coordinates. Left differentiation will differ from right differentiation. The appropriate definitions are (assuming for the moment that all the a_i are simple c-numbers):

$$\frac{\partial f}{\partial C_1} = \frac{\partial^L f}{\partial C_1} = a_1 + a_3 C_2 \,, \quad \frac{\partial f}{\partial C_2} = \frac{\partial^L f}{\partial C_2} = a_2 - a_3 C_1 \,. \tag{142}$$

Consistency of the above then requires that the derivative with respect to one coordinate *anti-commutes* with the other coordinate since from the 1st and 2nd equations we get

$$\frac{\partial}{\partial C_2}\left(\frac{\partial f}{\partial C_1}\right) = a_3 \,, \qquad \frac{\partial}{\partial C_1}\left(\frac{\partial f}{\partial C_2}\right) = -a_3 \,, \tag{143}$$

respectively, so that by summing we find

$$\left[\frac{\partial}{\partial C_2}\frac{\partial}{\partial C_1}+\frac{\partial}{\partial C_1}\frac{\partial}{\partial C_2}\right]f=0\,, \tag{144}$$

for the most general function f.

Of course, the more trivial case of double differentiation with respect to the same C_i gives zero for the most general f. For instance,

$$\frac{\partial}{\partial C_1}\frac{\partial f}{\partial C_1}=\frac{\partial}{\partial C_1}(a_1+a_3C_2)=0 \tag{145}$$

by virtue of the fact that the 2nd $\frac{\partial}{\partial C_1}$ has no C_1 on which to act.

Note that it is also possible to define right differentiation

$$\frac{\partial^R f}{\partial C_1}=a_1-a_3C_2\,, \tag{146}$$

but we will stick to the L differentiation definition, and will not explicitly write the L.

- **Note that**

$$C_1\frac{\partial f}{\partial C_1} = a_1 C_1 + a_3 C_1 C_2\,, \quad C_1 f = a_0 C_1 + a_2 C_1 C_2\,, \quad \frac{\partial}{\partial C_1}(C_1 f) = a_0 + a_2 C_2\,, \qquad (147)$$

implying that

$$\left(C_1\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1}C_1\right) f = f\,, \quad \text{or} \quad C_1\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1}C_1 = 1 \qquad (148)$$

as an operator identity.

- **Similarly,**

$$C_2\frac{\partial f}{\partial C_1} = a_1 C_2\,, \quad C_2 f = a_0 C_2 + a_1 C_2 C_1\,, \quad \frac{\partial}{\partial C_1}(C_2 f) = -a_1 C_2\,, \qquad (149)$$

implying that

$$\left(C_2\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1}C_2\right) f = 0\,, \quad \text{or} \quad C_2\frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1}C_2 = 0 \qquad (150)$$

as an operator identity.

- **Summarizing, we have in general**

$$\left\{C_i, \frac{\partial}{\partial C_j}\right\} = \delta_{ij}\,, \quad \left\{\frac{\partial}{\partial C_i}, \frac{\partial}{\partial C_j}\right\} = 0. \tag{151}$$

(Ryder has corrected the small error in the first case present in earlier editions.)

- **So much for differentiation; what about integration.**

  Clearly the integration differential dC_i also needs to be a Grassmann quantity, and so we take

$$\{dC_i, dC_j\} = 0\,. \tag{152}$$

For $i \neq j$, we will also have $\{C_i, dC_j\} = 0$, but we must be more careful for this latter when $i = j$. Multiple integrals are interpreted as iterated; for example,

$$\int dC_1 dC_2 f(C_1, C_2) \equiv \int dC_1 \left[\int dC_2 f(C_1, C_2)\right]\,. \tag{153}$$

But, what about $\int dC_1$ and $\int dC_1 C_1$? We have

$$\left(\int dC_1\right)^2 = \int dC_1 \int dC_2 = \int\int dC_1 dC_2 = -\int\int dC_2 dC_1 = -\left(\int dC_1\right)^2 \tag{154}$$

where the first equality follows from the standard use of a dummy integration variable and the other equalities follow from the previous identities. The result is that we must have $\left(\int dC_1\right)^2 = 0$ or $\int dC_1 = 0$. Obviously, then

$$\int dC_1 dC_2 = \int dC_1 \int dC_2 = 0 \tag{155}$$

since the two integrals are individually 0. As for the non-zero result of integration, we must be content with a formal definition,

$$\int dC_1 C_1 = 1 \tag{156}$$

and so forth. This is because integration cannot be defined simply as the inverse of differentiation due to the fact that $\left\{\frac{\partial}{\partial C_1}, \frac{\partial}{\partial C_1}\right\} = 0$.

Note that Eq. (156) implies that integration and differentiation give the same result!

The advantages of these definitions are that they preserve some important properties of integrals. In particular, suppose $F = a + bC_1$ is a pure c-number, implying that b is Grassmann. Our definitions imply that $\int dC_1 F = -b$ (we had to pass dC_1 past b). Now, if we translate C_1 by C_2, we find

$$\int dC_1 F(C_1 + C_2) = \int dC_1 (a + bC_1 + bC_2) = -b \tag{157}$$

so that the integral is translation invariant. This property is critical when it comes to fermionic path integrals where we will want to complete a square and then shift integration variables. Without this property, the shifting would not be possible.

Under a multiplicative change of variables, however, something unusual happens. Suppose we take $C_1 \to \tilde{C}_1 = a + bC_1$, where a is Grassmann and b is a regular c-number. Normally in the non-Grassmann context we would find

$$\int d\tilde{x} f(\tilde{x}) = \int dx \left(\frac{d\tilde{x}}{dx}\right) f(\tilde{x}(x))\,. \tag{158}$$

However, in the Grassmann case, let us consider a function $P(C_1) =$

$p_0 + p_1 C_1$, where $p_0 =$ regular, $p_1 =$ Grassmann. We have

$$\int d\tilde{C}_1 (p_0 + p_1 \tilde{C}_1) = \frac{d}{d\tilde{C}_1}(p_0 + p_1 \tilde{C}_1) = -p_1 \tag{159}$$

and

$$\int dC_1 (p_0 + p_1 \tilde{C}_1) = \int dC_1 p_1 b C_1 = -p_1 b \tag{160}$$

(recalling that b is a regular number). As a result, we have

$$\int d\tilde{C}_1 P(\tilde{C}_1) = \int dC_1 P(\tilde{C}_1(C_1)) \frac{1}{b} = \int dC_1 \left(\frac{d\tilde{C}_1}{dC_1} \right)^{-1} P(\tilde{C}_1(C_1)) \,, \tag{161}$$

i.e. we get the opposite of the commuting result; the Jacobian is the reverse of what we might have expected.

- We can generalize this to a case in which we have a bunch of C_i and redefine variables via $\widetilde{C}_i = b_{ij} C_j$. One then finds with $p(C_i) = p_1 C_1 C_2 \ldots C_n$ (only such a term will matter, other lesser products giving zero)

$$\int d\widetilde{C}_n \ldots d\widetilde{C}_1 p(\widetilde{C}_i) = \int dC_n \ldots dC_1 \left[\det \frac{d\widetilde{C}}{dC}\right]^{-1} p(\widetilde{C}(C)) \,. \quad (162)$$

To prove this, write $\widetilde{C}_1 \ldots \widetilde{C}_n = b_{1i_1} \ldots b_{ni_n} C_{i_1} \ldots C_{i_n}$, where the right-hand side is only non-zero if the $i_1 \ldots i_n$ are all different, in which case we have

$$\begin{aligned} \widetilde{C}_1 \ldots \widetilde{C}_n &= b_{1i_1} \ldots b_{ni_n} C_{i_1} \ldots C_{i_n} \\ &= b_{1i_1} \ldots b_{ni_n} \epsilon_{i_1 \ldots i_n} C_1 \ldots C_n \\ &= \det b \; C_1 \ldots C_n \end{aligned} \quad (163)$$

implying that

$$\int d\widetilde{C}_n \ldots d\widetilde{C}_1 p(\widetilde{C}_i) = (\det b)^{-1} \int dC_n \ldots dC_1 p(\widetilde{C}(C)) \,. \quad (164)$$

On the other hand, $\widetilde{C}_i = b_{ij} C_j$ implies that $\frac{d\widetilde{C}_i}{dC_j} = b_{ij}$, which is to say that $\det \frac{d\widetilde{C}_i}{dC_j} = \det b_{ij}$.

- The next generalization is to consider c and $\overline{c}$ to be two independent complex Grassmann quantities. They are independent in the sense that one can think of $c = \frac{1}{\sqrt{2}}(C_1 + iC_2)$ while $\overline{c} = \frac{1}{\sqrt{2}}(C_1 - iC_2)$, where C_1 and C_2 are two Grassmann numbers of the type we have been discussing that are independent of one another. In other words, to fully specify c or $\overline{c}$ you must specify C_1 *and* C_2.

  We will define a Grassmann algebra for them using procedures like those we have discussed. The Grassmann algebra is the following:

$$c\overline{c} + \overline{c}c = 0\,, \quad c^2 = 0\,, \quad \overline{c}^2 = 0\,, \tag{165}$$

  as consistent with the $C_{1,2}$ breakup.

  Of course, we also assume (consistent with the $C_{1,2}$ breakup) that

$$\{dc, d\overline{c}\} = \{dc, dc\} = \{d\overline{c}, d\overline{c}\} = 0\,. \tag{166}$$

  When it comes to integration, Itzykson and Zuber p. 440 write the following:

$$\int d\bar{c}\,\bar{c} = 1\,, \quad \int dc\,c = 1\,, \quad \int d\bar{c} = \int dc = 0\,. \tag{167}$$

But these rules are not completely consistent with the $C_{1,2}$ breakup. Rather, for consistency one should write

$$\int dc\,\bar{c} = 1\,, \quad \int d\bar{c}\,c = 1\,, \quad \int d\bar{c} = \int dc = 0\,. \tag{168}$$

However, all that matters in the end are that whatever rules we decide upon for integration be applied consistently. Following IZ and others, I will employ the IZ definitions for integration in what follows.

- Now define a (non-Grassmann) c-number function

$$p(\bar{c}, c) = p_0 + \bar{p}_1\bar{c} + p_1 c + p_{12} c\bar{c} \tag{169}$$

where $\bar{p}_1$ and p_1 are Grassmann and p_0 and p_{12} are regular c-numbers. Then,

$$\int d\bar{c} p(\bar{c}, c) = -\bar{p}_1 - p_{12} c = \bar{\partial} p \tag{170}$$

etc. Again, integration is equivalent to differentiation.

- Next, we have the result (A is a c-number)

$$\int d\bar{c}dc \exp\left[-\bar{c}Ac\right] = \int d\bar{c}dc \left[1 - \bar{c}Ac + 0\right] = 0 + A = A \qquad (171)$$

as contrasted to the usual pure c-number case where we would have gotten something proportional to $1/A$, assuming integration from $-\infty$ to $+\infty$. However, if we now include an additional factor of $c\bar{c}$, one gets

$$\int d\bar{c}dcc\bar{c} \exp\left[-\bar{c}Ac\right] = \int d\bar{c}dcc\bar{c} \left[1 - \bar{c}Ac + 0\right] = 1 + 0 = \frac{1}{A}A \,. \qquad (172)$$

The extra $\frac{1}{A}$ is what you would have anticipated in analogy with regular integration by rescaling $c \to \sqrt{A}c$ and $\bar{c} \to \sqrt{A}\bar{c}$.

- Now generalize this to a matrix situation:

$$\int \prod_{k=1}^{n} d\bar{c}_k dc_k \exp\left[-\sum_{kl} \bar{c}_k A_{kl} c_l\right] \,. \qquad (173)$$

We expand the exponential and consider what happens. I hope it is apparent that the only terms which survive are those which are *precisely* linear in each of the $\overline{c}_k$'s and c_l's. What do we get from these terms?

Consider the case of just $n = 2$.

$$\int d\overline{c}_1 dc_1 d\overline{c}_2 dc_2 \exp\left[-\overline{c}_1 A_{11} c_1 - \overline{c}_1 A_{12} c_2 - \overline{c}_2 A_{21} c_1 - \overline{c}_2 A_{22} c_2\right] . \tag{174}$$

The linear term in the expansion of the exponential is at most bilinear in the c_i and/or $\overline{c}_i$. Thus, for each term in this linear term there will always be two integrals which are zero.

Terms higher than 2nd order vanish by virtue of their always containing the square of one or more of the c_i or $\overline{c}_i$.

The only surviving terms from the expansion of the exponential come from the 2nd order term of the exponential expansion:

$$\begin{aligned} \exp(\ldots) &\sim \frac{1}{2!}\left[-\overline{c}_1 A_{11} c_1 - \overline{c}_1 A_{12} c_2 - \overline{c}_2 A_{21} c_1 - \overline{c}_2 A_{22} c_2\right]^2 \\ &\ni \frac{1}{2!}\left[2\overline{c}_1 A_{11} c_1 \overline{c}_2 A_{22} c_2 + 2\overline{c}_1 A_{12} c_2 \overline{c}_2 A_{21} c_1 + \ldots\right] \end{aligned} \tag{175}$$

where the $\ldots$ terms are actually 0 by virtue of the fact some c_i and/or $\overline{c}_i$ is squared.

Thus, we have, keeping careful track of $-$ signs as we bring the dc_i or $d\overline{c}_i$ next to its companion,

$$\begin{aligned} & \int d\overline{c}_1 dc_1 d\overline{c}_2 dc_2 \left[\overline{c}_1 A_{11} c_1 \overline{c}_2 A_{22} c_2 + \overline{c}_1 A_{12} c_2 \overline{c}_2 A_{21} c_1\right] \frac{2}{2!} \\ = \; & A_{11} A_{22} - A_{12} A_{21} = \det A \,. \end{aligned} \tag{176}$$

This generalizes; one always gets $\det A$. For a fairly complete proof, see Cheng and Li, p. 26 and following.

Another useful result is obtained in our example case by throwing in an extra factor of, for example, $c_1 \overline{c}_2$.

$$\begin{aligned} & \int d\overline{c}_1 dc_1 d\overline{c}_2 dc_2 \; c_1 \overline{c}_2 \exp\left[-\overline{c}_1 A_{11} c_1 - \overline{c}_1 A_{12} c_2 - \overline{c}_2 A_{21} c_1 - \overline{c}_2 A_{22} c_2\right] \\ = \; & \int d\overline{c}_1 dc_1 d\overline{c}_2 dc_2 \; c_1 \overline{c}_2 \left[-\overline{c}_1 A_{12} c_2 + \ldots\right] = -A_{12} = \det A A^{-1}_{12} \,. \end{aligned} \tag{177}$$

This, of course, generalizes to

$$\int \prod_{k=1}^{n} d\bar{c}_k dc_k c_i \bar{c}_j \exp\left[-\sum_{kl} \bar{c}_k A_{kl} c_l\right] = \det A \; A^{-1}_{ij} \,. \tag{178}$$

- In fact, it will be useful to consider the infinite dimensional version of these Grassmann games (that will be important when considering the infinite cell number limit of a cell division of space-time).

  Let us suppose that there is some matrix M_{ij} that depends upon the different space-time cells i, j in some non-trivial way, and that we need to have a convenient expression for $\det M$ in the continuum limit. Then, the generalization of the above is

$$\det M = \int [dc][d\bar{c}] \exp\left[-\int d^4x d^4y \bar{c}(x) M(x,y) c(y)\right] \tag{179}$$

  This notation is completely consistent with our functional notation. We are simply integrating over all possible choices for c and $\bar{c}$ at all space time locations (i.e. in all cells).

We can equally well rewrite the above with a phase (noting that convergence issues are not relevant in the case of the Grassmann integration definitions) in the form

$$\det(-iM) = \int [dc][d\overline{c}] \exp\left[i \int d^4x d^4y \overline{c}(x) M(x,y) c(y) \right] . \quad (180)$$

where we note that $\det(-iM) = (-i)^{\text{number of cells}} \det M$. The large power of i will not be important in the applications.

Fermi Field Theory

- To make the transition to field theory, we generalize the Grassmann algebra to a large number of Grassmann objects each corresponding to a different cell of space-time. We then take the continuum limit and define Grassmann functions that we might denote by $C(x)$. They will obey

$$
\begin{aligned}
&\{C(x), C(y)\} = 0\,, \\
&\frac{\partial C(x)}{\partial C(y)} = \delta^4(x-y)\,, \\
&\int dC(x) = 0\,, \\
&\int dC(x) C(y) = \delta^4(x-y)\,,
\end{aligned}
\tag{181}
$$

where the latter is consistent with integration being the same as differentiation. In the above, something like $\int dC(x)$ means to integrate over all possible values of $C(x)$ *at a given space time location* x. That is, in each of the last two expressions, x is fixed. A full mathematical justification for all these formulae and the procedures we will follow employing them has not been given, but you will see that everything works perfectly in the usual physicist's sense.

- We will now proceed in analogy with the scalar field case, but using the Grassmann quantities.

  We begin with the standard

$$\mathcal{L} = i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\overline{\psi}(x)\psi(x)\,, \tag{182}$$

and see what happens if we define our generating functional for free Dirac fields as

$$Z_0(\eta,\overline{\eta}) = \frac{1}{N}\int[d\overline{\psi}][d\psi]\exp\left\{i\int d^4x\left[\overline{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\right]\right\}\,, \tag{183}$$

where $\overline{\eta}(x)$ is a 4-component Grassmann source term for the 4-component Grassmann $\psi(x)$ in that $\psi(x)$ can be generated by $\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}$, and $\eta(x)$ is a 4-component source term for the 4-component $\overline{\psi}(x)$ in the sense that $\overline{\psi}(x)$ is generated by $\frac{1}{-i}\frac{\delta}{\delta\eta(x)}$. The extra minus sign in this last expression is because $\frac{\delta}{\delta\eta(x)}$ anticommutes with $\overline{\psi}(x)$ before it can act on $\eta(x)$ in the form $\overline{\psi}(x)\eta(x)$.

Of course, N is given by the numerator expression with $\eta = \overline{\eta} = 0$.

Note: $\overline{\psi}$ and ψ are *independent* Grassmann objects completely analogous to the $\bar{c}$ and c that we have discussed previously. This is in the spirit that there are two degrees of freedom in $\overline{\psi}$ and ψ related to two "real" Grassmann degrees of freedom. In the 2nd quantization via anticommutation game, this corresponds to the fact that the c and d operators are completely independent of one another.

- Let us write

$$\int d^4x\overline{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) = \int d^4x d^4y \overline{\psi}(x)\mathcal{O}(x,y)\psi(y) \tag{184}$$

with

$$\mathcal{O}(x,y) = (i\gamma\cdot\partial_x - m)\delta^4(x-y) \tag{185}$$

which results in the structure for the stuff under the integral in the exponential above of the form (not writing all the integrals in x, y and other temporary integrations – it is better to think of dividing up space time into cells anyway for this manipulation)

$$
\begin{aligned}
& [\overline{\psi}\mathcal{O}\psi + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\eta}\mathcal{O}^{-1}\eta] - \overline{\eta}\mathcal{O}^{-1}\eta \\
= \quad & (\overline{\psi} + \overline{\eta}\mathcal{O}^{-1})\mathcal{O}(\psi + \mathcal{O}^{-1}\eta) - \overline{\eta}\mathcal{O}^{-1}\eta \,. \qquad (186)
\end{aligned}
$$

We will shift to $\psi' = \psi + \mathcal{O}^{-1}\eta$ and $\overline{\psi}' = \overline{\psi} + \overline{\eta}\mathcal{O}^{-1}$ (it is here that our definition of Grassmann integration to preserve this shift property enters in a key way) so that we end up with

$$
\begin{aligned}
Z_0(\eta, \overline{\eta}) \quad = \quad & \frac{1}{N} \int [d\overline{\psi}'][d\psi'] \exp\left\{ i \int d^4x d^4y \left[\overline{\psi}'(x)\mathcal{O}(x-y)\psi'(y) - \overline{\eta}(x)\mathcal{O}^{-1}(x-y)\eta(y) \right] \right\} \\
= \quad & \frac{1}{N} \det(-i\mathcal{O}) \exp\left\{ -i \int d^4x d^4y \overline{\eta}(x)\mathcal{O}^{-1}(x-y)\eta(y) \right\} \,, \qquad (187)
\end{aligned}
$$

where we have converted to a notation that separates x and y as would be the case if we discretized space time, just as in the scalar field work.

Note that $\det(-i\mathcal{O})$ is the determinant of an infinite matrix with rows and columns labeled by the space-time cell index (when we discretize space-time into cells to define everything).

We could have derived this same result employing discretized space-time by expanding the exponential and proceeding in a systematic straightforward manner. I give the example for just two space time

cells.

$$
\begin{aligned}
&\int d\overline{\psi}_1 d\psi_1 d\overline{\psi}_2 d\psi_2 \exp\Big\{ i\Big[\overline{\psi}_1 \mathcal{O}_{11}\psi_1 + \overline{\psi}_2\mathcal{O}_{21}\psi_1 + \overline{\psi}_1\mathcal{O}_{12}\psi_2 + \overline{\psi}_2\mathcal{O}_{22}\psi_2 \\
&\qquad\qquad\qquad + \overline{\eta}_1\psi_1 + \overline{\eta}_2\psi_2 + \overline{\psi}_1\eta_1 + \overline{\psi}_2\eta_2\Big]\Big\} \\
= \quad & \int d\overline{\psi}_1 d\psi_1 d\overline{\psi}_2 d\psi_2 \Big[\frac{1}{2!} i^2 2\left(\overline{\psi}_1\mathcal{O}_{11}\psi_1\overline{\psi}_2\mathcal{O}_{22}\psi_2 + \overline{\psi}_1\mathcal{O}_{12}\psi_2\overline{\psi}_2\mathcal{O}_{21}\psi_1\right) \\
&+\frac{1}{3!} i^3 6\left(\overline{\psi}_1\mathcal{O}_{11}\psi_1\overline{\eta}_2\psi_2\overline{\psi}_2\eta_2 + \overline{\psi}_2\mathcal{O}_{21}\psi_1\overline{\eta}_2\psi_2\overline{\psi}_1\eta_1 + \overline{\psi}_1\mathcal{O}_{12}\psi_2\overline{\eta}_1\psi_1\overline{\psi}_2\eta_2 + \overline{\psi}_2\mathcal{O}_{22}\psi_2\overline{\eta}_1\psi_1\overline{\psi}_1\eta_1\right) \\
&\qquad\qquad + \frac{1}{4!} i^4 24\left(\overline{\eta}_1\psi_1\overline{\eta}_2\psi_2\overline{\psi}_1\eta_1\overline{\psi}_2\eta_2\right) + \text{terms that give } 0\Big] \\
= \quad & -\det\mathcal{O} + (i\overline{\eta}_2\mathcal{O}_{11}\eta_2 - i\overline{\eta}_2\mathcal{O}_{21}\eta_1 - i\overline{\eta}_1\mathcal{O}_{12}\eta_2 + i\overline{\eta}_1\mathcal{O}_{22}\eta_1) - \overline{\eta}_1\overline{\eta}_2\eta_1\eta_2 \\
= \quad & -\det\mathcal{O}\left(1 - i\overline{\eta}_2\left[\frac{\mathcal{O}_{11}}{\det\mathcal{O}}\right]\eta_2 - i\overline{\eta}_2\left[\frac{-\mathcal{O}_{21}}{\det\mathcal{O}}\right]\eta_1 - i\overline{\eta}_1\left[\frac{-\mathcal{O}_{12}}{\det\mathcal{O}}\right]\eta_2 - i\overline{\eta}_1\left[\frac{\mathcal{O}_{22}}{\det\mathcal{O}}\right]\eta_1 + \frac{\overline{\eta}_1\overline{\eta}_2\eta_1\eta_2}{\det\mathcal{O}}\right) \\
= \quad & \det(-i\mathcal{O})\left(1 - i\overline{\eta}_2\mathcal{O}^{-1}_{22}\eta_2 - i\overline{\eta}_2\mathcal{O}^{-1}_{21}\eta_1 - i\overline{\eta}_1\mathcal{O}^{-1}_{12}\eta_2 - i\overline{\eta}_1\mathcal{O}^{-1}_{11}\eta_1 + \overline{\eta}_1\overline{\eta}_2\eta_1\eta_2(\mathcal{O}^{-1}_{22}\mathcal{O}^{-1}_{11} - \mathcal{O}^{-1}_{21}\mathcal{O}^{-1}_{12})\right) \\
= \quad & \det(-i\mathcal{O})\exp\left(-i\overline{\eta}_2\mathcal{O}^{-1}_{22}\eta_2 - i\overline{\eta}_2\mathcal{O}^{-1}_{21}\eta_1 - i\overline{\eta}_1\mathcal{O}^{-1}_{12}\eta_2 - i\overline{\eta}_1\mathcal{O}^{-1}_{11}\eta_1\right) \\
= \quad & \det(-i\mathcal{O})\exp\left(-i\sum_{kl}\overline{\eta}_k\mathcal{O}^{-1}_{kl}\eta_l\right) \qquad (188)
\end{aligned}
$$

where we used $(a+b+c)^3 \ni 6abc$, $(a+b+c+d)^4 \ni 24abcd$, $\frac{1}{\det\mathcal{O}} = \det(\mathcal{O}^{-1})$ **and, e.g.,** $\overline{\eta}_1\overline{\eta}_2\eta_1\eta_2\mathcal{O}^{-1}_{22}\mathcal{O}^{-1}_{11} = -\overline{\eta}_2\mathcal{O}^{-1}_{22}\eta_2\overline{\eta}_1\mathcal{O}^{-1}_{11}\eta_1$.

The continuum limit of this is the result stated earlier using Grassmann variable shifting techniques.

- Since $\mathcal{O} = (i\not\partial_x - m)\delta^4(x-y)$ it is clear that $\mathcal{O}^{-1}$ should satisfy

$$\int d^4z(i\not\partial_x - m)\delta^4(x-z)\mathcal{O}^{-1}(z-y) = (i\not\partial_x - m)\mathcal{O}^{-1}(x-y) = \delta^4(x-y) \qquad (189)$$

which means that

$$\mathcal{O}^{-1}(x-y) = S_F(x-y) = -i\langle 0|T\{\psi(x), \overline{\psi}(y)\}|0\rangle\,. \qquad (190)$$

- We now note that $\mathcal{O}$ has no field dependence and that $N = \det(-i\mathcal{O})$, so that we end up with

$$Z_0(\eta, \overline{\eta}) = \exp\left[-i\int d^4x d^4y \overline{\eta}(x) S_F(x-y)\eta(y)\right]\,. \qquad (191)$$

- So, the above result is a purely algebraic result. Now we wish to compute $\langle 0|T\left\{\psi(w), \overline{\psi}(z)\right\}|0\rangle$.

According to the analogy with scalar field theory, we should have

$$\langle 0|T\{\psi(w),\overline{\psi}(z)\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(w)}\right]\left[\frac{1}{-i}\frac{\delta}{\delta\eta(z)}\right] Z_0(\eta,\overline{\eta})\Big|_{\eta=\overline{\eta}=0}, \quad (192)$$

using the generating function identities discussed earlier. (Note that Ryder is quite careless here and his expressions are not really correct.)

To compute the above, we first do the $\frac{\delta}{\delta\eta(z)}$. We note that it must pass through the $\overline{\eta}(x)$ in the expression for Z_0, which causes a $-$ sign. Thus,

$$\begin{aligned}\frac{1}{-i}\frac{\delta}{\delta\eta(z)}Z_0 &= \frac{1}{-i}\frac{\delta}{\delta\eta(z)}\exp\left[-i\int d^4x d^4y\overline{\eta}(x)S_F(x-y)\eta(y)\right] \\ &= -\int d^4x\overline{\eta}(x)S_F(x-z)Z_0\,. \quad (193)\end{aligned}$$

We then have

$$\begin{aligned}\langle 0|T\{\psi(w),\overline{\psi}(z)\}|0\rangle &= \left[\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(w)}\right]\left[\frac{1}{-i}\frac{\delta}{\delta\eta(z)}\right] Z_0\Big|_{\eta=\overline{\eta}=0} \\ &= \left[\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(w)}\right](-)\int d^4x\overline{\eta}(x)S_F(x-z)Z_0\Big|_{\eta=\overline{\eta}=0} \\ &= iS_F(w-z) \quad (194)\end{aligned}$$

as is indeed correct.

- Overall, you could say that we have replaced one "arbitrary" (except that causality, . . . seems to require it) assumption of the $\{\psi(x), \overline{\psi}(y)\}$ anti-commutation condition with another assumption — namely, Grassmann variable integration for fermionic degrees of freedom.

  Both are equivalent, as we have seen in the simple case above, and will, among other things, lead to a causal theory.

- One final note: we will employ this same formalism eventually to rewrite a determinant, that will arise during gauge theory 2nd quantization (as part of the gauge fixing process), in terms of an integral over (fictitious, but very useful) spinless, anticommuting scalar Grassmann objects c and $\overline{c}$.

Gauge Theory 2nd Quantization via Path Integrals

- **In the path integral approach to gauge theory quantization, we implement gauge fixing by restricting in some manner or other the path integral over gauge fields $\int [dA_\mu]$.**

  In other words we will write instead

$$\int [dA_\mu] \delta(\text{some gauge fixing condition}) e^{i\int d^4x \mathcal{L}(A_\mu)} \tag{195}$$

- **Let us see how the need for the gauge fixing condition arises in the path integral context.**

  Naively, we might start with $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual, and employ

$$\int [dA_\mu] \exp\left\{ i \int d^4x (\mathcal{L}(A_\mu) + J_\mu A^\mu) \right\} \tag{196}$$

as the generating function for the vacuum expectation values of time ordered products of the A_μ fields. Note that J_μ should be conserved ($\partial^\mu J_\mu = 0$) in order for the full expression $\mathcal{L}(A_\mu) + J_\mu A^\mu$ to be gauge invariant under the integral sign when $A^\mu \to A^\mu + \partial^\mu \Lambda$.

- Lets examine what happens more closely. We rewrite, using the usual partial integration technique and the antisymmetry of $F_{\mu\nu}$,

$$\int d^4x \mathcal{L}(A_\mu) = +\frac{1}{2} \int d^4x A_\mu(x) \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu(x) \,. \tag{197}$$

Let us define

$$K^{\mu\nu} = g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \,. \tag{198}$$

As in the scalar field case, we would like to perform the $\int [dA_\mu]$ by completing the square using $(K^{\mu\nu})^{-1}$.

The problem is that $(K^{\mu\nu})^{-1}$ does not exist. We have already discussed this, but let's just quickly review.

There are 4 different ways of checking that the inverse does not exist.

1. First, we can show that $K_{\mu\nu}K^\nu_\lambda = 2K_{\mu\lambda}\partial^2$ which means that $K_{\mu\nu}$ is a projection operator. A projection operator has no inverse.

Proof: Projection operator is defined (dropping indices for simplicity) by $P^2 = P$. Suppose that P^{-1} exists. Then $PP^{-1} = 1$. Multiply this equation by P from the left to obtain $P^2P^{-1} = P$. But then, since $P^2 = P$, this means $PP^{-1} = P$ or $1 = P$. i.e. the only projection operator with an inverse is the trivial $P = 1$. Our K operator is not trivial and thus has no inverse.

2. We would define K^{-1} through the equation

$$\int d^4z(g_\mu^\nu \partial_x^2 - \partial_{x\mu}\partial_x^\nu)\delta^4(x-z)K^{-1}_{\nu\alpha}(z-y) = \delta^4(x-y)g_{\mu\alpha}\,. \quad (199)$$

If we take the Fourier transform of the above, we obtain

$$-(g_\mu^\nu k^2 - k_\mu k^\nu)K^{-1}_{\nu\alpha}(k) = g_{\mu\alpha}\,. \quad (200)$$

The most general form of $K^{-1}_{\nu\alpha}(k)$ is

$$K^{-1}_{\nu\alpha}(k) = a(k^2)k_\nu k_\alpha + b(k^2)g_{\nu\alpha}\,. \quad (201)$$

Substituting this into the above equation gives no requirement on

$a(k^2)$ but the $b(k^2)$ terms must obey

$$-k^2 b(k^2) g_{\mu\alpha} + k_\mu k_\alpha b(k^2) = g_{\mu\alpha} \tag{202}$$

which has no solution.

3. Since K is a projection operator, $\det K = 0$ and K cannot have an inverse. This follows since

$$\begin{aligned} \det(KK) &= \det K, \quad \Rightarrow \\ (\det K)^2 &= \det K \end{aligned} \tag{203}$$

which has only the two solutions $\det K = 0$ or $\det K = 1$. The latter holds only for the trivial $K = 1$ choice.

4. Finally, that $\det K = 0$ can be also seen by noting that K has a zero eigenvalue for the trivial (gauge transform) function $\partial^\mu \Lambda(x)$:

$$(g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\partial^\nu \Lambda = \partial_\mu \partial^2 \Lambda - \partial_\mu \partial^2 \Lambda = 0\,. \tag{204}$$

Thus, in trying to complete the square, we would not be able to define an inverse and the $1/\sqrt{\det K}$ that might have emerged when we did this would be infinite!

- This is a true infinity related to the redundancy of being able to make gauge transforms without changing the physics. We must specify the gauge in some way that will get rid of this ∞.

  The language that will go with this is the phrase "gauge orbits". If we have a given physical choice for A_μ, there are many physically equivalent choices obtained by gauge transforms. The full set of all these equivalent A_μ choices is called the "gauge orbit" that corresponds to the given physical choice. The volume of the gauge orbit is infinite and since $\mathcal{L}$ is invariant on the gauge orbit the $\int [dA_\mu] \exp\left\{\int d^4x \mathcal{L}(A_\mu)\right\}$ has an infinity buried in it.

- The simplest way to get around this is to impose the gauge condition by hand in $\mathcal{L}$. If, for example, we impose $\partial_\mu A^\mu = 0$ directly we have

$$\begin{aligned}\int d^4x \mathcal{L} &= \int d^4x \frac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu \\ \to &\int d^4x \frac{1}{2} A_\mu \partial^2 g^{\mu\nu} A_\nu \,. \end{aligned} \tag{205}$$

  The new kernel $\partial^2 g^{\mu\nu}$ (the $\frac{1}{2}$ is always kept as a standard external factor) does have an inverse. The required equation in Fourier space takes the form:

$$-k^2 g^{\mu\nu} K^{-1}_{\nu\alpha}(k) = g^{\mu}_{\alpha} \tag{206}$$

with solution

$$K^{-1}_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2}\,, \tag{207}$$

which we recognize as the usual propagator form in Lorentz gauge.

- As we have seen this is equivalent to writing

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial^\mu A_\mu)^2 \tag{208}$$

since when we integrate and use parts integration we have

$$\begin{aligned} \int d^4x \left[-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}\right] &= \frac{1}{2}\int A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu\,, \quad \textbf{and} \\ \int d^4x \left[-\frac{1}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\right] &= \frac{1}{2}\int d^4x A_\mu \partial^\mu \partial^\nu A_\nu \end{aligned} \tag{209}$$

so that the gauge condition term cancels the offending part of the original stuff, leaving us with the simple form employed in the previous discussion.

- We can generalize to a more general gauge condition by using

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 \\ '' ='' &\ \frac{1}{2}A_\mu \left[\partial^2 g^{\mu\nu} + \left(\frac{1}{\alpha} - 1\right)\partial^\mu \partial^\nu\right] A_\nu \,, \end{aligned} \tag{210}$$

where $'' ='' $ means under $\int d^4x \mathcal{L}$.

Then, the kernel in question is (in Fourier transform space)

$$K_{\mu\nu}(k) = -k^2 g_{\mu\nu} - \left(\frac{1}{\alpha} - 1\right) k_\mu k_\nu \tag{211}$$

whose inverse (defined by $K_\mu^{\ \alpha}(k) K^{-1}_{\alpha\nu}(k) = g_{\mu\nu}$) is

$$K^{-1}_{\mu\nu}(k) \equiv D_{\mu\nu}(k) = -\frac{1}{k^2}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}\right] . \tag{212}$$

In the $\alpha \to 1$ case, we reduce back to the Feynman choice for the Lorentz gauge that we have discussed above.

Another often useful choice is $\alpha = 0$, the so-called "Landau" gauge.

- Having defined some appropriate kernel with an inverse, we go through the square completion process and end up with

$$Z_0[J] = \exp\left\{-\frac{i}{2}\int d^4z d^4w J^\rho(z) D_{\rho\sigma}(z-w) J^\sigma(w)\right\}, \tag{213}$$

where $D_{\rho\sigma}(z-w)$ is the inverse Fourier transform of one of the above inverse kernels. As always, in the above, we have normalized so that $Z_0[J=0]=1$.

In obtaining the above, it is important that the determinant $\det K$ that is generated when performing the $\int [dA_\mu]$ functional integration does not itself have any A_μ dependence.

We will see that this is not the case when we come to non-Abelian gauge theories.

- Just to review the square completion process, let us refer to our starting form Eq. (196), with $\mathcal{L}$ being the $\mathcal{L}$ after including gauge fixing. We would need to compute

$$\int [dA_\mu] \exp\left\{ i \int d^4x \left[\mathcal{L}(A_\mu(x)) + J^\mu(x) A_\mu(x) \right] \right\} \tag{214}$$

where we have seen above that

$$\int d^4x \mathcal{L}(A_\mu(x)) = \int d^4x d^4y \frac{1}{2} A^\mu(x) K_{\mu\nu}(x,y) A^\nu(y) \tag{215}$$

where $K_{\mu\nu}$ is invertible for the given gauge choice.

We would divide space-time up into cells with labels α for x and β for y. We would also have the Lorentz labels μ for the x-cell α and the label ν for the y-cell β. Putting the indices together, we could write a matrix in $n = (\alpha, \mu)$ and $m = (\beta, \nu)$ space. The resulting conversion is:

$$\int d^4x d^4y \left[\frac{1}{2} A^\mu(x) K_{\mu\nu}(x,y) A^\nu(y) + J^\mu(x) \delta^4(x-y) A_\mu(y) \right]$$

$$\rightarrow \sum_n \epsilon^4 \sum_m \epsilon^4 \left[\frac{1}{2} A_n K_{nm} A_m + J_n \frac{\delta_{nm}}{\epsilon^4} A_m \right]$$

$$
\begin{aligned}
&= \sum_n \epsilon^4 \sum_m \epsilon^4 \left[\frac{1}{2}(A + \frac{K^{-1}}{\epsilon^4}J)_n K_{nm}(A + \frac{K^{-1}}{\epsilon^4}J)_m - \frac{1}{2}J_n \frac{K^{-1}_{nm}}{\epsilon^8}J_m\right] \\
&= \sum_n \epsilon^4 \sum_m \epsilon^4 \left[\frac{1}{2}A'_n K_{nm} A'_m - \frac{1}{2}J_n \frac{K^{-1}_{nm}}{\epsilon^8}J_m\right] \qquad (216)
\end{aligned}
$$

where we used, for example,

$$
\begin{aligned}
(K^{-1}J)_n K_{nm} A_m &= K^{-1}_{nl} J_l K_{nm} A_m \\
&= J_l K^{-1}_{ln} K_{nm} A_m \quad \text{by symmetry of } K^{-1} \\
&= J_l \delta_{lm} A_m \qquad (217)
\end{aligned}
$$

and

$$
\begin{aligned}
(K^{-1}J)_n K_{nm} (K^{-1}J)_m &= K^{-1}_{nl} J_l K_{nm} A_m K^{-1}_{mp} J_p \\
&= J_l K^{-1}_{ln} K_{nm} K^{-1}_{mp} J_p \quad \text{by symmetry of } K^{-1} \\
&= J_l \delta_{lm} K^{-1}_{mp} J_p \\
&= J_m K^{-1}_{mp} J_p \,. \qquad (218)
\end{aligned}
$$

We then compute (shifting from dA_p to dA'_p)

$$\int \prod_p [dA'_p] \exp\left\{ i \sum_n \epsilon^4 \sum_m \epsilon^4 \left[\frac{1}{2} A'_n K_{nm} A'_m \right] \right\}$$
$$= \frac{1}{\sqrt{\det K_{nm}}} \prod_p \sqrt{\frac{2\pi i}{\epsilon^8}}, \tag{219}$$

so that we are left with

$$Z_0[J] \propto \exp\left\{ -i\frac{1}{2} \sum_n \epsilon^4 \sum_m \epsilon^4 \left[J_n \frac{K^{-1}_{nm}}{\epsilon^8} J_m \right] \right\}. \tag{220}$$

Like in the earlier scalar field case, K^{-1} is defined by

$$\sum_p \epsilon^4 K_{np} \frac{K^{-1}_{pm}}{\epsilon^8} = \frac{\delta_{nm}}{\epsilon^4} \tag{221}$$

which in the continuum limit becomes

$$\int d^4z K_\mu^\lambda(x-z) D_{\lambda\nu}(z-y) = \delta^4(x-y) g_{\mu\nu}\,, \tag{222}$$

so that $Z_0[J]$ takes the continuum limit form

$$Z_0[J] \propto \exp\left\{ -i\frac{1}{2} \int d^4x d^4y J^\mu(x) D_{\mu\nu}(x-y) J^\nu(y) \right\}. \tag{223}$$

- For the $\alpha = 1$ gauge choice, for example, we have

$$K_\mu^\lambda(x-z) = \partial_x^2 \delta^4(x-z) g_\mu^\lambda \tag{224}$$

and Eq. (222) takes the form

$$\partial_x^2 g_\mu^\lambda D_{\lambda\nu}(x-y) = \delta^4(x-y) g_{\mu\nu} \tag{225}$$

the solution to which is

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \left[\frac{-g_{\mu\nu}}{k^2}\right] , \tag{226}$$

a result we recognize from the previous quarter.

- In the generating function technique in the free-field case we would then compute

$$\begin{aligned} \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\frac{1}{i}\frac{\delta}{\delta J^\nu(y)} \exp\left\{-\frac{i}{2}\int d^4z d^4w J^\rho(z) D_{\rho\sigma}(z-w) J^\sigma(w)\right\}\Big|_{J=0} \\ &= iD_{\mu\nu}(x-y) , \end{aligned} \tag{227}$$

where $D_{\mu\nu}(x-y)$ is the result that we obtained for the gauge propagator (see above example) using the commutator 2nd quantization techniques. Note that we have obtained the correct sign and there is no factor of $\frac{1}{2}$ because each of the $\frac{\delta}{\delta J}$ derivatives can see either of the J's in the exponential.

- It is useful to give a heuristic derivation of this result that will be useful for understanding how to deal with non-Abelian gauge theories.

First, in the integral $\int [dA_\mu]$ over all field configurations, we write each A_μ as

$$A_\mu(x) \sim \overline{A}_\mu(x), \Lambda(x) \tag{228}$$

where $\overline{A}_\mu(x)$ is some choice on a given gauge orbit and all the different choices of $\Lambda(x)$ generate all the other forms of A_μ on that same gauge orbit.

Then, we would have

$$Z \propto \int [dA_\mu] e^{iS(A_\mu)} = \int [d\overline{A}_\mu] e^{iS(\overline{A}_\mu)} \int [d\Lambda] \tag{229}$$

where we have used the fact that $S(A_\mu) = S(\overline{A}_\mu)$ on a given gauge orbit. The latter factor is the infinity that gives us the problem. We modify it by instead writing

$$\int [d\Lambda] \to \int [d\Lambda] e^{\frac{-i}{2\alpha}(f(A_\mu))^2} . \tag{230}$$

For an appropriate choice of $f(A_\mu)$ (e.g. $f(A_\mu) = \partial^\mu A_\mu$) the integral would now converge but, as written, the result would be gauge dependent, i.e. it would depend upon how f depends upon $A_\mu = (\overline{A}_\mu, \Lambda)$.

So, instead, let us try the replacement

$$\int [d\Lambda] \to \int [df] e^{\frac{-i}{2\alpha} f^2} = \int [d\Lambda] \det \left(\frac{\partial f}{\partial \Lambda} \right) e^{\frac{-i}{2\alpha} f^2} . \tag{231}$$

By integrating over all possible values for the given function f at each space-time point, we have obviously eliminated any sensitivity to the value of f for a given field configuration A_μ.

You may ask what kind of determinant it is that appears above? Well, to understand you must recall that these functional integrations are really defined by dividing space-time up into cells. Since f typically contains a derivative, it is actually sensitive to differences between neighboring cells. Thus, what we really have is $\frac{\partial f_i}{\partial \Lambda_j}$ where the i, j indicate the cells considered. This creates a matrix that we shall call M_{ij} that records how f_i in cell i responds to a gauge transform in cell j. We will return to this M shortly.

For the moment, let us note that if we do the above, we obtain

$$Z[J] \propto \int [d\overline{A}_\mu] \int [d\Lambda] \exp \left[i \int d^4x \left(\mathcal{L} + JA - \frac{1}{2\alpha} f^2 \right) \right] \det \left(\frac{\partial f}{\partial \Lambda} \right) , \tag{232}$$

where we have made explicit the J source terms (JA is shorthand for $J_\mu A^\mu$) which are invariant under all the above manipulations so long as they are gauge invariant. In QED, for example, the J_μ should correspond to conserved currents $\partial^\mu J_\mu = 0$.

The above form is exactly what we wanted except for the addition of the $\det$ multiplicative factor.

In QED, this $\det$ factor does not depend upon the field A_μ (see below) and so can be simply discarded as part of the $Z[J=0]=1$ normalization process.

However, in the non-Abelian case we will instead proceed as follows. First, it is sufficient to compute $\frac{\partial f}{\partial \Lambda}$ by using a small gauge transformation under which $f \to f + M\Lambda$ so that $\frac{\partial f}{\partial \Lambda} = M$. In general M can be very complicated and can, in particular, depend upon the field A_μ. But it does not in QED. To see this, let us take $f = \partial^\mu A_\mu$ and let $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \Lambda$. Then $f(x) \to f(x) - \frac{1}{e}\partial_x^2 \Lambda(x)$ and

$$M(x,y) = \frac{\partial f(x)}{\partial \Lambda(y)} \;\; = \;\; -\frac{1}{e}\frac{\partial}{\partial \Lambda(y)}(\partial_x^2 \Lambda(x)) = -\frac{1}{e}\partial_x^2 \delta^4(x-y) \qquad (233)$$

Clearly there is no A_μ dependence of this result. This should also clarify the fact that M is really a matrix in the sense that had we divided space-time up into little cells, as we implicitly must do in order to define these path integrals, then, in this case, the determinant of the resulting "gauge response function" is diagonal in the cell index.

We now use the trick discussed earlier, Eq. (180), to write

$$\det M \propto \int [dc][d\bar{c}] \exp\left[i \int d^4x d^4y \bar{c}(x) M(x-y) c(y)\right] \tag{234}$$

where the c and $\bar{c}$ are the spinless anticommuting fields mentioned previously and are usually called Faddeev-Popov "ghost" fields.

The bottom-line result will then be

$$Z[J] \propto \int [dA_\mu][dc][d\bar{c}] \exp\left[i \int \left(\mathcal{L} + JA - \frac{1}{2\alpha} f^2 + \bar{c} M c\right)\right], \tag{235}$$

where we have used $\int [d\overline{A}_\mu][d\Lambda] \equiv \int [dA_\mu]$. Thus, the net effect is to replace $\mathcal{L}$ by

$$\mathcal{L}_{\rm eff} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}, \tag{236}$$

where GF refers to gauge-fixing and FPG to Faddeev-Popov ghosts.

In general, the implication of the above form is that we may compute a physical transition amplitude using a particular gauge choice for the vector field propagator only if we compensate for unphysical degrees of propagation by also including ghost fields that cancel (since they anti-commute, i.e. are fermionic — this will become clearer later, but for now think of the $-$ sign associated with a fermion loop) these unphysical propagations.

The ghost fields will enter into the Feynman rules as internal (never external since they do not correspond to a real degree of freedom) virtual objects having propagators and vertices (with other fields). We can effectively implement our gauge fixing in all generality in this convenient way.

Non-Abelian Gauge Theory (Yang-Mills)

I will employ a treatment that differs somewhat from Ryder's section 3, which focuses rather specifically on the case of $SU(2)$. Of course, $SU(2)$ is a very useful special case.

- An example of a non-Abelian group is $SU(2)$ (for example, the $SU(2)$ of isospin used in nuclear physics).

  The extra indices associated with the non-Abelian group have nothing to do with anything like spin or normal charge, they act on an internal intrinsic index associated with a given particle or field.

Gauge invariance of the 1st kind (global GI)

- Let us recall that for every conserved quantum number one can *construct* a transformation on the fields which leaves $\mathcal{L}$ invariant. The simple case we have dealt with so far is charge. Let us review.

  If we say that ϕ_i has charge q_i, this means that under the $U(1)$ group transformation

$$\phi_i(x) \to e^{-iq_i\theta}\phi_i(x)\,. \tag{237}$$

A term in $\mathcal{L}$ in which a number of these fields are multiplied together transforms as

$$\phi_1\ldots\phi_n \to e^{-i(q_1+\ldots+q_n)\theta}\phi_1\ldots\phi_n\,. \tag{238}$$

If $\mathcal{L}$ is invariant under the transformation, then $\sum_i q_i = 0$ is required, which is to say that the product is "neutral".

If θ does not depend upon x, then each of the kinetic energy terms $\partial_\mu\phi_i^\dagger\partial^\mu\phi_i$ is also invariant.

The corresponding infinitesimal transformation with $\theta = \epsilon \ll 1$ is

$$\delta\phi_i = -i\epsilon q_i\phi_i \tag{239}$$

Noether's theorem then says that

$$
\begin{aligned}
\delta\mathcal{L} &= 0 \\
&= \sum_j \frac{\partial\mathcal{L}}{\partial\phi_j}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j) \\
&= \sum_j \left[\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\right]\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j) \quad \text{by e.o.m.} \\
&= -i\epsilon\sum_j \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}q_j\phi_j\right], \qquad (240)
\end{aligned}
$$

i.e.

$$
\partial_\mu J^\mu = 0\,, \quad \text{with} \quad J^\mu = \sum_j i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}q_j\phi_j\,. \qquad (241)
$$

Further, $Q = \int d^3x J^0$ is a conserved charge operator, which generates the transformation: i.e.

$$
\delta\phi_i = -i\epsilon[Q, \phi_i] \qquad (242)
$$

by virtue of canonical commutation relations, or equivalently the operator forms of Q and ϕ. The canonical commutation proof is as follows. First, for the standard type of $\mathcal{L}$,

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_j)} = \partial_0 \phi_j \tag{243}$$

implying

$$Q = \sum_j \int d^3x \; i\partial_0 \phi_j(x) q_j \phi_j(x) \, . \tag{244}$$

Then,

$$\begin{aligned} [Q, \phi_i(y)] &= \sum_j \int d^3x q_j \phi_j(x) [\partial_0 \phi_j(x), \phi_i(y)] \\ &= \sum_j \int d^3x i q_j \phi_j(x) [-i\delta_{ij} \delta^3(x-y)] \\ &= q_i \phi_i(y) \, , \end{aligned} \tag{245}$$

implying

$$\delta\phi_i(y) = -i\epsilon [Q, \phi_i(y)] = -i\epsilon q_i \phi_i(y) \tag{246}$$

as claimed. Equivalently, one can use the operator form of Q:

$$Q = \sum_j \int d\tilde{p} q_j a_j^\dagger(\vec{p}) a_j(\vec{p}) \tag{247}$$

where $d\tilde{p}$ is the invariant measure for the particular conventions chosen, along with the operator form of ϕ_i to prove the same thing.

- A more complicated case is the non-Abelian case of "isospin".

  The fields come in multiplets representing $SU(2)$ in the sense that under a transformation

$$\phi \to e^{-i\vec{L}\cdot\vec{\theta}}\phi \tag{248}$$

  where we should think of ϕ as a column vector and each $\vec{L}$ as a matrix of the same dimension.

  For example, in the "doublet" ϕ representation ϕ is a two-component column vector and $\vec{L}$ is the 2×2 matrix $\vec{L} = \frac{1}{2}\vec{\tau}$ ($\vec{\tau}$ = Pauli matrices). (The $\frac{1}{2}$ is just a conventional normalization choice.)

  For the "triplet" representation, ϕ is a three-component column vector and the matrices $L^{i=1,2,3}$ are defined by $L^i_{jk} = -i\epsilon^{ijk}$.

The $SU(2)$ group is defined by the commutation relations of its generators:

$$[T^i, T^j] = i\epsilon^{ijk}T^k \tag{249}$$

and the above matrix sets have thus been chosen to satisfy these same defining $SU(2)$ commutation relations

$$[L^i, L^j] = i\epsilon^{ijk}L^k\,. \tag{250}$$

Infinitesimally, we would have

$$\delta\phi = -\frac{i}{2}\vec{\tau}\cdot\vec{\epsilon}\phi \quad \textbf{doublet}\,, \quad \delta\phi_i = \epsilon^{ijk}\epsilon^j\phi_k \quad \textbf{triplet}\,. \tag{251}$$

Another example of a gauge group is $SU(3)$. There, the group structure is defined by

$$[T^a, T^b] = ic^{abc}T^c \tag{252}$$

where a, b, c run over $1, 2, \ldots 8$. Ryder uses the notation $c^{abc} = f^{abc}$ for the $SU(3)$ case. The f^{abc} are given by:

$$f^{123} = 1\,,$$

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2},$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}. \tag{253}$$

The most fundamental representation of $SU(3)$ is the "triplet" representation in which there are 3 ϕ_i components where $i = r, g, b$ (red, green, blue) or r, w, b, depending upon the book. In the triplet representation, the T^a are represented by a set of 8 matrices

$$T^a = \frac{\lambda^a}{2} \tag{254}$$

where the λ^a are given in Eq. (3.180) of Ryder.

Gauge Invariance of the 2nd kind

- The $U(1)$ example is already familiar:

$$\phi_i(x) \to \phi'_i(x) = e^{-iq_i\theta(x)}\phi_i(x)\,, \quad \delta\phi_i(x) = -iq_i\theta(x)\phi_i(x) \tag{255}$$

but

$$\partial_\mu \phi_i(x) \to e^{-iq_i\theta(x)} \partial_\mu \phi_i(x) - iq_i \partial_\mu \theta(x) e^{-iq_i\theta(x)} \phi_i(x) \,. \tag{256}$$

Because of the 2nd term, $\delta\mathcal{L} \neq 0$ for KE terms of $\mathcal{L}$.

We must introduce the A_μ fields and generalize to

$$D_\mu \phi_i = (\partial_\mu + ieq_i A_\mu)\phi_i \tag{257}$$

and choose the A_μ transformation property such that

$$(\partial_\mu + ieq_i A'_\mu)\phi'_i(x) = e^{-iq_i\theta(x)}(\partial_\mu + ieq_i A_\mu)\phi_i(x) \tag{258}$$

in order to get $\delta\mathcal{L} = 0$. Working this out, one finds the requirement

$$-\partial_\mu \theta(x) + eA'_\mu(x) = eA_\mu(x) \tag{259}$$

independent of q_i, i.e.

$$\delta A_\mu = \frac{1}{e}\partial_\mu \theta(x) \,. \tag{260}$$

And, of course, we must introduce appropriate $\mathcal{L}$ for the A_μ fields of

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{261}$$

where $F_{\mu\nu}$ is invariant under (260).

- **Lie Group Basics**

We now wish to generalize what we have done to the non-Abelian group situation. Before proceeding, let's do a few group basics. (see Georgi's book or Cahn's book — much of the following is from the Peskin chapt 15 summary.)

The groups that have arisen so far in particle theory are all compact Lie groups. A Lie group is a group in which the elements are labeled by a set of continuous parameters with a multiplication law that depends smoothly on the parameters.

Basically, for application to gauge theories, the local symmetry is normally a unitary transformation of a finite set of fields. Thus, we are primarily interested in Lie algebras that have finite-dimensional

Hermitian representations, where the finite number of fields, which you can view as being arranged in a column vector for example, form the "**representation**" of the group. Also, the groups of interest have a finite number of generators. Such Lie algebras are called "compact" because these conditions imply that the Lie group is a finite-dimensional compact manifold, where "compact" refers to the global property of the group that the volume of the parameter space for the group is finite.

In other words, any representation of a compact Lie group is equivalent to a representation by unitary operators generated by a finite number of generators. A group element is obtainable from the identity by continuous changes and can be written as $e^{-i\theta^a L^a}$, where the θ^a are real parameters ($a = 1, \ldots, N$) and the L^a are linearly independent hermitian operators. The set of all linear combinations $\theta^a L^a$ is a vector space, and the L^a are a basis in the space.

The term "**group generator**" refers (interchangeably) to an arbitrary element of the vector space or specifically to the basis vectors L^a.

Do not confuse the space of the group generators (N dimensional) with the space on which the generators act, which is some as yet unspecified Hilbert space of a set of fields (arranged in a column vector for example). To repeat, for the compact Lie groups, we can always take the space on which the generators act to be finite dimensional, so that you can think of the L^a as finite hermitian matrices. There are different possible hermitian matrix sets acting on different possible Hilbert spaces that can be used to "represent" the Lie group.

Note

(a) Generators can be multiplied by numbers and added to obtain other generators.

(b) Generators satisfy simple commutation relations which determine the full structure of the group for group elements that can be continuously connected to the identity. (Large global transformations, such as reflection fall outside this class. For example, $SU(2)$ and $O(3)$ have the same commutation relations but different global structure since $O(3)$ includes the reflection operation.)

Consider the product

$$e^{-i\lambda L^b} e^{-i\lambda L^a} e^{i\lambda L^b} e^{i\lambda L^a} = 1 + \lambda^2 [L^a, L^b] + \dots \tag{262}$$

Because of the group property, the product of group elements is another group element and can be written as $e^{i\beta^c L^c}$. As $\lambda \to 0$, we must have

$$\lambda^2[L^a, L^b] \to i\beta^c L^c \quad \Rightarrow \beta^c = \lambda^2 c^{abc} \quad \Rightarrow [L^a, L^b] = ic^{abc}L^c \,. \tag{263}$$

The c^{abc} are called the "structure constants" of the group. You might worry that $[L^a, L^b] = ic^{abc}L^c$ would not be sufficient to guarantee the matching to still higher orders in λ. But, in fact, it is all you need.

By the definition and the expansion process above, the c^{abc} are antisymmetric in the first two indices. Further, *they are real* if the L^a are hermitian. Reality of the c^{abc} follows from $[L^a, L^b] = ic^{abc}L^c$ and the hermiticity of the L^a's by taking the hermitian conjugate of this relation:

$$\begin{gathered}([L^a, L^b] = ic^{abc}L^c)^\dagger \;\;\Rightarrow\;\; [L^b, L^a] = -i(c^{abc})^* L^c \\ \Rightarrow ic^{bac}L^c = -i(c^{abc})^* L^c \;\;\Rightarrow\;\; c^{bac} = -c^{abc} = -(c^{abc})^* \,.\end{gathered} \tag{264}$$

As you see, the c^{abc} are determined by the group multiplication law. They also determine the multiplication law as follows:

$$e^{-i\alpha^a L^a} e^{-i\beta^b L^b} \equiv e^{-i\delta^c L^c} \tag{265}$$

with δ^c given by

$$\delta^c = \alpha^c + \beta^c + \frac{1}{2}c^{abc}\alpha^a\beta^b + \ldots . \qquad (266)$$

This illustrates the remarkable fact that $[L^a, L^b] = ic^{abc}L^c$ allows us to obtain δ^c to any desired order. (In proving the above lowest order, not to mention still higher orders, you must very carefully expand the exponentials keeping the orderings straight.)

In words, $[L^a, L^b] = ic^{abc}L^c$ is equivalent to saying that the set of generators L^a must span the space of infinitesimal group transformations, and so the commutator of two L^a generators must be a linear combination of generators.

The generators also satisfy the following identity, called the Jacobi identity:

$$[L^a, [L^b, L^c]] + \text{cyclic permutations} = 0 . \qquad (267)$$

It is obvious for a representation, since then the L^a are just linear operators. But, in fact it is true for the abstract group generators. In terms of the structure constants, the above equation becomes

$$c^{bcd}c^{ade} + c^{abd}c^{cde} + c^{cad}c^{bde} = 0 . \qquad (268)$$

- **Classification of Lie algebras**

If one of the L^a commutes with all others, it generates an independent continuous Abelian group which, as we have learned, has the structure of the group of phase rotations, $e^{i\alpha}$, that we have been calling a $U(1)$ group.

If the algebra contains no such commuting elements, i.e. no $U(1)$ factors, then we call the algebra "semi-simple". If, in addition, the Lie algebra cannot be divided into two mutually commuting sets of generators, the algebra is "simple".

A general Lie algebra is the direct sum of non-Abelian simple components and additional Abelian generators.

The basic conditions that a Lie algebra be compact and simple turn out to be extremely restrictive. In the 19th century Killing and Cartan classified all possible compact simple Lie algebras. Almost all of these algebras belong to one of three infinite families, with only five exceptions.

The three infinite families are the algebras corresponding to the so-called "classical groups", whose structures are most conveniently defined in

terms of particular matrix representations.

The definitions of the 3 families of classical groups are as follows:

1. $\boxed{\text{Unitary transformations of } N\text{-dimensional vectors.}}$

   If ξ and η are complex N-vectors, then the general linear transformation is

$$\eta_a \to U_{ab}\eta_b\,, \quad \xi_a \to U_{ab}\xi_b \tag{269}$$

   and this is a unitary transformation if it preserves the inner product $\eta_a^*\xi_a$. The pure phase transformations $\xi^a \to e^{i\alpha}\xi^a$ form a $U(1)$ subgroup which commutes with all other unitary transformations. We therefore remove this subgroup to form a simple Lie group, called $SU(N)$ which consists of all $N \times N$ unitary transformations satisfying $\det(U) = 1$. The generators of $SU(N)$ are represented by the $N \times N$ Hermitian matrices t^a with $\mathrm{Tr}(t^a) = 0$ (so that they are orthogonal to the phase transformation) defining the $U(1)$. There are $N^2 - 1$ traceless hermitian matrices.

2. $\boxed{\text{Orthogonal transformations of } N\text{-dimensional vectors.}}$

   This is the subgroup of unitary $N \times N$ transformations that preserves the symmetric inner product $\eta_a E_{ab}\xi_b$ with $E_{ab} = \delta_{ab}$. This is the usual vector product and you know that this group is the rotation group $SO(N)$. There is an independent rotation corresponding to each plane in N dimensions and so $SO(N)$ has $N(N-1)/2$ generators. Adding the reflection transformation to $SO(N)$ gives the group $O(N)$.

3. **Symplectic transformations of N-dimensional vectors**

   This is the sub-group of unitary $N \times N$ transformations, for N even, that preserves the antisymmetric inner produce $\eta_a E_{ab} \xi_b$ with $E_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ where the elements of the matrix are $N/2 \times N/2$ blocks. This group is called $Sp(N)$ and has $N(N+1)/2$ generators.

Beyond these 3 families, there are the five more "exceptional" Lie algebras, denoted as G_2, F_4, E_6, E_7 and E_8. Of these, E_6 and E_8 have been employed as local symmetry groups in interesting unified models of the fundamental interactions and often emerge in the low-energy effective theories derived from string theory.

- **Representations**

So, we have learned that the generators and their commutation relations define the "Lie algebra" associated with the Lie group.

Typically, there is a set of fields that define a particular "**representation**" of the Lie algebra.

The generators in the representation, when exponentiated, give the operators of the group representation.

The "**dimension**" of a representation is the dimension of the vector space on which it acts (which is just simply the number of fields in our column vector of fields).

An arbitrary representation can generally be decomposed by finding a basis in which all representation matrices are simultaneously block-diagonal. Each block then forms an "**irreducible**" representation and the general representation is then said to be the direct sum of its irreducible component representations. We will implicitly discuss from now on only irreducible representations.

If we define a set of matrices T^a by $T^a_{bc} \equiv -ic^{abc}$ then Eq. (268) (the Jacobi identity in terms of the c^{abc}) becomes

$$[T^a, T^b] = ic^{abc}T^c\,, \tag{270}$$

which is to say that the structure constants themselves generate a representation of the group algebra. This representation is called the "**adjoint**" representation. Note that it is automatically hermitian if the c^{abc} structure constants are antisymmetric in their last two indices (as we will shortly demonstrate is the case in an appropriately defined basis):

$$(T^{a\,\dagger})_{bc} = (T^{a\,*})_{cb} = (-ic^{acb})^* = ic^{acb} = -ic^{abc} = T^a_{bc}\,. \tag{271}$$

clearly, the dimension of the adjoint representation is just the number of generators, which is the same as the number of real parameters necessary to describe a group element.

Of course, the structure constants depend on what basis we choose in the vector space of the generators. In general we can write

$$\mathrm{Tr}(L^a L^b) \equiv D^{ab}. \tag{272}$$

D is a real (since the L^a's are hermitian) symmetric (from the cyclic property of the trace) matrix, so we can diagonalize it by choosing appropriate real linear combinations of the L^a's.

In more detail, and using subscripts to simplify my typing, let us write $L'_a = M_{ab}L_b$. Then

$$\mathrm{Tr}(L'_a L'_b) = M_{ac}M_{bd}\mathrm{Tr}(L_c L_d) = M_{ac}D_{cd}(M^T)_{db} \tag{273}$$

and since D_{cd} is real and symmetric it can be diagonalized by an appropriate choice of M.

Suppose we have done this so that we obtain

$$\mathrm{Tr}(L^a L^b) = k^a \delta^{ab} \quad \textbf{no sum on } a\,. \tag{274}$$

In addition, since the L^a are hermitian (for the compact Lie groups of interest to us), all the k^a's are positive.[1] This is most simply seen by noting that any hermitian matrix can be diagonalized and that the diagonal will then contain real eigenvalues, λ_i ($i = 1, \ldots, n$). In the diagonal form for a given L^a, not all the λ_i can be zero (otherwise it would be equivalent to the 0 matrix) and we thus have

$$L^a L^a = \textbf{diag}(\lambda_1^2, \ldots, \lambda_n^2) \Rightarrow \mathrm{Tr}(L^a L^a) = \sum_{i=1,\ldots,n} \lambda_i^2 = k^a > 0\,. \tag{275}$$

We still have the freedom to rescale the generators, so the conventional

[1] Algebras in which some of the k^a's are negative have no nontrivial finite dimensional unitary representations (i.e. they are not compact). This does not mean they are not interesting. The Lorentz group is one such case. But, here we restrict ourselves to finite dimensional compact groups.

choice is to define the L^a basis so that all the k^a are equal, i.e. we define the basis so that D^{ab} is proportional to the unit matrix.

Once this is done for one irreducible representation, one can show that it is true for all irreducible representations.

In this basis, we then have

$$\mathrm{Tr}(L^a L^b) \equiv C(r)\delta^{ab}\,, \tag{276}$$

where $C(r)$ is a possibly representation (r) dependent constant. Our conventions are such that we have $C(r) = \frac{1}{2}$ for all representations.

It is in the above basis that the structure constants are completely antisymmetric, because we can write (for $C(r) = \frac{1}{2}$)

$$c^{abc} = -2i\mathrm{Tr}([L^a, L^b]L^c)\,, \tag{277}$$

which is completely antisymmetric because of the cyclic property of the trace.

Recall our earlier demonstration that in this basis, the generators in the adjoint representation are hermitian matrices.

Conjugate representation

For each irreducible representation r of the general group G, there is an associated "conjugate" representation $\overline{r}$. If we write for representation r

$$\phi \to (1 - i\theta^a L_r^a)\phi \tag{278}$$

the complex conjugate of this equation is

$$\phi^* \to (1 + i\theta^a (L_r^a)^*)\phi^* \tag{279}$$

implying that the object inside $(\ldots)$ must be the infinitesimal element of a representation of G, with representation vectors defined by the ϕ^* vectors. The conjugate representation to r then has representation matrices given by $L_{\overline{r}}^a = -(L_r^a)^* = -(L_r^a)^T$ (using hermiticity).

Since $\phi^*\phi$ is invariant under unitary transformations, it is clearly possible to combine fields transforming in the representations r and $\overline{r}$ so as to form a group invariant.

It is possible that the representation $\overline{r}$ is equivalent to r. This is the case if there is a unitary U such that $L^a_{\overline{r}} = U L^a_r U^\dagger$. In this case, the representation r is real and there is a matrix G_{ab} such that, if η and ξ belong to the representation r, then the combination $G_{ab}\eta_a\xi_b$ is an invariant.

It is sometimes useful to distinguish the case in which G_{ab} is symmetric from that in which G_{ab} is antisymmetric. In the former case, the representation is strictly real; in the latter case, it is pseudoreal. Both cases occur even in $SU(2)$: the invariant combination of two vectors is $v_a w_a$, so the vector is a real representation; the invariant combination of two spinors is $\epsilon^{\alpha\beta}\eta_\alpha\xi_\beta$, so the spinor is a pseudoreal representation.

In $SU(N)$, the basic irreducible representation (often called the fundamental representation) is the N-dimensional complex vector. For $N > 2$, this representation is complex, so there is a second, inequivalent, representation $\overline{N}$. In $SO(N)$, the basic N-dimensional representation is a strictly real representation. In $Sp(N)$, the N-dimensional vector is a pseudoreal representation.

Returning to the adjoint representation, since the structure constants are real and antisymmetric, $T^a = -(T^a)^*$, which means that the adjoint representation is always a real representation.

- **With this background, we can proceed to generalize global and local gauge invariance to the non-Abelian case.**

  We use general group definition of (here, I switch from a, b, c to i, j, k)

$$[T^i, T^j] = ic^{ijk}T^k \,. \tag{280}$$

  As before,

$$\phi(x) \to \phi'(x) = e^{-i\vec{L}\cdot\vec{\theta}}\phi(x) \sim (1 - i\vec{L}\cdot\vec{\theta}(x))\phi(x) \,, \tag{281}$$

  where ϕ is n-component column vector and each $\vec{L}$ is $n \times n$. Assume that we have constructed a $\mathcal{L}$ that is invariant under the global transformation, i.e. for $\theta =$constant.

  The question is the following: if we allow $\theta(x)$, then what gauge fields with what transformation properties must be introduced in order to get local invariance of $\mathcal{L}$.

  The trick is to define

$$D_\mu = \partial_\mu - ig\vec{L}\cdot\vec{A}_\mu(x) \tag{282}$$

(note one A^i_μ for each L^i) such that

$$D'_\mu \phi'(x) = (1 - i\vec{L}\cdot\vec{\theta}(x)) D_\mu \phi(x) \,. \tag{283}$$

Let's work out this requirement. We have

$$\begin{aligned} D'_\mu \phi' &= (\partial_\mu - igA'^{\,j}_\mu L^j)(1 - i\vec{L}\cdot\vec{\theta})\phi \\ &= (1 - i\vec{L}\cdot\vec{\theta})\partial_\mu \phi - i\vec{L}\cdot\left(\partial_\mu \vec{\theta}\right)\phi - igA'^{\,j}_\mu L^j (1 - i\vec{L}\cdot\vec{\theta})\phi \\ &= (1 - i\vec{L}\cdot\vec{\theta})(\partial_\mu - igA^j_\mu L^j)\phi \end{aligned} \tag{284}$$

which, after canceling common terms on the two sides of the equation becomes (since it must hold for arbitrary ϕ)

$$-i\vec{L}\cdot\partial_\mu\vec{\theta} - igA'^{\,j}_\mu L^j (1 - i\vec{L}\cdot\vec{\theta}) = -igA^j_\mu (1 - i\vec{L}\cdot\vec{\theta}) L^j \,. \tag{285}$$

We may verify that the solution is

$$A'^{\,j}_\mu = A^j_\mu - \frac{1}{g}\partial_\mu \theta^j + c^{jkl}\theta^k A^l_\mu \,. \tag{286}$$

Let us substitute and check. We require

$$-iL^j\partial_\mu\theta^j - ig(A^j_\mu - \frac{1}{g}\partial_\mu\theta^j + c^{jkl}\theta^k A^l_\mu)L^j(1-iL^n\theta^n) \stackrel{?}{=} -igA^j_\mu(1-iL^n\theta^n)L^j\,. \tag{287}$$

Neglecting terms of order θ^2 in the infinitesimal limit, this reduces to

$$\begin{aligned} &-iL^j\partial_\mu\theta^j - igA^j_\mu(1-iL^n\theta^n)L^j - igA^j_\mu[L^j, -iL^n\theta^n] + i\partial_\mu\theta^j L^j - igc^{jkl}\theta^k A^l_\mu L^j \\ &\stackrel{?}{=} -igA^j_\mu(1-iL^n\theta^n)L^j \end{aligned} \tag{288}$$

where our main manipulation was to commute two L's past one another on the left hand side. Canceling common terms, the above reduces to

$$-igA^j_\mu[L^j, -iL^n\theta^n] - igc^{jnl}\theta^n A^l_\mu L^j \stackrel{?}{=} 0\,. \tag{289}$$

But, we know that $[L^j, L^n] = ic^{jnl}L^l$ and the above reduces to

$$-igA^j_\mu(-i\theta^n ic^{jnl}L^l) - igc^{jnl}\theta^n A^l_\mu L^j \stackrel{?}{=} 0\,, \tag{290}$$

or, since we want this to hold for any θ^n choices,

$$c^{jnl}A^j_\mu L^l + c^{jnl}A^l_\mu L^j \stackrel{?}{=} 0\,. \tag{291}$$

Since j and l are dummy summation indices, we may switch them in the first term and the above becomes

$$c^{lnj} A^l_\mu L^j + c^{jnl} A^l_\mu L^j \stackrel{?}{=} 0\,, \tag{292}$$

and we see that the requirement is

$$c^{lnj} + c^{jnl} = 0\,, \tag{293}$$

which is a relation that is always satisfied by the structure constants of a non-Abelian Lie algebra.

Proof: from $[T^i, T^j] = ic^{ijk} T^k$ and the "orthogonality" property of $\mathrm{Tr}\{T^m T^n\} = \frac{1}{2}\delta^{mn}$ we find

$$\begin{aligned} ic^{ijk} &= 2\mathrm{Tr}\{T^k[T^i, T^j]\} = 2\mathrm{Tr}\{T^k T^i T^j - T^k T^j T^i\} \\ &= 2\mathrm{Tr}\{T^i T^j T^k - T^i T^k T^j\} \quad \text{by cyclic property} \\ &= -2\mathrm{Tr}\{T^i T^k T^j - T^i T^j T^k\} = -2\mathrm{Tr}\{T^i[T^k, T^j]\} \\ &= -ic^{kji}\,. \end{aligned} \tag{294}$$

- It is important to note that the required transformation from A^j_μ to ${A'}^j_\mu$ depends only on these c^{ijk}'s and not on any particular ϕ representation employed above.

- So, we must now construct an appropriate kinetic energy $\mathcal{L}$ for the non-Abelian $\vec{A}_\mu$ fields.

  To do so, it is most convenient to use a little "trick".

  So far, we have shown that

$$D'_\mu \phi' = (1 - i\vec{L}\cdot\vec{\theta}) D_\mu \phi \,, \tag{295}$$

  which is to say that $\hat{\phi} = D_\mu \phi$ transforms just like ϕ, implying that

$$D_\nu D_\mu \phi \to (1 - i\vec{L}\cdot\vec{\theta}) D_\nu D_\mu \phi \,, \tag{296}$$

  and

$$D_\mu D_\nu \phi \to (1 - i\vec{L}\cdot\vec{\theta}) D_\mu D_\nu \phi \,. \tag{297}$$

  Then, if we define

$$(D_\mu D_\nu - D_\nu D_\mu)\phi \equiv -igL^j F^j_{\mu\nu}\phi \tag{298}$$

then

$$L^j {F'_{\mu\nu}}^j \phi' = (1 - i\vec{L}\cdot\vec{\theta}) L^j F^j_{\mu\nu} \phi \tag{299}$$

which can be written as

$$L^j {F'_{\mu\nu}}^j (1 - i\vec{L}\cdot\vec{\theta})\phi = (1 - i\vec{L}\cdot\vec{\theta}) L^j F^j_{\mu\nu} \phi \tag{300}$$

which in turn implies that (assuming above must hold for arbitrary ϕ)

$$L^j {F'_{\mu\nu}}^j = (1 - i\vec{L}\cdot\vec{\theta}) L^j F^j_{\mu\nu} (1 + i\vec{L}\cdot\vec{\theta}) = U(\theta)\vec{L}\cdot\vec{F}_{\mu\nu} U^{-1}(\theta) \tag{301}$$

(dropping terms of order θ^2 and defining $\vec{L}\cdot\vec{F}_{\mu\nu} \equiv L^j F^j_{\mu\nu}$). Using this, we can show that

$$\mathrm{Tr}\left[(\vec{L}\cdot\vec{F}_{\mu\nu})(\vec{L}\cdot\vec{F}^{\mu\nu})\right] = \frac{1}{2} F^j_{\mu\nu} F^{j\,\mu\nu} \tag{302}$$

is invariant under gauge transformations.

Proof:

$$\frac{1}{2} F'^{\,j}_{\mu\nu} F'^{\,j\,\mu\nu} \quad = \quad \mathrm{Tr}\left[(\vec{L}\cdot\vec{F}'_{\mu\nu})(\vec{L}\cdot\vec{F}'^{\,\mu\nu})\right]$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[U(\vec{L}\cdot\vec{F}_{\mu\nu})U^{-1}U(\vec{L}\cdot\vec{F}^{\mu\nu})U^{-1}\right] \\
&= \mathrm{Tr}\left[(\vec{L}\cdot\vec{F}_{\mu\nu})(\vec{L}\cdot\vec{F}^{\mu\nu})\right] \\
&= \frac{1}{2}F^j_{\mu\nu}F^{j\,\mu\nu} \qquad (303)
\end{aligned}
$$

using $U^{-1}U = 1$ and the cyclic property of the trace.

- Returning to

$$
L^j {F'_{\mu\nu}}^j = (1 - i\vec{L}\cdot\vec{\theta})L^j F^j_{\mu\nu}(1 + i\vec{L}\cdot\vec{\theta}) \tag{304}
$$

we see that

$$
\delta F^j_{\mu\nu}L^j = -i\theta^m[L^m, F^k_{\mu\nu}L^k] = -i\theta^m F^k_{\mu\nu} i c^{mkj} L^j \tag{305}
$$

or

$$
\delta F^j_{\mu\nu} = c^{jmk}\theta^m F^k_{\mu\nu}\,. \tag{306}
$$

A vector that transforms in this way (i.e. using the group structure constants) is transforming like a member of an *adjoint* representation of the Lie algebra. Let us check this against our definition of the adjoint representation discussed earlier, according to which the transformation matrix is $T^m_{jk} = -ic^{mjk}$. We should have (using the infinitesimal expansion of $U \sim 1 - iL^m\theta^m$, but with $L^m = T^m$ in the adjoint representation)

$$\delta F^j_{\mu\nu} = -iT^m_{jk}\theta^m F^k_{\mu\nu} = -i(-ic^{mjk})\theta^m F^k_{\mu\nu} = -c^{mjk}\theta^m F^k_{\mu\nu} \quad (307)$$

which agrees after using $c^{mjk} = -c^{jmk}$.

- All this can be generalized to the non-infinitesimal case of $\phi' = U(\theta)\phi$ where $U(\theta) = e^{-i\vec{L}\cdot\vec{\theta}}$.

  The basic requirement is analogous to that of the infinitesimal transformation case:

$$D'_\mu \phi'(x) = U(\theta) D_\mu \phi \,, \Rightarrow D'_\mu U(\theta) = U(\theta) D_\mu \,. \quad (308)$$

  Using the form $D_\mu = \partial_\mu - ig\vec{L}\cdot\vec{A}_\mu$, this becomes

$$(\partial_\mu - ig\vec{L}\cdot\vec{A}'_\mu)U = U(\partial_\mu - ig\vec{L}\cdot\vec{A}_\mu) \,, \quad (309)$$

which is equivalent to

$$(\partial_\mu U) + U\partial_\mu - ig\vec{L}\cdot\vec{A}'_\mu U = U\partial_\mu - igU\vec{L}\cdot\vec{A}_\mu \tag{310}$$

which we can multiply (from the right) by U^{-1} to obtain

$$\vec{L}\cdot\vec{A}'_\mu = U\vec{L}\cdot\vec{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}. \tag{311}$$

Our definition of $\vec{F}_{\mu\nu}$ remains the same as before:

$$(D_\mu D_\nu - D_\nu D_\mu)\phi = -igF_{\mu\nu}\phi\,, \tag{312}$$

where $F_{\mu\nu} = \vec{L}\cdot\vec{F}_{\mu\nu}$. Further, we have $D'_\mu D'_\nu \phi' = UD_\mu D_\nu\phi$ and $D'_\nu D'_\mu\phi' = UD_\nu D_\mu\phi$. (Remember the argument? One takes $D_\nu\phi = \hat{\phi}$ and we know that $D'_\mu\hat{\phi}' = UD_\mu\hat{\phi}$, which is to say that $D'_\mu[D'_\nu\phi'] = UD_\mu[D_\nu\phi]$.) As a result we see that

$$-igF'_{\mu\nu}\phi' = (D'_\mu D'_\nu - D'_\nu D'_\mu)\phi' = U(D_\mu D_\nu - D_\nu D_\mu)\phi \tag{313}$$

reduces to (using on the left $\phi' = U\phi$)

$$-igF'_{\mu\nu}U\phi = U(-igF_{\mu\nu})\phi \tag{314}$$

or, since this must hold for arbitrary ϕ,

$$F'_{\mu\nu} = UF_{\mu\nu}U^{-1}\,. \tag{315}$$

This is what we had in the infinitesimal case before. Following exactly the earlier proof, we can show that

$$\mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] = \mathrm{Tr}\left[(\vec{L}\cdot\vec{F}_{\mu\nu})(\vec{L}\cdot\vec{F}^{\mu\nu})\right] = \frac{1}{2}F^j_{\mu\nu}F^{j\,\mu\nu} \tag{316}$$

is invariant under the gauge transformation.

- It is important to note that a $\mathcal{L}$ contribution of the form

$$\frac{1}{2}m^2 A^j_\mu A^{\mu j} = \mathrm{Tr}[m^2(\vec{L}\cdot\vec{A}_\mu)(\vec{L}\cdot\vec{A}^\mu)] \tag{317}$$

would not be invariant under the transform of $\vec{L} \cdot \vec{A}_\mu$ given in Eq. (311) because of the extra term that is not simply $U \vec{L} \cdot \vec{A}_\mu U^{-1}$.

As a result, we cannot *explicitly* give a mass to the gauge fields without violating the fundamental gauge invariance. The consequence of violating the fundamental gauge invariance would be that the theory would be non-renormalizable.

Thus, if we want a massive gauge field in the non-Abelian context, we must find an alternative mechanism to explicit mass introduction. This alternative mechanism is *spontaneous symmetry breaking*. We will not pursue this in detail here; it is the topic of 245B. However, I will make some additional qualitative remarks.

Nature has chosen non-Abelian gauge groups for the Standard Model. One of them corresponds to an unbroken gauge theory. This is QCD (Quantum Chromodynamics), the gauge group of the strong interactions which is the non-Abelian group $SU(3)$. Sometimes it is referred to as the "color" group. Here, the particles corresponding to the gauge field are called gluons and they are massless since the "color" group remains a good symmetry that is not even spontaneously broken. The unbroken nature of the color group, combined with its being $SU(3)$, will, we shall discover in the renormalization course, imply that interactions become very strong (leading to quark confinement) at long distances, while becoming weaker at short distances (higher energy scales).

There is another aspect of the masslessness of the gluons that is important. This is the fact that a massless vector particle can have only two (transverse) polarization states (just like the photon). If one computes gluon-gluon scattering for just transverse polarizations (which we will discuss at the end of this quarter) one discovers that the amplitude is well-behaved in that it does not grow with s. This means that the theory a) is renormalizable, in that loop diagrams are sufficiently under control and b) does not violate unitarity. Unitarity (the analogue of the optical theorem in non-relativistic quantum mechanics) is a statement of probability conservation. One requirement from unitarity is that if one decomposes the scattering amplitude into partial wave amplitudes a_J of given J, then, for example,

$$|\mathrm{Re} a_0| < \frac{1}{2} \tag{318}$$

is required. This is satisfied by QCD since the amplitudes do not grow with s, and instead approach a constant value.

The structure of the rest of the SM is not so simple. In particular, we know experimentally that there are three massive gauge bosons, W^+, W^- and Z. That there were $W^\pm$ gauge bosons with large mass was known long before their actual discovery; they are required to make sense of the weak interactions, in which, for example, we have decays like

$$u \to d e^+ \nu_e \,. \tag{319}$$

You will learn, or perhaps have already learned, that at low energies such interactions are well-described by the Fermi-theory in which the Lagrangian for the above interaction is written as

$$\mathcal{L}_{weak} = \frac{G_F}{\sqrt{2}} \overline{\psi}_d \gamma_\mu P_L \psi_u \overline{\psi}_\nu \gamma^\mu P_L \psi_e \,, \tag{320}$$

where $G_F \sim 10^{-5}\ \mathrm{GeV}^{-2}$ is a dimensionful coupling constant. If one computes scattering of fermions using the Feynman rule deriving from $\mathcal{L}_{weak}$, then one finds violation of unitarity. This goes hand-in-hand with the fact that 1-loop and higher order graphs are not "renormalizable" because of the nature of the divergences. The "fix" proposed was the existence of the $W^\pm$ gauge bosons so that at high energy the correct

picture is

$$u \to dW^{+*} \quad \text{with} \quad W^{+*} \to e^+\nu_e\,, \tag{321}$$

where the $*$ implies that the W is being exchanged as a virtual particle. Because the W^+ propagator behaves as $\frac{1}{q^2-m_W^2}$ (assuming the W is massive) the high q (large momentum) behavior of fermionic scattering processes is controlled while at low q the propagator simply behaves as a multiplicative factor giving $G_F \propto \frac{g_W^2}{-m_W^2}$, where g_W is the strength of the udW and $We\nu$ interaction vertices. Thus, by assuming the existence of a massive W we can simultaneously have the Fermi theory of weak interactions at low energy while avoiding bad high energy behavior for fermionic scattering.

Along the way, there was a choice of weak interaction theories. One theory that could have been correct had no Z boson in addition to the $W^\pm$. Another, chosen by nature, had an additional massive Z. Experiment told us that the 2nd theory was the correct one (implying the $SU(2) \times U(1)$ group structure of the weak and electromagnetic interactions in the SM). The first experimental evidence for the Z was the discovery of "neutral currents" in deep-inelastic scattering that could only be explained by exchange of a virtual massive neutral gauge boson. And, of course, eventually at LEP and SLD the Z was produced directly

and its properties checked against the predictions of the SM.

However, the massiveness of the W and Z gauge bosons presents new problems. How do they acquire mass? We have argued above that gauge invariance of the 2nd kind is violated by simply introducing mass "by hand" in the Lagrangian. Instead, one should turn to spontaneous symmetry breaking, which in the SM is associated with the existence of a Higgs boson. In fact, all of this is again related to bad high energy behavior, unitarity, and renormalizability. Very roughly, the argument goes as follows.

– If we introduce mass by hand, the W, in particular, acquires a longitudinal (W_L) polarizaton state in addition to the two transverse polarizations that it would have if massless.
– If one simply computes $W_L W_L \to W_L W_L$ scattering (there are a bunch of diagrams involving W and Z exchanges in various channels), one discovers that it violates unitarity for energies above $\sim 1~\mathrm{TeV}$.

  This bad high energy behavior would also imply that the mass-by-hand theory would be non-renormalizable (uncontrolled loop corrections).
– If there is a Higgs boson (coming from not giving mass by hand, but rather by spontaneous symmetry breaking), then there are additional diagrams involving exchange of the Higgs, h. These cancel the bad

high energy behavior of the gauge-boson exchange diagrams and this cancellation occurs early enough to avoid unitarity violation provided $m_h \lesssim 1~\mathrm{TeV}$.

It is the fact that $W_L W_L \to W_L W_L$ scattering violates unitarity at (center-of-mass) energies above $\sim 1~\mathrm{TeV}$ and that the Higgs boson that can potentially cure this problem must have mass below $\sim 1~\mathrm{TeV}$ in order to actually do so that motivates the design of the LHC.

- In order to reach cm energy of $1~\mathrm{TeV}$ in the colliding $W_L W_L$ cms, one must have much higher energy for the colliding protons.
  This is because one cannot construct W_L beams. One must start with protons, each of which can then "radiate" a W_L from one of its constituent quarks.
  Each quark has lower energy than its parent proton and the radiated W_L will have only a fraction of the quark's momentum.
- Some detailed work shows that only if the protons each have $\sim 7~\mathrm{TeV}$ will there be adequate probability for there to be two W_L's with roughly $0.5~\mathrm{TeV}$ each that can collide.
  In fact, not only do you need large proton cms energies, you also need many colliding protons in order to have an adequate number of $W_L W_L$ collisions. This determines the required "luminosity" for the colliding

protons at the LHC.

– This is not to say that other signals of physics beyond the SM cannot emerge at lower energies and lower luminosities, *e.g.* a light Higgs boson, extra dimensions, ..., all of which provide a cure to the $W_L W_L$ unitarity problem.

  It is just that if you want to be certain to see how the fundamental $W_L W_L$ unitarity problem is cured, you need to probe $W_L W_L$ scattering itself with a sufficiently high event rate at sufficiently high energy.

– If a light Higgs is present, $W_L W_L$ scattering at $1~\mathrm{TeV}$ will be perfectly well-behaved, *e.g.* obey unitarity and be perturbative in nature.

  This is the SM vision and also the vision of supersymmetric models.

– But, maybe this vision wrong and we will see “strong” interactions (just barely obeying unitarity) in the $W_L W_L$ sector.

– The possibilities are endless and are only limited by the number of theorists and the time they have to develop new ideas before the LHC turns on.

– We expect that the LHC data will very quickly narrow the possibilities along a fairly well-defined path. We just don't know ahead of time what this path will be. This is what makes the next few years so very exciting for both theorists and experimentalists in high energy.

Summary

- For correct normalization (like the $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ of QED) for each of the A^j_μ fields we write

$$\mathcal{L}_A = -\tfrac{1}{4}F^j_{\mu\nu}F^{\mu\nu\,j} = -\frac{1}{2}\mathrm{Tr}[\vec{L}\cdot\vec{F}_{\mu\nu}\vec{L}\cdot\vec{F}^{\mu\nu}] = -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \tag{322}$$

where $F_{\mu\nu} = \vec{L}\cdot\vec{F}_{\mu\nu} = L^j F^j_{\mu\nu}$ was defined by

$$F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = \frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right]\,, \tag{323}$$

which yields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{324}$$

where $A_\nu = \vec{L}\cdot\vec{A}_\nu$.

Obviously, it is useful to write out $F_{\mu\nu}$ in component form. From Eq. (324), we have (using $[L^j, L^k] = ic^{jki}L^i$)

$$L^i F^i_{\mu\nu} = L^i(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu) - ig[L^j A^j_\mu, L^k A^k_\nu]$$
$$= L^i \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gc^{jki} A^j_\mu A^k_\nu\right) , \tag{325}$$

which implies [using $c^{jki} = -c^{ikj}$, as derived earlier, see Eq. (294)]

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - gc^{ikj} A^j_\mu A^k_\nu . \tag{326}$$

Note: Even though we used a specific representation (imagined a matrix set, L^i) to derive the form of $F^i_{\mu\nu}$, the final form of $F^i_{\mu\nu}$ depends only on the A^i_μ fields and the fundamental group structure constants c^{ijk}.

In fact, the above form for $F^i_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ by virtue of the fact that $c^{ikj} = -c^{ijk}$, which also implies that we may write

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gc^{ijk} A^j_\mu A^k_\nu . \tag{327}$$

To prove $c^{ijk} = -c^{ikj}$ we note that $c^{ijk} = -c^{kji} = c^{jki} = -c^{ikj}$, where the first equality is the Eq. (294) antisymmetry used above, the

2nd equality follows from $c^{ijk} = -c^{jik}$ (as required by the definition $[T^i, T^j] = ic^{ijk}T^k$ and the antisymmetry of the commutator), and the last equality is a repeat of the Eq. (294) antisymmetry. (Clearly, the net result is that c^{ijk} is antisymmetric under interchange of any pair of indices.)

- **Equations of motion**

The equations of motion are easily obtained from $\mathcal{L}_A$ above. It is easiest to start from first principles and consider

$$\begin{aligned}
0 &= \delta \int d^4x \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \\
&= 2\int d^4x \mathrm{Tr}\left[F_{\mu\nu}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu - ig[\delta A^\mu, A^\nu] - ig[A^\mu, \delta A^\nu])\right] \\
&= 4\int d^4x \mathrm{Tr}\left[F_{\mu\nu}(-\partial^\nu \delta A^\mu - ig\delta A^\mu A^\nu + igA^\nu \delta A^\mu)\right] . \qquad (328)
\end{aligned}$$

We now use the cyclic properties of the trace to write

$$\mathrm{Tr}[F_{\mu\nu}\delta A^\mu A^\nu] = \mathrm{Tr}[\delta A^\mu A^\nu F_{\mu\nu}]$$

$$\mathrm{Tr}[F_{\mu\nu}A^\nu \delta A^\mu] = \mathrm{Tr}[\delta A^\mu F_{\mu\nu} A^\nu] \tag{329}$$

and also do parts integration on the ∂^μ to write

$$\int d^4x \mathrm{Tr}[F_{\mu\nu}(-\partial^\nu \delta A^\mu)] = \int d^4x \mathrm{Tr}[(\partial^\nu F_{\mu\nu})\delta A^\mu] \tag{330}$$

so that we can extract the coefficient of δA^μ and set it equal to zero. The result is simply

$$\partial^\nu F_{\mu\nu} - ig[A^\nu, F_{\mu\nu}] = -[D^\nu, F_{\nu\mu}] = 0\,. \tag{331}$$

Note that these are a set of highly non-linear equations.

Also note that if A^μ is a solution of this equation so is any of its gauge transforms, since under any gauge transform we know that $D'_\mu = U D_\mu U^{-1}$ and $F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$, implying that we get

$$[D^{\nu\,\prime}, F'_{\nu\mu}] = U[D^\nu, F_{\nu\mu}]U^{-1} = 0\,. \tag{332}$$

- There are many interesting things to explore about a non-Abelian gauge theory. We will have time for only a few.

First, an important theorem is:

$F_{\mu\nu} = \vec{L} \cdot \vec{F}_{\mu\nu} = 0 \Leftrightarrow$ there exists a U such that $\vec{L} \cdot \vec{A}_\mu(x) = -\frac{i}{g}(\partial_\mu U)U^{-1}$

Proof: Direction 1

One direction is easy. We show that $F_{\mu\nu} = 0$ if $\vec{L} \cdot \vec{A}_\mu(x) = -\frac{i}{g}(\partial_\mu U)U^{-1}$.

For this, we need to first return to our definition of $F_{\mu\nu}$ in terms of $D_\mu = \partial_\mu - ig\vec{L} \cdot \vec{A}_\mu$:

$$-igF_{\mu\nu}\phi = (D_\mu D_\nu - D_\nu D_\mu)\phi \,. \tag{333}$$

Writing this out and removing the arbitrary ϕ, we get

$$-igF_{\mu\nu} = \partial_\mu(-ig\vec{L} \cdot \vec{A}_\nu) - \partial_\nu(-ig\vec{L} \cdot \vec{A}_\mu) - g^2[\vec{L} \cdot \vec{A}_\mu, \vec{L} \cdot \vec{A}_\nu] \tag{334}$$

which we rewrite as

$$F_{\mu\nu} = \partial_\mu(\vec{L} \cdot \vec{A}_\nu) - \partial_\nu(\vec{L} \cdot \vec{A}_\mu) - ig[\vec{L} \cdot \vec{A}_\mu, \vec{L} \cdot \vec{A}_\nu] \,. \tag{335}$$

Into this, we substitute the ansatz $\vec{L} \cdot \vec{A}_\mu(x) = -\frac{i}{g}(\partial_\mu U)U^{-1}$ to obtain

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu[-\frac{i}{g}(\partial_\nu U)U^{-1}] - \partial_\nu[-\frac{i}{g}(\partial_\mu U)U^{-1}] \\ &\quad -ig\left(\frac{-i}{g}\right)^2 \left[(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1} - (\partial_\nu U)U^{-1}(\partial_\mu U)U^{-1}\right] \\ &= -\frac{i}{g}\left[(\partial_\nu U)(\partial_\mu U^{-1}) - (\partial_\mu U)(\partial_\nu U^{-1}) - (\partial_\mu U)(-\partial_\nu U^{-1}) + (\partial_\nu U)(-\partial_\mu U^{-1})\right] = 0\,. \end{aligned} \tag{336}$$

In order to get the form after the 2nd $=$, we used two identities for the last two terms of the first form. For example, from

$$0 = \partial_\nu(1) = \partial_\nu(UU^{-1}) = (\partial_\nu U)U^{-1} + U(\partial_\nu U^{-1})\,, \tag{337}$$

implying

$$(\partial_\nu U)U^{-1} = -U(\partial_\nu U^{-1}) \tag{338}$$

which in turn implies

$$(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1} = (\partial_\mu U)U^{-1}[-U(\partial_\nu U^{-1})] = -(\partial_\mu U)(\partial_\nu U^{-1})\,. \tag{339}$$

Proof: Direction 2

In a certain sense, the proof of the theorem in the opposite direction is simple. One simply takes the $A_\mu \equiv \vec{L}\cdot\vec{A}_\mu$ that gives $F_{\mu\nu}=0$ and writes

$$U(x) = P\exp\left[ig\int_C dy^\mu A_\mu(y)\right] \tag{340}$$

where C is a space-time path from, say, the origin at $y=0$ to the location $y=x$ and P is the "path ordering" instruction (sort of like time ordering) that in the expansion of the exponential one always orders the $A_\mu(y)$ values appearing in this expansion in the order they appear along the path with the A_μ nearest the origin being furthest to the right.

From this form, we compute

$$-\frac{i}{g}(\partial_\mu U)U^{-1} = -\frac{i}{g}\left(igA_\mu(x)P\exp\left[ig\int_C dy^\mu A_\mu(y)\right]\right)\left(P\exp\left[ig\int_C dy^\mu A_\mu(y)\right]\right)^{-1} = A_\mu\,. \tag{341}$$

But, obviously, this would work for any A_μ. The point is that the above definition, Eq. (340), only gives a *unique* result for U (i.e. one independent of path) if $F_{\mu\nu}=0$. The requirement for this to be true is clearly that

$$U' = P\exp\left[ig\int_{C'} dy^\mu A_\mu(y)\right] = 1 \tag{342}$$

where C' is the closed contour that is formed by the two different paths that we require to give the same answer for U. To explore this, it is sufficient to assume a very small path for the moment and expand U' to 2nd order in the form

$$
\begin{aligned}
U' &= 1 + ig\int_{C'} dy^\mu A_\mu(y) + \frac{(ig)^2}{2}\int_{C'}\int_{C'} P\left[dy^\mu A_\mu(y) dz^\nu A_\nu(z)\right] \\
&= 1 + ig\int_{C'} dy^\mu A_\mu(y) + (ig)^2\int_{C'}\int_{C'} \left[dy^\mu A_\mu(y) dz^\nu A_\nu(z)\right]_{y>z}, \qquad (343)
\end{aligned}
$$

where $y > z$ means y is further along the curve C' than z. Well, there is still work to do to show that the analogue of Stoke's theorem gives

$$\int_{C'} dy^\mu A_\mu(y) = \int_{S'} dy^\lambda \wedge dy^\alpha (\partial^y_\lambda A_\alpha(y) - \partial^y_\alpha A_\lambda(y)) \,, \qquad (344)$$

where S' is a surface spanned by the closed curve C'. This is fairly easy to understand since Stoke's theorem in 3-d says $\int_{C'} d\vec{x}\cdot\vec{M} = \int_{S'} \vec{\nabla}\times\vec{M}\cdot d\vec{A}$. In 4-d, $\vec{\nabla}\times\vec{M} \to \epsilon^{\mu\nu\lambda\alpha}\partial_\lambda M_\alpha$, while $d\vec{A} \to dy^\delta dy^\beta \epsilon_{\delta\beta\mu\nu}$. Using

$$\epsilon_{\delta\beta\mu\nu}\epsilon^{\mu\nu\lambda\alpha} = 2(g^\lambda_\delta g^\alpha_\beta - g^\lambda_\beta g^\alpha_\delta) \qquad (345)$$

the 3-d result converts to (with $M \to A$)

$$\int_{C'} dy^\mu A_\mu = 2\int_{S'}[dy^\lambda dy^\alpha - dy^\alpha dy^\lambda]\partial_\lambda A_\alpha$$
$$= \int_{S'}[dy^\lambda dy^\alpha - dy^\alpha dy^\lambda](\partial_\lambda A_\alpha - \partial_\alpha A_\lambda)\,. \quad (346)$$

Meanwhile, the form

$$\int_{C'}\int_{C'}[dy^\mu A_\mu(y) dz^\nu A_\nu(z)]_{y>z} \quad (347)$$

can be developed by thinking of it in the form

$$\int_{C'} dy^\mu A_\mu(y)\left(\int_0^y dz^\nu A_\nu(z)\right) = \int_{C'} dy^\mu B_\mu(y)\,. \quad (348)$$

We then apply the above Stoke's technique to convert this to

$$\int_{S'}[dy^\lambda dy^\alpha - dy^\alpha dy^\lambda](\partial_\lambda B_\alpha - \partial_\alpha B_\lambda)$$

$$\sim \int_{S'} [dy^\lambda dy^\alpha - dy^\alpha dy^\lambda](A_\alpha A_\lambda - A_\lambda A_\alpha)\,, \tag{349}$$

dropping terms that are of order $(area)^2$. In short, the quadratic term reduces to

$$-\int_{S'} dy^\lambda \wedge dy^\alpha (A_\lambda(y) A_\alpha(y) - A_\alpha(y) A_\lambda(y)) \tag{350}$$

As a result, we find

$$U' \sim 1 + ig \int_{S'} dy^\lambda \wedge dy^\alpha F_{\lambda\alpha}(y)\,, \tag{351}$$

where

$$F_{\lambda\alpha} = \partial_\lambda A_\alpha - \partial_\alpha A_\lambda - ig[A_\lambda, A_\alpha] \tag{352}$$

is the usual definition of F as given in Eq. (335). Since we have assumed $F_{\lambda\alpha} = 0$ (throughout all of space), $U' \sim 1$.

The difference between two longer paths can be realized by the difference between many smaller paths, or equivalently the surface filling in the

closed contour C' can be subdivided into many little surfaces, and the proof can be applied to each little surface.

This type of A_μ that can be written as

$$A_\mu(x) = -\frac{i}{g}(\partial_\mu U)U^{-1} \tag{353}$$

is referred to as a "pure gauge" form of A_μ.

Connection Between Local Gauge Invariance and Geometry

I will employ a treatment that is a bit different than found in Ryder, Sec. 3.6. Presumably by looking at both you will learn the most.

- In a curved space, to compare a vector field at two different space time points, $V_\mu(x')$ vs. $V_\mu(x)$, you must first parallel transport $V_\mu(x)$ from x to x', and then reference both of them to a given coordinate system defined at x'. The result will take the form

$$V_\mu(x') = (V_\mu(x) + \delta V_\mu) + DV_\mu \tag{354}$$

where δV_μ would be the result of simply parallel transporting the original vector, and DV_μ accounts for the rest of the difference. In a curved space-time, it is DV_μ that is the only appropriate definition of the 'true' differential change in V_μ, since the only way of comparing a vector at one point with a vector at a different point in the curved space-time is by using parallel transport along the natural geodesics of the curved space-time.

(By parallel transport, we mean, for instance in a two-dimensional picture, keeping V_μ at a fixed angle to the tangent to the trajectory.)

Writing $V_\mu(x') - V_\mu(x) = dV_\mu$ we have

$$DV_\mu = dV_\mu - \delta V_\mu \,. \tag{355}$$

This is illustrated in Fig. 2.

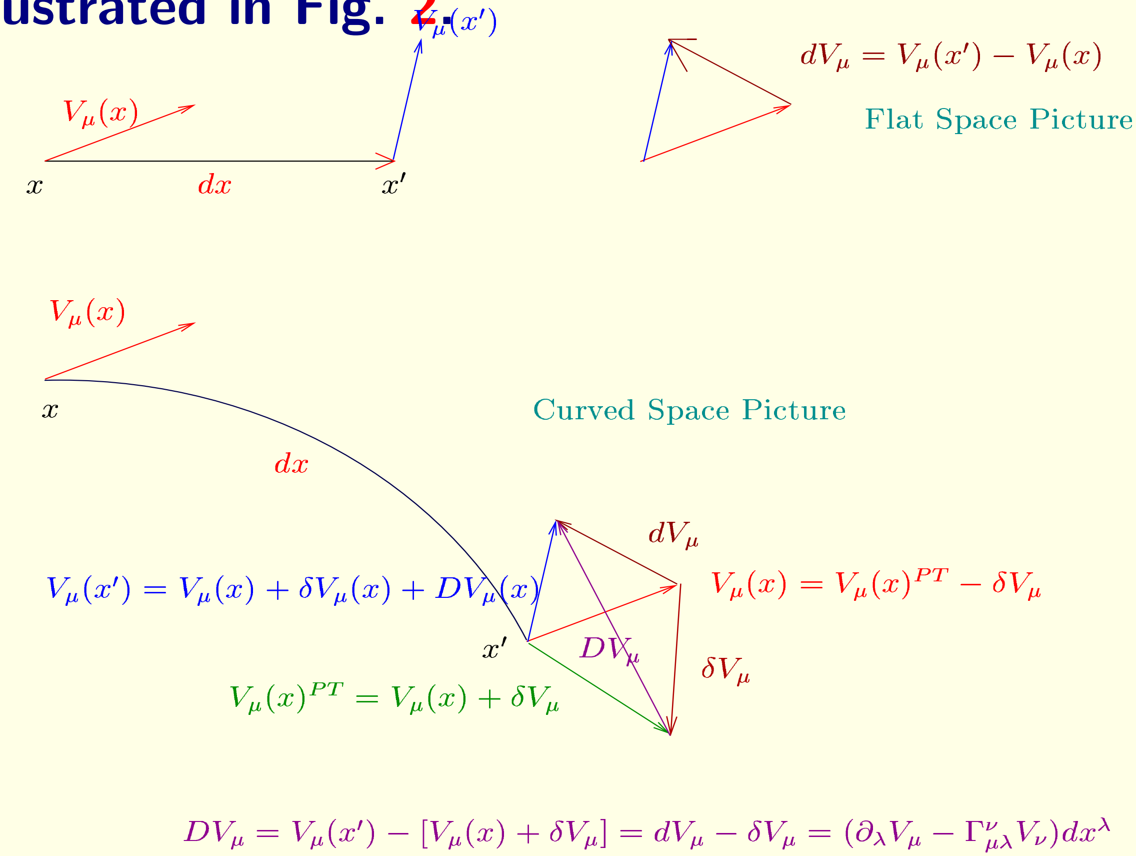


Figure 2: Comparing two vectors in a curved space.

- **In general, we write**

$$\delta V^\mu = -\Gamma^\mu_{\nu\lambda} V^\nu dx^\lambda\,, \quad \delta V_\mu = +\Gamma^\nu_{\mu\lambda} V_\nu dx^\lambda\,, \tag{356}$$

where $\Gamma^\mu_{\nu\lambda}$ is the *Affine connection* **or** *Christoffel symbol* **familiar from general relativity. Since**

$$V^\mu(x') - V^\mu(x) = dV^\mu = \partial_\lambda V^\mu dx^\lambda\,, \tag{357}$$

we have

$$DV^\mu = (\partial_\lambda V^\mu + \Gamma^\mu_{\nu\lambda} V^\nu) dx^\lambda \tag{358}$$

The signs and conventions of Eq. (356) are chosen so that

$$\begin{aligned} 0 = \delta(V_\mu V^\mu) &= V_\mu \delta V^\mu + \delta V_\mu V^\mu \\ &= V_\mu(-\Gamma^\mu_{\nu\lambda} V^\nu dx^\lambda) + (\Gamma^\nu_{\mu\lambda} V_\nu dx^\lambda) V^\mu \\ &= V_\mu V^\nu dx^\lambda (-\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\nu\lambda}) \end{aligned} \tag{359}$$

is obeyed (when $dV_\mu = 0$).

- Another concept in curved space is the curvature tensor. Consider a parallelogram in curved space, defined by 4 points P, P_1, P_2 and P_3. We go from point P to point P_2 via two alternate paths along this parallelogram.

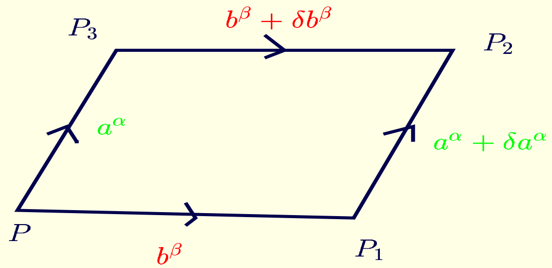


Figure 3: Parallelogram for curvature computation.

First, we go from P to P_1 via b^β and then from P_1 to P_2 via $a^\alpha + \delta a^\alpha$.

Second, we go from point P to point P_3 by a small a^α and then from P_3 to point P_2 via $b^\beta + \delta b^\beta$.

Now consider the round trip, $P \to P_1 \to P_2 \to P_3 \to P$. The net change in a vector due just to parallel transport is

$$\Delta V_\mu = \delta V_\mu^{PP_1P_2} - \delta V_\mu^{PP_3P_2}\,. \tag{360}$$

Let's write this out.

We have

$$\begin{aligned}
\delta V_\mu^{PP_1P_2} &= (\Gamma^\nu_{\mu\beta}V_\nu)_P b^\beta + (\Gamma^\nu_{\mu\alpha}V_\nu)_{P_1}(a^\alpha + \delta a^\alpha) \\
\delta V_\mu^{PP_3P_2} &= (\Gamma^\nu_{\mu\alpha}V_\nu)_P a^\alpha + (\Gamma^\nu_{\mu\beta}V_\nu)_{P_3}(b^\beta + \delta b^\beta)\,. \qquad (361)
\end{aligned}$$

Now we need to expand around the P_1 and P_3 points.

$$\begin{aligned}
(\Gamma^\nu_{\mu\alpha}V_\nu)_{P_1} &= (\Gamma^\nu_{\mu\alpha} + \partial_\beta\Gamma^\nu_{\mu\alpha}b^\beta)_P(V_\nu + \Gamma^\sigma_{\nu\beta}V_\sigma b^\beta)_P \\
(\Gamma^\nu_{\mu\beta}V_\nu)_{P_3} &= (\Gamma^\nu_{\mu\beta} + \partial_\alpha\Gamma^\nu_{\mu\beta}a^\alpha)_P(V_\nu + \Gamma^\sigma_{\nu\alpha}V_\sigma a^\alpha)_P\,. \qquad (362)
\end{aligned}$$

From now on, the P subscript is omitted for quantities evaluated at point P. We also use

$$\begin{aligned}
(a^\alpha + \delta a^\alpha) &= a^\alpha - \Gamma^\alpha_{\tau\eta}a^\tau b^\eta \\
b^\beta + \delta b^\beta &= b^\beta - \Gamma^\beta_{\tau\eta}b^\tau a^\eta\,. \qquad (363)
\end{aligned}$$

We obtain

$$\Delta V_\mu = a^\alpha b^\beta V_\nu\left(\partial_\beta\Gamma^\nu_{\mu\alpha} + \Gamma^\lambda_{\mu\alpha}\Gamma^\nu_{\lambda\beta} - \partial_\alpha\Gamma^\nu_{\mu\beta} - \Gamma^\lambda_{\mu\beta}\Gamma^\nu_{\lambda\alpha}\right)$$

$$\equiv \quad a^\alpha b^\beta V_\nu R^\nu_{\mu\alpha\beta}, \tag{364}$$

where the contributions from the δa^α and δb^β terms in Eq. (363) ended up canceling by virtue of the symmetry $\Gamma^\sigma_{\tau\eta} = \Gamma^\sigma_{\eta\tau}$.

In general relativity $R^\nu_{\mu\alpha\beta}$ would be called the Riemann-Christoffel curvature tensor. The idea is that if the transport around the closed path ends up producing a change in V_μ then the space-time truly is curved and the non-zero value of $R^\nu_{\mu\alpha\beta}$ signals the presence of true curvature. In particular, it is possible for the connection $\Gamma^\lambda_{\mu\alpha}$ to have a non-zero value even if there is no true curvature.

- Now let us establish the manner in which gauge theory is analogous to the above structures. Here, the curvature tensor and connections will largely involve some internal index space rather than real space-time, but a very close analogy can be constructed.

  We begin by comparing the covariant derivative

$$DV^\mu = (\partial_\lambda V^\mu + \Gamma^\mu_{\nu\lambda} V^\nu) dx^\lambda \tag{365}$$

to the non-Abelian gauge theory covariant derivative expression

$$D_\mu \phi^j = \partial_\mu \phi^j - ig(\vec{L})^j{}_k \cdot \vec{A}_\mu \phi^k \,. \tag{366}$$

From this comparison we see that the gauge group connection is

$$\Gamma^j_{k\mu} = -ig(\vec{L})^j{}_k \cdot \vec{A}_\mu = -ig(A_\mu)^j{}_k \tag{367}$$

in the *internal charge space*, i.e. nothing to do with space-time for the first two indices. Thus, it is as if the gauge fields establish a curved system of internal coordinates that must be used in computing how an internal space vector changes as one moves in real space-time. The gauge field provides the instruction as to how to compensate for the change of the local internal index 'frame' in going between different space-time points and

$$\delta\phi^j = -\Gamma^j_{k\mu}\phi^k dx^\mu = ig(\vec{L})^j{}_k \cdot \vec{A}_\mu dx^\mu \tag{368}$$

under parallel transport. With this identification, the curvature tensor becomes

$$R^j_{k\alpha\beta} \;=\; -\left(\partial_\alpha \Gamma^j_{k\beta} - \partial_\beta \Gamma^j_{k\alpha} + \Gamma^l_{k\beta}\Gamma^j_{l\alpha} - \Gamma^l_{k\alpha}\Gamma^j_{l\beta}\right)$$

$$
\begin{aligned}
&= +ig(\partial_\alpha A_\beta - \partial_\beta A_\alpha - ig[A_\alpha, A_\beta])^j{}_k \\
&= ig(F_{\alpha\beta})^j{}_k\,.
\end{aligned} \tag{369}
$$

We conclude that the gauge theory analogue of the curvature tensor is non-zero only if the gauge field is not a pure gauge, i.e. only if there is a non-zero $F_{\mu\nu}$. The presence of a physically non-trivial field configuration "distorts" the relation between the internal space of the gauge group indices and the Lorentz index of real space-time. The 'coordinate axes' for the internal indices are changing as one moves about in real space-time, and these changes are truly physical if the $F_{\mu\nu}$ is non-zero.

- We can gain further intuition regarding the gauge transformation as follows.

  Consider at each x a gauge transform by $\theta(x)$ of some matter field representation $\phi(x)$:

$$
\begin{aligned}
\phi(x) &\to \phi'(x) = U(\theta(x))\phi(x) \\
\phi(x') &\to \phi'(x') = U(\theta(x'))\phi(x')\,.
\end{aligned} \tag{370}
$$

Now, under parallel transport ($D\phi = 0$), define

$$\begin{aligned} \phi^j(x) &\rightarrow \phi^j(x+dx) \\ &= \phi^j(x) + \delta\phi^j(x) = \phi^j(x) + ig(\vec{L})^j{}_k \cdot \vec{A}_\mu dx^\mu \phi^k(x) \\ &\equiv P^j{}_k(x+dx, x)\phi^k(x) \,. \end{aligned} \tag{371}$$

For a finite interval, the expression for the parallel transport matrix P becomes

$$P(x', x) = P \exp\left\{ ig \int_x^{x'} \vec{L} \cdot \vec{A}_\mu(y) dy^\mu \right\} . \tag{372}$$

We further note that we can write ($\overline{\phi}$ is the matrix hermitian conjugate, a row vector, of the column vector ϕ)

$$\overline{\phi}(x+dx)\phi(x+dx) = \overline{\phi}(x+dx)P(x+dx, x)\phi(x) \tag{373}$$

under parallel transport. In order to keep $\overline{\phi}(z)\phi(z)$ invariant under gauge transform, this then means that we must keep the product

$\overline{\phi}(x+dx)P(x+dx,x)\phi(x)$ invariant under a gauge transform. Thus, we require

$$\begin{aligned}
\overline{\phi}'(x+dx)\phi'(x+dx) &\sim \overline{\phi}'(x+dx)P'(x+dx,x)\phi'(x) \\
&= \overline{\phi}(x+dx)U^{-1}(x+dx)P'(x+dx,x)U(x)\phi(x) \\
&= \overline{\phi}(x+dx)P(x+dx,x)\phi(x) \\
&= \overline{\phi}(x+dx)\phi(x+dx)\,, \qquad (374)
\end{aligned}$$

where the line-2 equality follows from $\overline{\phi}' = \overline{\phi}U^\dagger = \overline{\phi}U^{-1}$ as a result of the unitarity of U. This implies the requirement

$$P'(x+dx,x) = U(x+dx)P(x+dx,x)U^{-1}(x)\,, \qquad (375)$$

or, using the standard notation $A_\mu = \vec{L}\cdot\vec{A}_\mu$,

$$(1+igA'_\mu dx^\mu) = (U(x)+\partial_\mu U(x)dx^\mu)(1+igA_\mu dx^\mu)U^{-1}(x)\,. \qquad (376)$$

Isolating the coefficient of dx^μ this becomes

$$igA'_\mu = (\partial_\mu U)U^{-1} + U(igA_\mu)U^{-1}\,, \qquad (377)$$

our usual gauge transformation equation for A_μ! In short, gauge invariance of the concept of parallel transport for a matter field determines the gauge transformation for the vector field.

- Finally, we note without proof (see Ryder for details) that the non-Abelian group Jacobi identity

$$[[T^i,T^j],T^k]+[[T^j,T^k],T^i]+[[T^k,T^i],T^j]=0 \tag{378}$$

(which follows simply from the definition of the commutator and leads to the structure constant relation $c^{ijm}c^{mkl}+c^{jkm}c^{mil}+c^{kim}c^{mjl}=0$) is equivalent to

$$[D_\rho,[D_\mu,D_\nu]]+[D_\mu,[D_\nu,D_\rho]]+[D_\nu,[D_\rho,D_\mu]]=0 \tag{379}$$

which in turn has the general relativity analogue

$$D_\rho R^\kappa_{\lambda\mu\nu}+D_\mu R^\kappa_{\lambda\nu\rho}+D_\nu R^\kappa_{\lambda\rho\mu}=0 \tag{380}$$

known as the Bianci identity. To see the analogy of course requires remembering that $[D_\mu,D_\nu]\propto F_{\mu\nu}$ and that $(F_{\mu\nu})^i{}_k$ is the analogue of $R^\kappa_{\lambda\mu\nu}$.

Path Integral 2nd Quantization for Non-Abelian Gauge Theory

Illustrative analogue example of gauge fixing on gauge orbit

- Consider

$$Z = \int dx dy e^{iS(x,y)} = \int d\vec{r} e^{iS(\vec{r})} . \tag{381}$$

- Suppose $S(\vec{r}) = S(\vec{r}_\phi)$, where $\vec{r} = (r, \theta)$ and $\vec{r}_\phi = (r, \theta + \phi)$.

  For example, $S = S(r)$.

- In this case, it is clear that there is a factoring "volume" factor $\int d\theta = 2\pi$.

  We will divide out this factor in a fancy way that will be closely analogous to what we do in the gauge theory.

1. Write $1 = \int d\phi \delta(\theta - \phi)$.
2. Insert this into Z:

$$Z = \int d\phi \int d\vec{r} e^{iS(\vec{r})} \delta(\theta - \phi) = \int d\phi Z_\phi , \tag{382}$$

where

$$Z_\phi = \int d\vec{r}\delta(\theta - \phi)e^{iS(\vec{r})}\,. \tag{383}$$

3. Now, $Z_\phi = Z_{\phi'}$ due to invariance of $S(\vec{r})$ so that

$$Z = \int d\phi Z_\phi = Z_\phi \int d\phi = 2\pi Z_\phi\,. \tag{384}$$

4. More generally, a more complicated constraint can be chosen: $g(\vec{r}) = 0$ with function g such that $g(\vec{r})$ intersects each orbit of constant r *only once*.

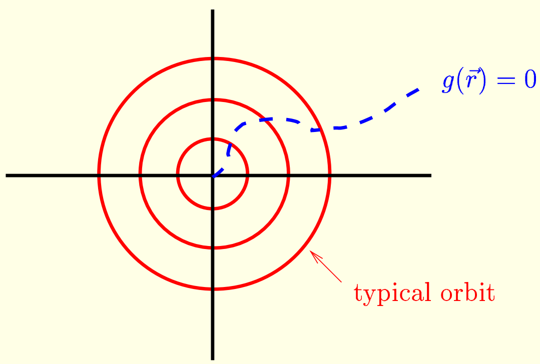


Figure 4: Illustration of gauge orbit 2-d analogue.

Define

$$[\Delta_g(\vec{r})]^{-1} = \int d\phi\delta[g(\vec{r}_\phi)] = \left[\left.\frac{\partial g(\vec{r})}{\partial\theta}\right|_{g=0}\right]^{-1}. \tag{385}$$

Note that this can be rewritten as

$$1 = \Delta_g(\vec{r}) \int d\phi \delta[g(\vec{r}_\phi)] \,. \tag{386}$$

5. Δ is invariant since

$$\begin{aligned} [\Delta_g(\vec{r}_{\phi'})]^{-1} &= \int d\phi \delta[g(\vec{r}_{\phi+\phi'})] \\ &= \int d\phi'' \delta[g(\vec{r}_{\phi''})] \\ &= [\Delta_g(\vec{r})]^{-1} \end{aligned} \tag{387}$$

where we simply shifted integration variables to $\phi'' = \phi + \phi'$.

6. Continuing on, we write

$$Z = \int d\phi Z_\phi \,, \quad \text{with} \quad Z_\phi = \int d\vec{r} e^{iS(\vec{r})} \Delta_g(\vec{r}) \delta[g(\vec{r}_\phi)] \tag{388}$$

where we have inserted "1" using Eq. (386).

7. As before, Z_ϕ is rotationally invariant.

To prove this, let us define $\vec{r}_{\phi'} = (r, \theta + \phi')$, with $\phi \neq \phi'$. We then have, from the defining equation for Z_ϕ above,

$$\begin{aligned} Z_{\phi'} &= \int d\vec{r} e^{iS(\vec{r})} \Delta_g(\vec{r}) \delta[g(\vec{r}_{\phi'})] \\ &= \int d\vec{r}' e^{iS(\vec{r}')} \Delta_g(\vec{r}') \delta[g(\vec{r}'_{\phi})] \\ &= Z_\phi \quad \text{since } \vec{r}' \text{ is a dummy integration variable}, \end{aligned} \tag{389}$$

where $\vec{r}' = (r, \theta + \phi' - \phi)$ and $\vec{r}'_\phi = (r, \theta + \phi') = \vec{r}_{\phi'}$, and we used the $S(\vec{r}') = S(\vec{r})$ and $\Delta_g(\vec{r}') = \Delta_g(\vec{r})$ invariances, and the invariance of the integration measure $\int d\vec{r} = \int d\vec{r}'$.

- **So, the net result is that we can write**

$$Z = \int d\phi Z_\phi = (2\pi) Z_\phi \,, \tag{390}$$

where

$$Z_\phi = \int d\vec{r} e^{iS(\vec{r})} \Delta_g(\vec{r}) \delta[g(\vec{r}_\phi)] \tag{391}$$

can be defined with any choice of ϕ we like.

- The final very crucial point is that Z_ϕ is not only independent of ϕ, but it is also independent of the form of g (so long as $g(\vec{r}_\phi) = 0$ defines a unique point on each orbit).

  The proof begins with the definition

$$1 = \Delta_g(\vec{r}) \int d\phi \delta[g(\vec{r}_\phi)] = \Delta_g(\vec{r}) \left[\frac{\partial g(\vec{r}_\phi)}{\partial \phi}\right]^{-1}_{g=0} \tag{392}$$

  which obviously implies that

$$\Delta_g(\vec{r}) = \left[\frac{\partial g(\vec{r}_\phi)}{\partial \phi}\right]_{g=0} . \tag{393}$$

  To proceed, we use (i) $d\vec{r} = r\, dr d\theta$, (ii) $S(\vec{r}) = S(r)$, (iii) $\int d\theta = \int d(\theta + \phi)$ and (iv) $\left[\frac{\partial g(r,\theta+\phi)}{\partial \phi}\right]_{g=0} = \left[\frac{\partial g(r,\theta+\phi)}{\partial(\theta+\phi)}\right]_{g=0}$ — all that matters is the $g = 0$ instruction for where to evaluate the derivative — to obtain

$$\begin{aligned} Z_\phi &= \int r\, dr d\theta e^{iS(\vec{r})} \left[\frac{\partial g(\vec{r}_\phi)}{\partial \phi}\right]_{g=0} \delta[g(\vec{r}_\phi)] \\ &= \int r\, dr d(\theta+\phi) e^{iS(r)} \left[\frac{\partial g(r,\theta+\phi)}{\partial(\theta+\phi)}\right]_{g=0} \delta[g(r,\theta+\phi)] \\ &= \int r\, dr e^{iS(r)} \end{aligned} \tag{394}$$

where the last equality simply follows from the chain rule

$$\int d(\theta+\phi) \left[\frac{\partial g(r,\theta+\phi)}{\partial(\theta+\phi)}\right]_{g=0} \delta[g(r,\theta+\phi)] = \int dg \delta[g] = 1\,. \tag{395}$$

Thus, Z_ϕ is clearly independent of the form of g. The only requirement in all the above manipulations was that $\left[\frac{\partial g(r,\theta+\phi)}{\partial \phi}\right]_{g=0} \neq 0$, i.e. that the g curve of the figure never be parallel to the orbit at constant r.

Back to Gauge Theory

- **We have an action that is gauge invariant: that is, it is constant on the orbit of the gauge group formed out of all the $A_\mu^{\vec{\theta}}$, where $\vec{\theta}$ is the set of gauge parameters specifying the gauge transformation $U(\theta) = e^{-i\vec{L}\cdot\vec{\theta}}$, obtained starting with some fixed A_μ and allowing $\vec{\theta}$ to vary over all elements of the group.**

  In what follows, we temporarily assume the $SU(2)$ group with 3 A_μ^a ($a = 1, 2, 3$) denoted by $\vec{A}_\mu$ and three gauge parameters denoted by $\vec{\theta}$, where both objects are three component vectors.

  We must restrict the path integral to a hypersurface which intersects each orbit *only once***.**

  For example, we could try to define an appropriate hypersurface using the equations (3 equations are required for the $SU(2)$ group since there are 3 gauge parameters specifying a gauge transformation)

$$f_a(\vec{A}_\mu(x)) = 0\,, \quad a = 1, 2, 3 \quad \text{for } SU(2) \text{ group}\,. \tag{396}$$

Then, the set of equations

$$f_a(\vec{A}_\mu^{\vec{\theta}}(x)) = 0\,, \quad a = 1, 2, 3 \tag{397}$$

must have a unique solution, $\vec{\theta}$*, for any given starting* $\vec{A}_\mu$ *at every space-time point* x*.*

- To use the formalism developed in our simple example, we also need to define the integration measure over the group space. We do this by referencing to the infinitesimal form

$$U(\theta) = 1 - i\vec{L}\cdot\vec{\theta} + \mathcal{O}(\theta^2) \tag{398}$$

and defining

$$d\theta = \prod_{a=1}^{3} d\theta_a\,. \tag{399}$$

This exhibits an important invariance, namely

$$d(\theta\theta') = d\theta' \tag{400}$$

since

$$d(\theta\theta') = \prod_{a=1}^{3} d(\theta_a + \theta'_a) = \prod_{a=1}^{3} d\theta'_a \,, \tag{401}$$

as required for consistency with $U(\theta\theta') = U(\theta)U(\theta')$ and the exponential expansion

$$U(\theta)U(\theta') \sim 1 - i\vec{L} \cdot (\vec{\theta} + \vec{\theta}') + \ldots \,. \tag{402}$$

- Now, we must remember that $\vec{\theta} = \vec{\theta}(x)$ in the local gauge transformation case. So the appropriate definition of the Δ factor in the gauge theory case is

$$\Delta_f^{-1}[\vec{A}_\mu] = \int [d\vec{\theta}(x)] \prod_{a,x} \delta[f_a(\vec{A}^{\vec{\theta}}_\mu(x))] \,, \tag{403}$$

which is to say that we must integrate over all gauge transformations *as a function of* x (which means we have something very much in the nature of a functional integral) and fix all these with enough δ functions. So, Ryder calls the above a "delta functional" since it is really a product of δ functions with one δ function for each group generator a *and* each space-time point x (or in the case where we divide space-time up into cells, there is a δ function for each space-time cell).

It is conventional to write the complicated product of δ functions using the shorthand notation:

$$\delta[f_a(\vec{A}_\mu)] \equiv \prod_{a,x} \delta[f_a(\vec{A}_\mu(x))]\,, \tag{404}$$

where the left-hand side is the notation for the "delta functional".

The implication of the above formula can perhaps better be appreciated by writing it in the form

$$\Delta_f[\vec{A}_\mu] = \det M_f \tag{405}$$

where

$$(M_f)_{ab}(x,y) = \left.\frac{\delta f_a(\vec{A}_\mu^{\vec{\theta}}(x))}{\delta\theta_b(y)}\right|_{f_a=0,\quad a=1,2,\ldots}\,. \tag{406}$$

The determinant is thus a determinant over all a, b and over all (x, y) — clearly something that can only be defined by dividing space-time up into many cells in the usual fashion.

Another phraseology is that M_f is the response of the $f_a[\vec{A}_\mu]$ to the infinitesimal gauge transformation away from $f_a[\vec{A}_\mu] = 0$.

- **To understand better the determinant origin, let us simplify to one space-time cell and to a two-parameter gauge group. Then, we would have two f_a functions ($a = 1, 2$) and for an infinitesimal transformation away from the $f_1 = 0, f_2 = 0$ solutions we can write**

$$
\begin{aligned}
&\int [d\theta(x)] \prod_{a,x} \delta[f_a(\vec{A}_\mu^{\vec{\theta}}(x))] \\
= &\int d\theta^1 d\theta^2 \delta(M^{11}\theta^1 + M^{12}\theta^2)\delta(M^{21}\theta^1 + M^{22}\theta^2)\,, \qquad (407)
\end{aligned}
$$

where the first δ function is for $f_{a=1}$ and the 2nd is for $f_{a=2}$. We now perform the θ^1 integral using the first δ function to get

$$
\begin{aligned}
= &\ \frac{1}{M^{11}} \int d\theta^2 \delta\left(M^{21}\left[\frac{-M^{12}\theta^2}{M^{11}}\right] + M^{22}\theta^2 \right) \\
= &\int d\theta^2 \delta\left([M^{22}M^{11} - M^{21}M^{12}]\theta^2 \right) \\
= &\ \frac{1}{\det M}\,. \qquad (408)
\end{aligned}
$$

This same simplified example applies equally well to a one-parameter

gauge group and a division of space-time into two cells.

I hope that it is obvious that it generalizes.

- Let us move on to the example of $SU(2)$.

  There, we have (for small θ^a's)

$$A_\mu^{a\,\theta} = A_\mu^a + \epsilon^{abc}\theta^b A_\mu^c - \frac{1}{g}\partial_\mu \theta^a \tag{409}$$

and the response function appears in the expansion

$$f_a[\vec{A}_\mu^{\vec{\theta}}(x)] = f_a[\vec{A}_\mu(x)] + \int d^4y [M_f(x,y)]_{ab}\theta_b(y) + \mathcal{O}(\theta^2)\,. \tag{410}$$

The unique solution requirement is equivalent to $\det M_f \neq 0$.

- Let us see how this works in a particularly simple case: The Axial Gauge.

1. We choose $f_a = A_3^a = 0$ ($a = 1,2,3$ for $SU(2)$) to define the gauge.

2. We write

$$\begin{aligned} f_a(\vec{A}^{\vec{\theta}}_\mu(x)) &= A^a_3(x) + \epsilon^{abc}\theta^b(x)A^c_3(x) - \frac{1}{g}\partial_3\theta^a(x) \\ &= -\frac{1}{g}\partial_3\theta^a(x) \quad \textbf{since } \vec{A}_3 = 0\,. \end{aligned} \tag{411}$$

3. Matching this to the general form of Eq. (410), implies that

$$[M_f(x,y)]_{ab} = -\frac{1}{g}\partial^x_3\delta^4(x-y)\delta_{ab}\,. \tag{412}$$

4. The axial gauge picks out a unique element on each gauge trajectory provided the fields vanish fast enough at ∞.
   The logic is as follows.
   (a) Gauge transforms, U, which preserve $\vec{A}_3 = 0$ are independent of x^3.
   (b) If the fields vanish at ∞ then U must keep the fields vanishing at ∞, implying that $U \to 1$ at ∞ in all directions.
   (c) If $U \to 1$ at ∞ and U is independent of x^3 *everywhere*, then $U = 1$ everywhere.

(d) **Thus, any U that attempts to move from the axial gauge to another version of the axial gauge, while keeping fields vanishing at ∞, is trivial, implying that the axial gauge defines a unique point on every gauge orbit.**

In any case, that $\det M_f \neq 0$ is clear from the explicit diagonal (and non-zero) form of Eq. (412).

The axial $\vec{A}_3 = 0$ gauge is, in fact, a very useful gauge not only because $\det M_f \neq 0$ but most importantly because M_f is independent of the gauge field. This makes it a very simple gauge to use in many applications.

In fact, in this gauge it is possible to show that canonical quantization and path-integral quantization are completely equivalent.

Then, since physical results are independent of gauge, we conclude that any canonical quantization will give the same physical results as any path-integral quantization (with gauge fixing incorporated in both techniques).

We probably do not have time to go through this proof and so I will skip it for now. It can be found in Cheng and Li, with a related proof in a different gauge in Abers and Lee.

- Let us now return to the general case, and show that $\Delta_f[\vec{A}_\mu]$ is gauge invariant. We have:

$$\Delta_f[\vec{A}_\mu]^{-1} = \int [d\theta'(x)]\delta[f_a(\vec{A}_\mu^{\vec{\theta}'}(x))] \tag{413}$$

and

$$\begin{aligned}
\Delta_f[\vec{A}_\mu^{\vec{\theta}}]^{-1} &= \int [d\theta'(x)]\delta[f_a(\vec{A}_\mu^{(\vec{\theta'\theta})}(x))] \\
&= \int d[\theta'(x)\theta(x)]\delta[f_a(\vec{A}_\mu^{(\vec{\theta'\theta})}(x))] \\
&= \int d[\theta''(x)]\delta[f_a(\vec{A}_\mu^{(\vec{\theta''})}(x))] \\
&= \Delta_f[\vec{A}_\mu]^{-1}, \qquad (414)
\end{aligned}$$

where we simply shifted to the net gauge transformation specified by $\vec{\theta''} \equiv \vec{\theta'\theta}$ and used the crucial invariance of the gauge parameter integration measure emphasized earlier around Eq. (400).

In words, what is happening is the following. In the computation of

$\Delta_f[\vec{A}_\mu]^{-1}$ we start at some field value $\vec{A}_\mu$ and apply θ' transforms until we reach the point where (for every a and every x) $f_a(\vec{A}_\mu^{\vec{\theta}'}(x)) = 0$. In the computation of $\Delta_f[\vec{A}_\mu^{\vec{\theta}}]^{-1}$ we start at $\vec{A}_\mu^{\theta}$ and apply θ''s until we reach $f_a(\vec{A}_\mu^{(\vec{\theta'\theta})}(x)) = 0$. *The solution point in vector potential space is the same, even though we started from different points on the gauge orbit, and so the Jacobian obtained by examining the response to small deviations about the solution point does not change!*

- We now proceed as in the simple example and substitute "1" into the path integral using the above and see what happens.

  We begin with the general path integral (analogous to earlier Z of simple example)

$$
\begin{aligned}
&\int [d\vec{A}_\mu] \exp\left\{ i \int d^4x \mathcal{L}(x) \right\} \\
= &\int [d\vec{\theta}(x)][d\vec{A}_\mu(x)] \Delta_f[\vec{A}_\mu] \delta[f_a(\vec{A}_\mu^{\vec{\theta}})] \exp\left\{ i \int d^4x \mathcal{L}(A_\mu(x)) \right\} \\
= &\int [d\vec{\theta}(x)][d\vec{A}_\mu^{\vec{\theta}}(x)] \Delta_f[\vec{A}_\mu^{\vec{\theta}}] \delta[f_a(\vec{A}_\mu^{\vec{\theta}})] \exp\left\{ i \int d^4x \mathcal{L}(A_\mu^{\theta}(x)) \right\}
\end{aligned}
$$

$$= \int [d\vec{\theta}(x)][d\vec{A}_\mu(x)]\Delta_f[\vec{A}_\mu]\delta[f_a(\vec{A}_\mu)] \exp\left\{ i \int d^4x \mathcal{L}(A_\mu(x)) \right\} \tag{415}$$

where in the one step we used the gauge invariance of Δ_f, of $\mathcal{L}$ and of the integration measure $\int [d\vec{A}_\mu]$ [2] and in the next step we simply shifted to $\vec{A}^{\vec{\theta}}_\mu \equiv \vec{A}'_\mu$ and then dropped the prime.

- We can now factor out the infinite quantity $\int [d\vec{\theta}(x)]$, since everything else has no dependence on $\vec{\theta}$, and define

$$Z_f[J] = \int [d\vec{A}_\mu(x)]\Delta_f[\vec{A}_\mu]\delta[f_a(\vec{A}_\mu)] \exp\left\{ i \int d^4x [\mathcal{L}(A_\mu(x)) + \vec{J}_\mu \cdot \vec{A}^\mu] \right\} \tag{416}$$

and it is always useful to keep in mind the equivalences (ignoring indices):

$$\Delta_f[\vec{A}_\mu] = \det M_f = \det \left| \frac{\delta f}{\delta \theta} \right|_{f=0} . \tag{417}$$

[2] If we are integrating over all vector field values, that is equivalent to integrating over all vector field values transforming by some given θ. Either way, the coverage of vector field values is complete.

Eq. (416) defines the Faddeev-Popov ansatz. $Z_f[J]$ will be finite since we have removed the gauge invariance infinity.

- **The proof of the equivalence of canonical quantization with path-integral quantization using the FP ansatz in the axial gauge would thus begin with [3]**

$$Z_f[J] = \int [d\vec{A}_\mu] \prod_a \delta[A_3^a] \exp\left\{ i \int d^4x [\mathcal{L}(A_\mu(x)) + \vec{J}_\mu \cdot \vec{A}^\mu] \right\} \quad (418)$$

As stated, we do not have time for this proof. I must ask you to assume that such a proof exists. You can attempt to penetrate the version given in Cheng and Li and the related one in Abers and Lee. (Proof in my notes is found in 245B gray notebook, a summary of which appears in an Appendix which follows this section.)

- **Assuming that the equivalence between the 2nd quantization and path-integral approaches can be established in axial gauge, the only remaining**

[3] In the axial gauge, we can drop $\Delta_f[\vec{A}_\mu]$ since it is field-independent. As we shall see, in general we cannot drop this factor since it will depend on $\vec{A}_\mu$. The ability to drop $\Delta_f[\vec{A}_\mu]$ is the reason why the equivalence between 2nd-quantization and the Faddeev-Popov path-integral procedure is relatively straightforward to establish in axial gauge.

item is to show that the path integral approach itself is gauge independent.

If so then the path integral approach is equivalent to 2nd quantization regardless of the gauge employed for the path integral approach.

We thus wish to show that (dropping the generator indices a for simplicity in the shorthand notation below)

$$Z_f = \int [dA_\mu]\delta[f]\det\left|\frac{\delta f}{\delta\theta}\right|_{f=0} e^{iS[A]} \tag{419}$$

is independent of f.

Proof:

1. Write $[dA_\mu]$ as follows.
   Consider the A_μ which are equivalent by gauge transform (i.e. those on the same gauge orbit).
   Choose a $\widehat{A}_\mu$ on each orbit.
   Then, all of the A_μ are generated from the collection of $\widehat{A}_\mu$ by $U(\vec{\theta})$'s.
   This means that
   $$[dA_\mu] = [d\widehat{A}_\mu][d\vec{\theta}] \tag{420}$$

2. Using this decomposition we can write

$$Z_f = \int [d\widehat{A}_\mu][d\vec{\theta}]\delta[f(\widehat{A}_\mu^{\vec{\theta}})] \det \left|\frac{\delta f}{\delta\theta}\right|_{f=0} e^{iS[\widehat{A}]} \tag{421}$$

where we used the fact that $S[\widehat{A}_\mu^{\vec{\theta}}] = S[\widehat{A}_\mu]$ by virtue of gauge invariance of the action. (Keep remembering that we are suppressing all the group generator labels a, that are actually present in $\delta[\ldots]\det[\ldots]$ and $[d\widehat{A}_\mu]$.)

3. Now notice that

$$\int [d\vec{\theta}]\delta[f(\widehat{A}_\mu^{\vec{\theta}})] \det \left|\frac{\delta f}{\delta\theta}\right|_{f=0} = \int df \delta[f] = 1\,, \tag{422}$$

i.e. we can simply change variables, *which is always possible provided that* $\det\left|\frac{\delta f}{\delta\theta}\right| \neq 0$. This latter is equivalent to the requirement that $f(\widehat{A}_\mu^{\vec{\theta}}) = 0$ specifies a unique $\vec{\theta}$.

To clarify this really very complicated variable change a little bit more, let us remember that to define all this we must divide space-time up into cells, labeled by γ and α in the equation below. And we also expose the group indices a. Thus, what the above equation really says is that

$$\int \prod_{\gamma} \prod_{b} [d\theta^b_\gamma] \prod_{\alpha} \prod_{a} \delta \left[\left(f_a(\vec{\widehat{A}_\mu}^{\vec{\theta}}) \right)_\alpha \right] \det \left| \frac{\delta f^\alpha_a}{\delta \theta^\gamma_b} \right|_{f=0} = \prod_{a} \prod_{\alpha} \int df^\alpha_a \delta[f^\alpha_a] = 1 \,, \tag{423}$$

which I hope you see is clearly true by definition of the δ function.

Thus, regardless of what gauge we employ to set up the path integral, everything reduces to

$$Z = \int [d\widehat{A}_\mu] e^{iS[\widehat{A}_\mu]} \tag{424}$$

which is a finite integral in which we are able to employ the same sample $\widehat{A}_\mu$ from each gauge orbit *regardless of what choice we make for the gauge functions* $f_a(A_\mu)$.

I hope that you recall the simple analogue result that we had in the 2-d circular orbit picture. There we showed that the gauge-fixed "path" integral specified by a function g was actually independent of the choice of g (so long as g specified a unique location on each of the circular orbits) and that for any g one obtained $\int r\, dr e^{iS(r)}$ when $S(\vec{r}) = S(r)$ was independent of location on the orbit.

- **Another perspective on the Faddeev-Popov procedure is obtained from the following mathematical methods type argument.**

Consider the integral

$$G(\mathbf{A}) = \int_{-\infty}^{\infty} dx_1 \ldots dx_N e^{-x_i A_{ij} x_j} \tag{425}$$

where $\mathbf{A}$ is a real symmetric $N \times N$ matrix with elements A_{ij}. Obviously, we can write

$$x_i A_{ij} x_j = X^T \mathbf{A} X\,, \quad \textbf{with} \quad \mathbf{A}^T = \mathbf{A}\,, \tag{426}$$

and $\mathbf{A}$ can be diagonalized by means of a rotation:

$$\mathbf{A} = \mathbf{R}^T \mathbf{D} \mathbf{R}\,, \quad \textbf{where} \quad \mathbf{R}^T \mathbf{R} = \mathbf{R}\mathbf{R}^T = 1\,, \tag{427}$$

and $\mathbf{D}$ is a diagonal matrix with entries $d_1, \ldots, d_N$. Then,

$$\begin{aligned} G(A) &= \int dx_1 \ldots dx_N e^{-X^T R^T D R X} \\ &= \int dy_1 \ldots dy_N e^{-Y^T D Y}\,, \quad \textbf{with} \quad Y = RX \quad \textbf{(The Jacobian is 1.)} \end{aligned}$$

$$
\begin{aligned}
&= \pi^{N/2}(d_1 \dots d_N)^{-1/2} \quad \text{provided all } d_i > 0 \\
&= \pi^{N/2}(\det A)^{-1/2} \,. \qquad (428)
\end{aligned}
$$

These formulae are valid when the determinant does not vanish. If it does, it means that one or more of the d_i are equal to zero, leading to infinities from integrating over the associated infinite intervals. But, we can get a sensible answer even if the determinant vanishes. We simply have to remove the culprit infinite integral(s). Assume there are n zero eigenvalues. We define the restricted Gaussian integral

$$
G_{\rm rest}(A) = \int dy_1 \dots dy_{N-n} e^{-X^T(y) A X(y)} \,, \qquad (429)
$$

where we integrate only over the variables corresponding to a non-zero eigenvalue of A. This definition of $G_{\rm rest}$ is awkward since it depends on the right system of coordinates y. To make up for this, we invent new variables $y_{N-n+1}, \dots y_N$ (that depend on the x coordinates but are not necessarily those corresponding to the 0 eigenvalues above) and rewrite the previous equation in the form:

$$
G_{\rm rest}(A) = \int dy_1 \dots dy_{N-n} dy_{N-n+1} \dots dy_N \delta(y_{N-n+1}) \dots \delta(y_N) e^{-X^T(y) A X(y)} \,. \qquad (430)
$$

Now change variables from y to x, using the Jacobi formula

$$dy_1 \ldots dy_N = dx_1 \ldots dx_N \det \left| \frac{\partial y}{\partial x} \right| \tag{431}$$

to obtain

$$G_{\text{rest}}(A) = \int \left(\prod_{i=1}^{N} dx_i \right) \det \left| \frac{\partial y}{\partial x} \right| \prod_{a=N-n+1}^{N} \delta(y_a(x)) e^{-X^T(y) A X(y)} . \tag{432}$$

This integral is perfectly well-defined. The y_a are arbitrary functions of the x_i, and the extra factors $\det \left| \frac{\partial y}{\partial x} \right| \prod \delta(y)$ in the measure effectively restrict the integration from an N-dimensional space to an $N - n$ dimensional one. As the construction has shown, $G_{\text{rest}}(A)$ does not depend on the specific form of the $y_a(x)$ so long as they are sufficiently cleverly chosen so as to do the job, i.e., restrict the integration region; if they do not, the Jacobian $\det \left| \frac{\partial y}{\partial x} \right|$ will be zero.

Appendix: Equivalence of 2nd Quantization and Path Integrals in Axial Gauge

This appendix is currently under construction. What follows is still incomplete.

- We have seen that the path integral generating function in axial gauge can be written as:

$$\begin{aligned} Z[\vec{J}] &= \int [d\vec{A}_\mu]\delta(\vec{A}_3)\exp\{iS(\vec{J})\} \\ &= \int [dA_0][dA_1][dA_2]\exp\{iS(\vec{A}_\mu,\vec{J})\} \end{aligned} \tag{433}$$

  with

$$S(\vec{A}_\mu,\vec{J}) = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu\,a} + \vec{J}_\mu\cdot\vec{A}^\mu\,. \tag{434}$$

  We will shorthand and write $S(\vec{J})$ for the above in what follows.

- **It is convenient to rewrite this by introducing another integration**

$$Z'[\vec{J}] = \int [dF_{\mu\nu}][dA_\mu]\delta(\vec{A}_3)\exp\{iS'(\vec{J})\} \tag{435}$$

where

$$S'(\vec{J}) = \int d^4x \left[\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} - \frac{1}{2}F^{\mu\nu\, a}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu\right) + J^a_\mu A^{\mu\, a}\right] \tag{436}$$

If we integrate over $F^a_{\mu\nu}$ using square completion we see that $Z'[\vec{J}]$ is equivalent to $Z[\vec{J}]$ (dropping irrelevant constant multiplying factors as usual).

- **We will now work on 2nd quantization to see if it gives the same Feynman rules as does the generating function Z' obtained above in the path integral viewpoint.**

  We start by doing 2nd quantization of exactly the same S' (but without the $\vec{J}$ stuff for the moment) and show that this is equivalent, and then later verify that 2nd quantization of S' is the same as 2nd quantization of the original S (always in axial gauge).

- **To proceed, let us first rewrite $\mathcal{L}'$ (defined by $S'(J) = \int d^4x \mathcal{L}'(x)$) in a convenient way assuming $\vec{A}_3 = 0$ axial gauge. We have**

$$\begin{aligned} \mathcal{L}' &= \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \frac{1}{2} F^{ij\, a} \left(\partial_i A^a_j - \partial_j A^a_i + g c^{abc} A^b_i A^c_j \right) \\ &\quad - \frac{1}{2} F^{0i\, a} \left(\partial_0 A^a_i - \partial_i A^a_0 + g c^{abc} A^b_0 A^c_i \right) - F^{i3\, a}(-\partial_3 A^a_i) + F^{03\, a}(-\partial_3 A^a_0) \end{aligned} \tag{437}$$

where $i, j = 1, 2$!

- **The Euler Lagrange equations**

$$\partial^\lambda \frac{\delta \mathcal{L}'}{\delta(\partial^\lambda F^a_{\mu\nu})} = \frac{\delta \mathcal{L}'}{\delta F^a_{\mu\nu}}, \qquad \partial^\lambda \frac{\delta \mathcal{L}'}{\delta(\partial^\lambda A^a_\mu)} = \frac{\delta \mathcal{L}'}{\delta A^a_\mu} \tag{438}$$

give rise to the following constraint equations (i.e. ones with no time derivatives):

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g c^{abc} A^b_i A^c_j \tag{439}$$

$$F^a_{i3} = -\partial_3 A^a_i \tag{440}$$

$$F^a_{03} = -\partial_3 A^a_0 \qquad (441)$$

$$\partial^i F^a_{0i} - \partial^3 F^a_{03} = -g c^{abc} F^b_{0i} A^{i\,c} \qquad (442)$$

and the dynamical equations (containing time derivatives)

$$F^a_{0i} = \partial_0 A^a_i - \partial_i A^a_0 + g c^{abc} A^b_0 A^c_i \qquad (443)$$

$$\partial^\mu F^a_{\mu i} = -g c^{abc} \left(F^b_{ij} A^{j\,c} + F^b_{0i} A^{0\,c} \right) . \qquad (444)$$

Then, all the F's other than F^a_{0i} (i.e. F^a_{ij}, F^a_{i3} and F^a_{03}) have no time derivatives and neither does A^a_0. Thus, under the rules of 2nd quantization, we are instructed to use the above constraint equations to eliminate these non-dynamical variables from S' to obtain $S''(A^a_i, F^a_{0i})$ ($i = 1, 2$). The appropriate generating functional for the canonical 2nd quantization procedure is then

$$Z''[\vec{J}] = \int [dF_{01}][dF_{02}][dA_1][dA_2] \exp\{iS''(\vec{J})\} \qquad (445)$$

as we shall verify later. For now, we wish to first show that this result is equivalent to $Z'[\vec{J}]$ obtained earlier from the path integral point of view and defined in Eq. (435)

- For this we need a **Lemma**: If (and here we are talking about the path integral approach) a non-dynamical variable ($F_{ij,03,i3}$ and A_0 in our case) appears at most quadratically (with constant coefficient — A_0 requires special consideration since its coefficient is not constant) in S, then integrating over the variable is the same as eliminating it from the action by the Euler-Lagrange equation.

  Proof:

$$\begin{aligned}\int [d\phi] \exp[iS(\phi)] &= \int [d\phi] \exp\left\{ i \int d^4x \left[\frac{1}{2} a\phi^2(x) + f(x)\phi(x) \right] \right\} \\ &= \exp\left\{ \frac{-i}{2a} \int d^4x [f(x)]^2 \right\} . \end{aligned} \tag{446}$$

  On the other hand, the Euler-Lagrange equation $\Rightarrow a\phi(x) + f(x) = 0$ and substitution into S gives $S = -\frac{1}{2a} \int d^4x [f(x)]^2$, which would yield exactly the result above.

- So, we now use this Lemma to rewrite our path integral form by using it for the $[dF_{ij}]$, $[dF_{03}]$, $[dF_{i3}]$ and $[dA_0]$ integrations appearing in $Z'[\vec{J}]$ defined earlier in Eq. (435). The F integrals are of exactly the form assumed for the Lemma. So, it is clear that the path integral over them just gives a version of $Z'[J]$ where the 2nd quantization type substitutions from the equations of motion have been made.

However, the A_0 integral requires a bit of discussion since it enters in the structure $F^{i0}(\partial_i A_0 - \partial_0 A_k)$. So, we must reexamine the Lemma for the $[dA_0]$ integration.

To do so, we rewrite the above structure by doing a partial integration to move the ∂_i onto F^{i0} and then apply the Lemma to the A_0 integral. But, even before this we must do the $[dF_{03}]$ integral because square completion of this integral (or substitution ala the Lemma using Eq. (441)) brings in more A_0 dependence. After the F_{03} integral (or substitution) we have the structure, contained as part of $\mathcal{L}'$ given in Eq. (437),

$$\mathcal{L} \ni F^{0i\,a}\left(\partial_0 A_i^a - \partial_i A_0^a + gc^{abc}A_0^b A_i^c\right) - \frac{1}{2}(\partial_3 A_0^a)^2 \qquad (447)$$

where it is the last term that comes from the F^{03} integral (equivalently Lemma substitution of $F_{03}^a = -\partial_3 A_0^a$). So now we must do (after doing the previously mentioned parts integrations to get $\partial_i F^{0i\,a}$ and also doing a parts integration to get $\partial_3 A_0^a \partial_3 A_0^a \to -A_0^a \partial_3^2 A_0^a$) the following path integral:

$$
\begin{aligned}
&\int [dA_0] \exp\left\{ i \int d^4x \left[\frac{1}{2} A_0^a \partial_3^2 A_0^a + \left(\partial_i F^{0i\,a} \delta^{ab} + g F^{0i\,a} c^{abc} A_i^c \right) A_0^b \right] \right\} \\
\equiv\ & \int [dA_0] \exp\left\{ i \int d^4x \left[\frac{1}{2} A_0^a \mathcal{O} \delta^{ab} A_0^b + J^b A_0^b \right] \right\} \\
\Rightarrow\ & \exp\left\{ -i \frac{1}{2} \int dx' dx'' J^a(x') \mathcal{O}^{-1}_{ab}(x' - x'') J^b(x'') \right\} ,
\end{aligned}
\tag{448}
$$

where $\mathcal{O} = \partial_3^2$, $J^b = \partial_i F^{0i\,a} \delta^{ab} + g F^{0i\,a} c^{abc} A_i^c$ and $\mathcal{O}^{-1}$ is defined by $\partial_3^2 \mathcal{O}^{-1} = \delta^4(x - y)$.

This is exactly the form you would have gotten by eliminating A_0 in the starting form Eq. (447) using the constraint equation, Eq. (442),

$$
\partial^i F_{0i}^a - \partial^3 (F_{03}^a) = \partial^i F_{0i}^a - \partial^3(-\partial_3 A_0^a) = -g c^{abc} F_{0i}^b A^{i\,c} . \tag{449}
$$

To see this, rewrite Eq. (447) in the form one obtains after the above partial integrations (and a switch of the dummy summation indices

$a \leftrightarrow b$)

$$\begin{aligned}\mathcal{L} &\ni (\partial_i F^{0i\,a} + gF^{0i\,b}c^{bac}A_i^c)A_0^a + \frac{1}{2}A_0^a\partial_3^2\delta^{ab}A_0^b \\ &\equiv J^a A_0^a + \frac{1}{2}A_0^a\partial_3^2 A_0^a \quad \text{(same } J^a \text{ as defined previously)}\end{aligned} \tag{450}$$

and then substitute the constraint equation in the form

$$\partial_3^2 A_0^a = -\left(\partial^i F_{0i}^a + gc^{bac}F_{0i}^b A^{i\,c}\right) = -J^a \tag{451}$$

to obtain

$$\int d^4x\mathcal{L} \ni \int d^4x \frac{1}{2}J^a(x)A_0^a(x) = \int d^4x d^4y \left[-\frac{1}{2}J^a(x)\mathcal{O}_{ab}^{-1}(x-y)J^b(y)\right] \tag{452}$$

where we used $A_0^a(x) = -\int d^4y\mathcal{O}_{ab}^{-1}(x-y)J^b(y)$ as follows from the constraint equation written in the form Eq. (451) when solved for A_0 in integral equation form. The above result is that obtained by path integral techniques as displayed in Eq. (448).

So, at this point, we have reduced the path integral generating function $Z'[J]$ of Eq. (435) to the path integral defining $Z''[J]$ in Eq. (445) where $S''(J)$ is obtained from $S'(J)$ by making 2nd quantization-like substitutions for all but F_{0i} and A_i ($i = 1, 2$) and only path integrals over these variables need to be done.

- The final element of the proof is to work more on the 2nd quantization form. In particular, recall that the Hamiltonian formulation of 2nd quantization is equivalent to the Hamiltonian Path Integral formalism when we have independent fields and their canonically conjugate momenta.

  In the simplest scalar case this was the statement that using

$$Z[J] \propto \int [d\phi d\pi] \exp\left\{ i \int d^4x [\pi\dot{\phi} - \mathcal{H}(\pi, \phi) + J\phi] \right\} \quad (453)$$

  to develop Wick's theorem results was the same as using 2nd quantization to obtain the Wick's theorem contractions.

  In the present case this is like using the $[dF_{01} dA_1][dF_{02} dA_2]$ form Eq. (445), where F_{01} and F_{02} are, indeed, the appropriate canonical momenta associated with the transverse A_1 and A_2 field components , i.e. they are given by $\frac{\delta \mathcal{L}}{\delta \dot{A}_1}$ and $\frac{\delta \mathcal{L}}{\delta \dot{A}_2}$.

- Of course, we never explicitly showed this in the case of a vector field, but we could. The sequence of steps (that I will not go through in detail) is the following. (We employ abelian field notation, but the A_1 and A_2 gauge field components act independently for each superscript gauge index, which index I will suppress in the ensuing discussion.)

1. Write the starting $\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ in detail with the input $A_3 = 0$ gauge condition put in.
2. Check that the only time derivatives present are $\dot{A}_1$ and $\dot{A}_2$ and that the conjugate momenta are

$$\pi^i = -(\partial^0 A^i - \partial^i A^0) = -F^{0i} \quad \text{for } i = 1, 2. \tag{454}$$

Construct the Hamiltonian form

$$\mathcal{H}_{eff} = \pi^i \dot{\phi}_i - \mathcal{L} = -F^{0i}\partial_0 A_i - \mathcal{L} \tag{455}$$

where all occurrences of $\partial_0 A_i$ are to be eliminated using the definition of π_i in the form $\partial_0 A_i = F_{0i} + \partial_i A_0$.

3. **At this point, you will find that the effective Lagrangian is (implied summation over $i, j = 1, 2$)**

$$\begin{aligned} \mathcal{L}_{eff}(x) &= \pi^i \dot{A}_i - \mathcal{H}(\pi^i, A^i) \\ &= -\tfrac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) \\ &\quad -\frac{1}{2}\partial_3 A_i \partial^3 A^i - \frac{1}{2}F_{0i}F^{0i} - \frac{1}{2}\partial_3 A^0 \partial^3 A^0 \end{aligned} \tag{456}$$

and we know that the canonical procedure is equivalent to doing the path integral of this canonical form

$$\int [dF_{01}][dA_1][dF_{02}][dA_2] \exp\left\{ i \int d^4x \mathcal{L}_{eff}(x) \right\}, \tag{457}$$

with $\mathcal{L}_{eff}$ as given above.
We need now only verify that the above $\mathcal{L}_{eff}$ is the same as that appearing in our earlier $Z''[J]$ of Eq. (445). Recall that $S''[J]$ is obtained by substituting the constraint equations into the form $\mathcal{L}'$ given in Eq. (437). If you carry this out, you will find that

$$\mathcal{L}'(\text{after constraint substitutions}) = \mathcal{L}_{eff}\,. \tag{458}$$

This completes the final element needed in our proof that path integral quantization and canonical quantization are equivalent in axial gauge.

- **Perhaps a very concise summary of what was done will help.**

In the canonical procedure

1. **We derived 4 constraint equations and 2 dynamical equations.**
2. **The constraint equations were used to eliminate F^a_{ij}, F^a_{i3}, F^a_{03} and A^a_0 so as to obtain from our starting general Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ the appropriate $S'' = S''(A^a_i, F^a_{0i})$ (where only $i = 1, 2$ are relevant in the axial gauge as dynamical degrees of freedom).**
3. **We verified that the F^a_{0i} are the conjugate momenta for the A^a_i. This implied (using the usual 2nd quantization $\leftrightarrow$ path integral equivalence a la scalar fields) that a good generating function was (dropping gauge indices)**

$$Z''[J] = \int [dF_{01}][dA_1][dF_{02}][dA_2] \exp\{iS''(J)\}\,. \tag{459}$$

In the path integral approach

1. We converted

$$Z[\vec{J}] \;=\; \int [d\vec{A}_\mu]\delta(\vec{A}_3)\exp\{iS(\vec{J})\} = \int [dA_0][dA_1][dA_2]\exp\{iS(\vec{A}_\mu,\vec{J})\} \tag{460}$$

to a seemingly more complicated form $Z'[\vec{J}]$ by introducing the $F_{\mu\nu}$ quantities. At this time S converted to S' which involved these auxiliary quantities.

2. We thus had

$$Z'[J] = \int [dF_{\mu\nu}][dA_\mu]\delta(\vec{A}_3)\exp\{iS'(J)\}\,. \tag{461}$$

3. We established a Lemma that integrating over $[dF_{ij}]$, $[dF_{03}]$, and $[dF_{i3}]$, all of which appeared quadratically with a constant coefficient in $S'(J)$, is equivalent to substituting for them using their equation of motion (the constraint equations).
4. This almost gave us the form $S''(J)$, the only difference being that we

were left with

$$Z'[J] \to \int [dF_{01}][dA_1][dF_{02}][dA_2][dA_0] \exp\{iS'(J, \text{with } F\text{'s substituted})\}\,. \tag{462}$$

5. But, we were able to show that performing the $\int [dA_0]$ converted $S'(J, \text{with } F\text{'s substituted})$ into $S''(J)$. Equivalently, we showed that substituting for the A_0^a using the A_0^a constraint equations, but very carefully treated (we had to use $\mathcal{O}^{-1}$ to complete the substitution), gave us $S''(J)$.

QCD Path Integral Quantization

Possibly useful references beyond Ryder for the material presented here are:

1. Cheng and Li p. 248 and following;
2. Ramond Chapter VIII;
3. Itzykson and Zuber Chapter 12.

I will not actually follow any of these exactly. There are small differences between the references and now conventional notations. Cheng and Li are the closest but there are a few factors of i mislaid.

- We begin with an extension of our gauge condition by generalizing the $f_a(\vec{A}_\mu) = 0$ gauge condition to $f_a(\vec{A}_\mu) = C_a(x)$. For example, the Lorentz gauge would be generalized to $\partial^\mu A^a_\mu(x) = C^a(x)$. (There is no significance to up vs. down location of the index a.)

  At the end of the last section, we showed that the path integral result after implementing FP gauge fixing is independent of the choice of the gauge fixing function f_a, so using $f'_a(\vec{A}_\mu) = f_a(\vec{A}_\mu) - C_a(x)$ does not alter the result for the path integral.

In order to complete our procedure, we will want to change our expression for $Z_f[J]$ in Eq. (416) which reads

$$Z_f[J] = \int [d\vec{A}_\mu(x)] \Delta_f[\vec{A}_\mu] \delta[f_a(\vec{A}_\mu)] \exp\left\{ i \int d^4x [\mathcal{L}(x) + \vec{J}_\mu \cdot \vec{A}^\mu] \right\} \tag{463}$$

to

$$Z_f[J] = \int [d\vec{A}_\mu(x)] \Delta_f[\vec{A}_\mu] \delta[f_a(\vec{A}_\mu) - C_a(x)] \exp\left\{ i \int d^4x [\mathcal{L}(x) + \vec{J}_\mu \cdot \vec{A}^\mu] \right\} \tag{464}$$

where you will notice that I did not change $\Delta_f[\vec{A}_\mu]$. That is, the above change assumes that the response function M_f is independent of this shift of the gauge condition. In fact, what is important is that *the form* of $\Delta_f[\vec{A}_\mu]$ as a function of $\vec{A}_\mu$ is independent of the shift of the gauge condition.

For the particular change of the gauge-fixing condition we are discussing here this is indeed the case — since $C_a(x)$ is independent of $\vec{A}_\mu$ this shift generalization does not affect the $\vec{A}_\mu$ dependence of the operator M_f, which is the local response of the gauge condition to a small gauge transformation. This local response is obtained by expanding using a small $\vec{\theta}$ gauge transformation and this local response does not change as a function of $\vec{A}_\mu$ just because you add a $\vec{A}_\mu$-independent piece $C_a(x)$ to the gauge condition.

Still, you might worry that the local response M_f somehow depends upon the point on the gauge orbit at which you compute it, which is being shifted by the $C_a(x)$ addition to the gauge fixing condition. However, this possibility is taken into account automatically when we write $\Delta_f[\vec{A}_\mu]$ *inside the integral* — it will automatically then be evaluated at the location in $\vec{A}_\mu$ space where the generalized gauge condition is satisfied.

- So, now I hope you are convinced that we can write

$$Z_f[J] = \int [d\vec{A}_\mu(x)]\Delta_f[\vec{A}_\mu]\delta[f_a(\vec{A}_\mu) - C_a(x)] \exp\left\{i\int d^4x[\mathcal{L}(A_\mu(x)) + \vec{J}_\mu \cdot \vec{A}^\mu]\right\} \quad (465)$$

with $\Delta_f[\vec{A}_\mu]$ as defined just above. We now go one step further.

We know that Z can be multiplied by any overall ($\vec{A}_\mu$-independent) factor without consequence (provided we eventually normalize so that $Z[J=0]=1$). Let us multiply by $\exp\left[-i\frac{1}{2\alpha}\sum_a \int d^4x C_a^2(x)\right]$.

Now, we have shown above that the path integral is completely

independent of the choice of $C_a(x)$, so long as $\vec{J}_\mu$ corresponds to a conserved current, or equivalently so long as we compute a functional derivative combination corresponding to a physical and therefore gauge invariant observable. In this case, we can equally well consider integrating over a series of gauge choices specified by different C_a choices, using the above exponential weight.

The net result is to replace, inside the expression for $Z[J]$,

$$\begin{aligned} \prod_{x,a} \delta[f_a(\vec{A}_\mu(x)) - C_a(x)] &\rightarrow \int \prod_d [dC_d] \exp\left[-i\frac{1}{2\alpha}\int d^4x \sum_a C_a^2(x)\right] \prod_{x,a} \delta[f_a(\vec{A}_\mu(x)) - C_a(x)] \\ &= \exp\left[-i\frac{1}{2\alpha}\int d^4x \sum_a f_a^2(\vec{A}_\mu(x))\right], \end{aligned} \tag{466}$$

yielding

$$Z[J] = \int \prod_d [dA_\mu^d] \det M_f[\vec{A}_\mu] \exp\left\{i\int d^4x \left[\mathcal{L}(A_\mu(x)) + \sum_a J^{a\,\mu} A_\mu^a - \frac{1}{2\alpha}\sum_a f_a^2(\vec{A}_\mu)\right]\right\}. \tag{467}$$

where I have made explicit the (possible) $\vec{A}_\mu$ dependence of $\det M_f$.

- **The next manipulation is to use our trick for writing $\det M_f$, see**

Eq. (180):

$$\det M_f[\vec{A}_\mu] \propto \int \prod_d \left([dc^d][d\overline{c}^d]\right) \exp\left\{ i \int d^4x \int d^4y \sum_{ab} \overline{c}^a(x)(M_f)_{ab}(x,y)c^b(y) \right\}, \tag{468}$$

where

$$(M_f)_{ab}(x,y) = \frac{\delta f_a(\vec{A}_\mu^{\vec{\theta}}(x))}{\delta \theta_b(y)}. \tag{469}$$

Note that we do **not** specify a particular location (e.g. $f_a = 0$) at which to evaluate the M_f. As stated a few pages ago, what is important is the form of M_f as a function of $\vec{A}_\mu$ and we are integrating this form over all choices of $\vec{A}_\mu(x)$.

The result is that we obtain

$$\begin{aligned} Z[J] \quad = \quad & \int \prod_d \left([dA_\mu^d][dc^d][d\overline{c}^d]\right) \exp\left\{ i \int d^4x \left[\mathcal{L}(A_\mu(x)) + \sum_a J^{a\,\mu} A_\mu^a - \frac{1}{2\alpha} \sum_a f_a^2(\vec{A}_\mu) \right] \right. \\ & \left. + i \int d^4x \int d^4y \sum_{ab} \overline{c}^a(x)(M_f)_{ab}(x,y)c^b(y) \right\}. \end{aligned} \tag{470}$$

- Eventually we will want to introduce some source currents for the ghost fields $\overline{c}^a$ and c^a, and use generating function techniques for them. But, first let us construct M_f for some important standard gauge choices.

The first very important choice is the covariant (Lorentz) gauge which is defined by taking

$$f_a(\vec{A}_\mu) = \partial^\mu A^a_\mu \,, \quad a = 1, \ldots, N \,. \tag{471}$$

Note: We don't take $\partial^\mu A^a_\mu = 0$ since we are using the integration over the C_a functions with $\partial^\mu A^a_\mu = C_a$.

For small θ^a's, we have (using the notation f^{abc} for the group structure constants)

$$A^{a\,\theta}_\mu(x) = A^a_\mu(x) + f^{abc}\theta^b(x)A^c_\mu(x) - \frac{1}{g}\partial_\mu\theta^a(x) \tag{472}$$

so that

$$\partial^\mu A^{a\,\theta}_\mu = \partial^\mu A^a_\mu + \partial^\mu \left[f^{abc}\theta^b(x)A^c_\mu(x) - \frac{1}{g}\partial_\mu\theta^a \right] \,. \tag{473}$$

We now identify the response function from the expansion

$$f_a[\vec{A}^{\vec{\theta}}_\mu(x)] = f_a[\vec{A}_\mu(x)] + \int d^4y[M_f(x,y)]_{ab}\theta_b(y) + \mathcal{O}(\theta^2)\,. \quad (474)$$

Comparing, we find that

$$M_f(x,y)_{ab} = -\frac{1}{g}\partial^\mu \left\{ \left[\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu \right] \delta^4(x-y) \right\}. \quad (475)$$

Before proceeding, let us note that this response function has a *form* **(as a function of $\vec{A}_\mu$) that is in fact independent of the value of $\partial^\mu A^a_\mu$ from which one starts as promised by our general proofs.**

- **Substituting this into Eq. (470), we obtain**

$$\begin{aligned} Z[J] \quad = \quad & \int \prod_d \left([dA^d_\mu][dc^d][d\bar{c}^d] \right) \exp\left\{ i \int d^4x \left[\mathcal{L}(A_\mu(x)) + \sum_a J^{a\,\mu} A^a_\mu - \frac{1}{2\alpha}\sum_a f^2_a(\vec{A}_\mu) \right.\right. \\ & \left.\left. - \frac{1}{g}\sum_{ab} \bar{c}^a(x)\partial^\mu \left\{ \left[\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu \right] c^b(x) \right\} \right] \right\}. \quad (476) \end{aligned}$$

- However, there is one more little simplification: we can rescale the $\overline{c}^a$ and c^a to absorb the $1/g$ factor in the exponential above. (Note that both terms in the ghost Lagrangian contain one c^a field and one $\overline{c}^a$ field. Rescaling is also possible for the A fields stuff but the tri-linear and quartic terms do not scale like the quadratic terms, implying that such rescaling is not necessarily helpful.) This changes $Z[J]$ by some factor of many powers of g, but as usual we don't care. We simply have to make sure in the end that $Z[\text{sources} = 0] = 1$. With such a rescaling, and introducing sources $\overline{\xi}^a$ and ξ^a for the c^a and $\overline{c}^a$ ghost functions, we obtain

$$Z[J, \overline{\xi}, \xi] = \int \prod_d \left([dA^d_\mu][d\overline{c}^d][dc^d] \right) \exp \left\{ i \int d^4x \overline{\mathcal{L}}(x) \right\} \tag{477}$$

with

$$\begin{aligned} \overline{\mathcal{L}}(x) \quad &= \quad -\tfrac{1}{4} F^{a\,\mu\nu} F^a_{\mu\nu} - \frac{1}{2\alpha} \left(\partial^\mu A^a_\mu \right)^2 \\ &\quad\; -\overline{c}^a \partial^\mu \left\{ \left[\delta^{ab} \partial_\mu - g f^{abc} A^c_\mu \right] c^b \right\} \end{aligned}$$

$$+\sum_a \left(J^{\mu\, a} A^a_\mu + \overline{\xi}^a c^a + \overline{c}^a \xi^a\right) \tag{478}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,. \tag{479}$$

In the above, everything is a function of x and repeated indices a, b, c and μ, ν in the $\overline{\mathcal{L}}$ expression are summed over.

Of course, this is for the pure gauge theory only. Fermions and other particles are brought in as needed.

- In the above, the ghost fields are fully parallel in function to the gauge fields. The Feynman rules must include propagators and vertices involving the ghost fields as well as the gauge fields.

  We will derive the relevant Feynman rules using the functional techniques, but at this point they can equally well be read off of the Lagrangian using "naive" procedures where we imagine all the gauge and ghost fields to be 2nd quantized operators. In other words, we can use the path-integral technique to develop the required additional ghost fields and the associated Lagrangian form and then treat all fields as 2nd quantized operator fields.

The QCD Feynman Rules

- We begin by separating the full $\overline{\mathcal{L}}$ given above into the free-particle and the interaction components.

  We then compute the $Z_0[J,\overline{\xi},\xi]$ for the free-particle components and then bring in the interactions via the perturbative expansion.

- We have

$$
\begin{aligned}
Z_0[J,\overline{\xi},\xi] &= Z_0[J]Z_0[\overline{\xi},\xi] \\
&= \int \prod_b [dA^b_\mu] \exp\left\{ i\int d^4x \left[-\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + J^{\mu\, a} A^a_\mu \right] \right\} \\
&\quad \times \int \prod_b [d\overline{c}^b][dc^b] \exp\left\{ -i\int d^4x \left[\overline{c}^a \partial^2 c^a - \overline{c}^a \xi^a - \overline{\xi}^a c^a \right] \right\} . \qquad (480)
\end{aligned}
$$

- The interactions will come from

$$
\exp\left[iS_I \left(\frac{\delta}{i\delta J^{\mu\, a}}, \frac{\delta}{-i\delta \xi^a}, \frac{\delta}{i\delta \overline{\xi}^a} \right) \right] \qquad (481)
$$

where

$$S_I\left(A_\mu^a, \overline{c}^a, c^a\right) = \int d^4x \Big[-\frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g c^{abc} A^{\mu\, b} A^{\nu\, c} - \tfrac{1}{4} g^2 c^{abc} c^{ade} A_\mu^b A_\nu^c A^{\mu\, d} A^{\nu\, e} + \overline{c}^a g c^{abc} \partial^\mu (A_\mu^c c^b) \Big] \tag{482}$$

We will return to the interaction S_I once we have given the expression for Z_0.

- $\boxed{\text{The free particle } Z_0\text{'s}}$

- To compute Z_0, *we can separately perform the square completion process for the* A_μ^b *fields and the* $\overline{c}^b, c^b$ *fields.*

  For the former, we just have the addition of a group index to carry around with our cell and Lorentz indices. The full collection is α, μ, a, where α is the cell label.

  However, you can see that the quadratic part of $\overline{\mathcal{L}}$ is completely diagonal in the group index a. Thus, it should not surprise you that we get the same old result for the inverse Kernel with the addition of a multiplying δ function in the group generator indices.

More explicitly, we do the usual partial integration stuff and we are then left with the computation of

$$\begin{aligned} Z_0[J] &= \int \prod_d [dA^d_\mu] \exp\left\{ i \int d^4x \left[\frac{1}{2} A^a_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \left(1 - \frac{1}{\alpha}\right)\right) \delta_{ab} A^b_\nu + J^a_\mu A^{\mu\, a} \right] \right\} \\ &\equiv \int \prod_d [dA^d_\mu] \exp\left\{ \int d^4x d^4y \left[\frac{1}{2} A^a_\mu(x) K^{\mu\nu}_{ab}(x-y) A^b_\nu(y) + J^a_\mu(x) \delta^4(x-y) A^{\mu\, a}(y) \right] \right\} \\ &= \exp\left\{ -\frac{i}{2} \int d^4x' d^4y' J^{a'}_{\mu'}(x') D^{\mu'\nu'}_{a'b'}(x'-y') J^{b'}_{\nu'}(y') \right\} , \end{aligned} \quad (483)$$

where

$$K^{\mu\nu}_{ab}(x-y) = \left[g^{\mu\nu}\partial^2_x - \left(1 - \frac{1}{\alpha}\right)\partial^\mu_x\partial^\nu_x \right] \delta^4(x-y)\delta_{ab} \quad (484)$$

has the inverse $D^{\mu\nu}_{ab}$ defined by

$$\int d^4z K^{\mu\lambda}_{ab}(x-z) D^{bc}_{\lambda\nu}(z-y) = g^\mu_\nu \delta^c_a \delta^4(x-y) \quad (485)$$

with solution

$$D^{\mu\nu}_{ab}(x-y) = \delta_{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left[-\frac{g^{\mu\nu}}{k^2} + \frac{k^\mu k^\nu}{k^4}(1-\alpha) \right] . \quad (486)$$

Perhaps it is useful to explicitly check the above claim.

$$\begin{aligned}
& \int d^4z K^{\mu\lambda}_{ab}(x-z) D^{bc}_{\lambda\nu}(z-y) = g^\mu_\nu \delta^c_a \delta^4(x-y) \\
= & \int d^4z \left[g^{\mu\lambda}\partial^2_x - \left(1-\frac{1}{\alpha}\right)\partial^\mu_x\partial^\lambda_x\right]\delta^4(x-z)\delta_{ab}\delta_{bc}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(z-y)}\left[-\frac{g_{\lambda\nu}}{k^2}+\frac{k_\lambda k_\nu}{k^4}(1-\alpha)\right] \\
= & \ \delta_{ac}\int\frac{d^4k}{(2\pi)^4}\left[g^{\mu\lambda}\partial^2_x - \left(1-\frac{1}{\alpha}\right)\partial^\mu_x\partial^\lambda_x\right]e^{-ik\cdot(x-y)}\left[-\frac{g_{\lambda\nu}}{k^2}+\frac{k_\lambda k_\nu}{k^4}(1-\alpha)\right] \\
= & \ \delta_{ac}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left[-k^2g^{\mu\lambda}+\left(1-\frac{1}{\alpha}\right)k^\mu k^\lambda\right]\left[-\frac{g_{\lambda\nu}}{k^2}+\frac{k_\lambda k_\nu}{k^4}(1-\alpha)\right] \\
= & \ \delta_{ac}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left[g^\mu_\nu - \frac{k^\mu k_\nu}{k^2}(1-\alpha) - \frac{k^\mu k_\nu}{k^2}\left(1-\frac{1}{\alpha}\right) + \frac{k^\mu k_\nu}{k^2}(1-\alpha)\left(1-\frac{1}{\alpha}\right)\right] \\
= & \ \delta_{ac}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\left[g^\mu_\nu - 0\right] \\
= & \ \delta_{ac} g^\mu_\nu \delta^4(x-y)\,,
\end{aligned} \tag{487}$$

as hoped.

Going to momentum space, the above form of $Z_0[J]$ generates the Feynman rule

$$\langle 0|T\left\{A^a_\mu(x)A^b_\nu(y)\right\}|0\rangle = \left\{\frac{\delta}{i\delta J^{\mu\,a}(x)}\frac{\delta}{i\delta J^{\nu\,b}(y)}Z_0[J]\right\}_{J=0}$$

$$= \quad iD^{ab}_{\mu\nu}(x-y) \tag{488}$$

with momentum space Feynman rule

$$iD^{ab}_{\mu\nu}(k) \quad = \quad i\delta_{ab}\frac{1}{k^2+i\epsilon}\left[-g^{\mu\nu}+\frac{k^\mu k^\nu}{k^2}(1-\alpha)\right]. \tag{489}$$

- **We will now work on the c^a and $\overline{c}^a$. We want**

$$Z_0[\overline{\xi},\xi] \quad = \quad \int \prod_b [d\overline{c}^b][dc^b] \exp\left\{-i\int d^4x \left[\overline{c}^a \partial^2 c^a - \overline{c}^a \xi^a - \overline{\xi}^a c^a\right]\right\}. \tag{490}$$

Referring back to Eqs. (183) and (191), we had (for the case of $\mathcal{G} = i\partial\!\!\!/ - m$, but the result is clearly general)

$$\begin{aligned} Z_0(\eta,\overline{\eta}) \quad &= \quad \frac{1}{N}\int [d\overline{\psi}][d\psi] \exp\left\{i\int d^4x \left[\overline{\psi}(x)\mathcal{G}\psi(x) + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\right]\right\} \\ &= \quad \exp\left[-i\int d^4x' d^4y' \overline{\eta}(x')\mathcal{G}^{-1}(x'-y')\eta(y')\right]. \end{aligned} \tag{491}$$

Here, we replace ψ by c^b and $\overline{\psi}$ by $\overline{c}^a$, i.e. we have to bring in the group indices, and our $\mathcal{G}$ is simply $-\partial^2\delta^{ab}$ (note that the $-$ sign must

be included in this for consistency with the above general structure). We then obtain (with the conversion $\overline{\eta} \to \overline{\xi}^c$ and $\eta \to \xi^d$)

$$Z_0[\overline{\xi}, \xi] = \exp\left[-i \int d^4x' d^4y' \overline{\xi}^c(x') \Delta^{cd}(x'-y') \xi^d(y')\right], \quad (492)$$

where

$$-\partial^2 \delta^{ac} \Delta^{cb}(x-y) = \delta^{ab} \delta^4(x-y), \quad (493)$$

the solution to which is

$$\Delta^{ab}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 + i\epsilon} \delta^{ab}. \quad (494)$$

We can now compute the associated propagator as

$$\begin{aligned}
\langle 0|T\{c_a(x)\overline{c}_b(y)\}|0\rangle &= \left(\frac{1}{i}\frac{\delta}{\delta\overline{\xi}^a(x)}\right)\left(\frac{1}{-i}\frac{\delta}{\delta\xi^b(y)}\right) Z_0[\overline{\xi}, \xi]\Big|_{\overline{\xi}=\xi=0} \\
&= \Bigg[\left(\frac{1}{i}\frac{\delta}{\delta\overline{\xi}^a(x)}\right)\left(\frac{1}{-i}\right)(-)\left(-i\int d^4x'' \overline{\xi}^{c'}(x'') \Delta^{c'b}(x''-y)\right) \\
&\quad \times \exp\left[-i \int d^4x' d^4y' \overline{\xi}^c(x') \Delta^{cd}(x'-y') \xi^d(y')\right]\Bigg]_{\overline{\xi}=\xi=0}
\end{aligned}$$

$$= \quad i\Delta^{ab}(x-y)\,, \tag{495}$$

where the explicit $-$ sign after the 2nd equality comes from anticommuting the $\frac{\delta}{\delta\xi^b(y)}$ past the $\overline{\xi}^c$ in taking the derivative of the exponential argument.

The above leads to the Feynman rule for the ghost field propagator in momentum space:

$$i\Delta^{ab}(k) = \frac{i}{k^2+i\epsilon}\delta^{ab}\,. \tag{496}$$

Note that the sign of this propagator is different in some treatments but it is actually physically irrelevant so long as the *relative* sign of the propagator and the ghost ghost gluon vertex remains unchanged. This is because any Feynman graph computation will always contain one propagator for every ghost-ghost-gluon vertex. The propagator sign and the ghost-ghost-gluon vertex sign are simultaneously changed by changing the (irrelevant) sign of the response function M_f^{ab}.

- So now we come to the $\boxed{\text{Interactions}}$

$$\exp\left[iS_I\left(\frac{\delta}{i\delta J^{\mu\, a}}, \frac{\delta}{-i\delta\xi^a}, \frac{\delta}{i\delta\overline{\xi}^a}\right)\right] \tag{497}$$

where

$$S_I\left(A^a_\mu, \overline{c}^a, c^a\right) = \int d^4x \Big[-\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) g c^{abc} A^{\mu\, b} A^{\nu\, c} - \tfrac{1}{4} g^2 c^{abc} c^{ade} A^b_\mu A^c_\nu A^{\mu\, d} A^{\nu\, e} + \overline{c}^a g c^{abc} \partial^\mu (A^c_\mu c^b) \Big] \quad (498)$$

We want to develop the Feynman rules associated with these interactions using the lowest order expansion of S_I.

Higher orders will be built up by using the lowest order interactions within higher-order Feynman graphs. That is the perturbative approach.

So, we will write

$$\exp[iS_I] \approx 1 + iS_I \quad (499)$$

In S_I we see:

1. **a 3-gluon (here I use the QCD language) interaction;**
2. **a 4-gluon interaction;**
3. **a gluon-ghost-antighost interaction.**

1. **The 3-gluon interaction**

We rewrite $iS_I^{3-gluon}$ using the antisymmetry of the structure constant c^{abc} and relabeling the dummy summations to d, e, f:

$$-i\int d^4x \frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g c^{abc} A^{\mu\, b} A^{\nu\, c} = -ig\int d^4x \left(\partial_\mu^x A_\nu^d(x)\right) c^{def} A^{\mu\, e}(x) A^{\nu\, f}(x)\,. \tag{500}$$

This means that $Z[J] = \exp[iS_I(\frac{\delta}{i\delta J})]Z_0[J] \simeq [1 + iS_I(\frac{\delta}{i\delta J})]Z_0[J]$ contains a term that looks like

$$\begin{aligned} iS_I Z_0[J] &= \left[-ig\int d^4x \left(\partial_\mu^x \frac{\delta}{i\delta J^{\nu\, d}(x)}\right) c^{def} \left(\frac{\delta}{i\delta J_\mu^e(x)}\right)\left(\frac{\delta}{i\delta J_\nu^f(x)}\right)\right] \\ &\quad \times \exp\left[-\frac{i}{2}\int d^4y_1 d^4y_2 J_\rho^s(y_1) D_{st}^{\rho\lambda}(y_1 - y_2) J_\lambda^t(y_2)\right]\,. \end{aligned} \tag{501}$$

We need to figure out what term we need to keep in the resulting expression. For this we need to recall what we are after.
As I shall explain later, to compute the vertex, we want to compute

$$\langle 0|T\{A^{\alpha\, a}(y) A^{\beta\, b}(z) A^{\gamma\, c}(w)\}|0\rangle = \left[\left(\frac{\delta}{i\delta J_\alpha^a(y)}\right)\left(\frac{\delta}{i\delta J_\beta^b(z)}\right)\left(\frac{\delta}{i\delta J_\gamma^c(w)}\right)(iS_I Z_0[J])\right]_{J=0}\,. \tag{502}$$

Since we set $J = 0$ at the end, this means that each of the $\frac{\delta}{\delta J}$

derivatives must have something to act on and that all the multiplicative J dependence must then be gone so that when we set $J = 0$ all that happens is that the exponential argument $\to 0$.

What this means is that in $iS_I Z_0[J]$ we let each of the $\frac{\delta}{\delta J}$'s act on the exponential and bring down the usual one-sided J integral. Then at the 2nd stage of Eq. (502) we let each one of the $\frac{\delta}{\delta J}$'s associated with the "external" A fields act on one of the J's in the one-sided J integrals that we developed in the $iS_I Z_0[J]$ expression.

Thus, the relevant portion of $iS_I Z_0[J]$ is (where we used $\frac{-i}{i^3} = +1$)

$$\begin{aligned} iS_I Z_0[J] \quad \sim \quad & \int d^4x g c^{def} \left(-i \int d^4y_2 \partial^x_\mu D^{\lambda}_{\nu\, dt}(x-y_2) J^t_\lambda(y_2)\right) \left(-i \int d^4y'_2 D^{\mu\lambda'}_{et'}(x-y'_2) J^{t'}_{\lambda'}(y'_2)\right) \\ & \times \left(-i \int d^4y''_2 D^{\nu\lambda''}_{ft''}(x-y''_2) J^{t''}_{\lambda''}(y''_2)\right) \exp\Big[\ldots\Big]. \end{aligned} \tag{503}$$

We now follow with the operation of Eq. (502). There are $3! = 6$ ways of performing the three final $\frac{\delta}{\delta J}$'s. We examine in detail one of the terms and develop the others by symmetry. We focus on the term in which the derivatives operate as follows:

$$\alpha, a, y \to \lambda, t, y_2\,; \quad \beta, b, z \to \lambda', t', y_2'\,; \quad \gamma, c, w \to \lambda'', t'', y_2''\,. \tag{504}$$

This yields the result

$$\begin{aligned}
\langle 0|T\{A^{\alpha\,a}(y)A^{\beta\,b}(z)A^{\gamma\,c}(w)\}|0\rangle \quad &\ni \quad \int d^4x g c^{def}\left(-\int d^4y_2 \partial^x_\mu D^{\lambda}_{\nu\,dt}(x-y_2)\delta^4(y_2-y)\delta^{at}g^{\alpha}_{\lambda}\right) \\
&\quad \times \left(-\int d^4y_2' D^{\mu\lambda'}_{et'}(x-y_2')\delta^4(y_2'-z)\delta^{bt'}g^{\beta}_{\lambda'}\right) \\
&\quad \times \left(-\int d^4y_2'' D^{\nu\lambda''}_{ft''}(x-y_2'')\delta^4(y_2''-w)\delta^{ct''}g^{\gamma}_{\lambda''}\right) \\
&= \quad -gc^{def}\int d^4x \left(\partial^x_\mu D^{\alpha}_{\nu\,da}(x-y)\right) D^{\mu\beta}_{eb}(x-z)D^{\nu\gamma}_{fc}(x-w)\,. \qquad (505)
\end{aligned}$$

We now write the Fourier transforms of the D's.

$$\begin{aligned}
D^{\alpha}_{\nu\,da}(x-y) &= \int d^4\tilde{p}e^{ip\cdot(x-y)}(-i)\left[iD^{\alpha}_{\nu\,da}(p)\right] \\
D^{\mu\beta}_{eb}(x-z) &= \int d^4\tilde{q}e^{iq\cdot(x-z)}(-i)\left[iD^{\mu\beta}_{eb}(q)\right] \\
D^{\nu\gamma}_{fc}(x-w) &= \int d^4\tilde{r}e^{ir\cdot(x-w)}(-i)\left[iD^{\nu\gamma}_{fc}(r)\right]\,, \qquad (506)
\end{aligned}$$

where the p, q, r are the momenta associated with the 3 gluons in an outgoing sense, and we have used the notation $\int d^4\tilde{p} \equiv \int \frac{d^4p}{(2\pi)^4}$.

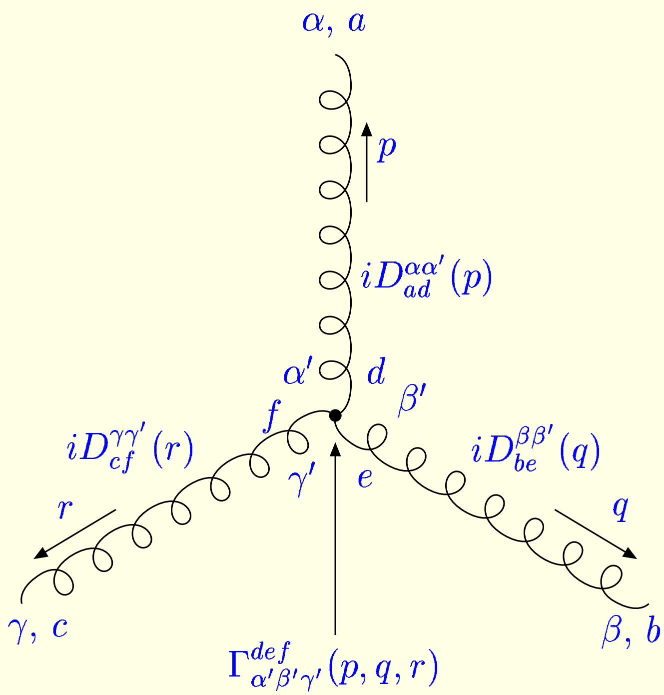
We can now perform the $\int d^4x$ to obtain

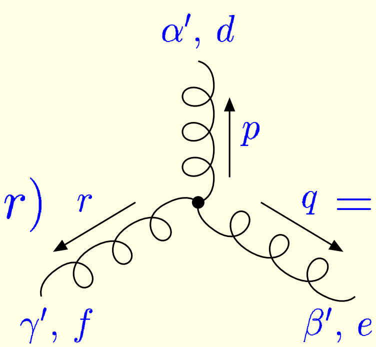
$$
\begin{aligned}
& \langle 0|T\{A^{\alpha\, a}(y)A^{\beta\, b}(z)A^{\gamma\, c}(w)\}|0\rangle \\
\ni \quad & (2\pi)^4\delta^4(p+q+r)(-gc^{def})(-i)^3(+ip_\mu)\left[iD^{\alpha}_{\nu\, da}(p)\right]\left[iD^{\mu\beta}_{eb}(q)\right]\left[iD^{\nu\gamma}_{fc}(r)\right] \\
\equiv \quad & [iD^{\alpha\alpha'}_{ad}(p)][iD^{\beta\beta'}_{be}(q)][iD^{\gamma\gamma'}_{cf}(r)]\Gamma^{def}_{\alpha'\beta'\gamma'}(p,q,r). \qquad (507)
\end{aligned}
$$

from which we find the required form of Γ for the particular one of the 6 terms on which we are focusing.

$$
\Gamma^{def}_{\alpha'\beta'\gamma'}(p,q,r) = gc^{def}(2\pi)^4\delta^4(p+q+r)g_{\gamma'\alpha'}p_{\beta'}\,. \qquad (508)
$$

The other terms will be generated by symmetry considerations as we shall describe.
Pictorially, we can display this process as in Fig. 5.





$$\Gamma^{def}_{\alpha'\beta'\gamma'}(p,q,r) = gc^{def}[g_{\gamma'\beta'}(r-q)_{\alpha'} + g_{\alpha'\gamma'}(p-r)_{\beta'} + g_{\beta'\alpha'}(q-p)_{\gamma'}]$$

Figure 5: The $\mathbf{3}$-gluon vertex diagram and answer to be obtained.

In extracting the above Γ, we left behind a variety of things:

$$\int d^4\tilde{p}e^{-ip\cdot y}\int d^4\tilde{q}e^{-iq\cdot z}\int d^4\tilde{r}e^{-ir\cdot w}[iD^{\alpha\alpha'}_{ad}(p)][iD^{\beta\beta'}_{be}(q)][iD^{\gamma\gamma'}_{cf}(r)]\,. \tag{509}$$

These factors disappear in relating

$$\langle 0|T\{A^{\alpha\,a}(y)A^{\beta\,b}(z)A^{\gamma\,c}(w)\}|0\rangle \tag{510}$$

to the actual physical probability amplitude connecting the vacuum state to the state of three outgoing gluons

$$\langle p,a,s;q,b,s';r,c,s''|0\rangle \tag{511}$$

using the reduction formalism that we did not go through. Removing the items that we did corresponds to going from the fields of the time ordered product to just the creation and annihilation operators contained in those fields that are used to define the 3-gluon state in the equation just above.
Thus, it is conventional to redefine the labels to go with the original external α, a, β, b and γ, c so that we write for the vertex (without

external propagators)

$$\Gamma^{abc}_{\alpha\beta\gamma}(p,q,r) = gc^{abc}(2\pi)^4\delta^4(p+q+r)g_{\gamma\alpha}p_\beta\,. \qquad (512)$$

If external gluons are attached to this vertex, one would simply multiply by the appropriate $\epsilon^\alpha(p), \ldots$.

Next, let me return for a moment as to why I said that p was the outgoing momentum associated with the a, α field. This is because of the way I defined $D^{\alpha}_{\nu\, da}(x-y)$ in terms of $e^{ip\cdot(x-y)}$. The $e^{+ip\cdot x}$ goes with an $a^\dagger(p)$ in the standard 2nd-quantized field expansion, corresponding to creation (i.e. which means outgoing) particle at the interaction vertex location defined by x.

Now, we must generate the other 5 terms that I said we could get by symmetry.

(a) First, we note that every term must be proportional to c^{abc} (or some permutation thereof that we can always write as c^{abc}).
The group indices can only appear here. This c^{abc} will be multiplied by some expression involving the momenta and Lorentz indices of the gauge particles.

(b) Next, since the gauge bosons are “bosons”, the vertex should be symmetric under interchange of the gauge bosons (gluons in SU(3) color case).

(c) Since c^{abc} is antisymmetric under interchange of any two gluons, that means that the structure multiplying it must itself be antisymmetric under the interchange of any two gluons.

The only structure involving momentum and Lorentz indices only that has the required antisymmetry and is consistent with the one term that we have derived is:

$$[g_{\gamma\alpha}(p-r)_\beta + g_{\gamma\beta}(r-q)_\alpha + g_{\alpha\beta}(q-p)_\gamma] \, . \tag{513}$$

One can check this structure for the required antisymmetries. For example, under $a, \alpha, p \leftrightarrow b, \beta, q$, the 1st term turns into $-$ the 2nd term, and vice versa, while the 3rd term is antisymmetric on its own when $p \leftrightarrow q$.

Thus, our final expression for the three-gluon (or more generally three-gauge-boson vertex) is

$$gc^{abc}(2\pi)^4\delta^4(p+q+r)\,[g_{\gamma\alpha}(p-r)_\beta + g_{\gamma\beta}(r-q)_\alpha + g_{\alpha\beta}(q-p)_\gamma] \, , \tag{514}$$

or, after removing the $(2\pi)^4\delta^4(p+q+r)$ in defining the $\mathcal{M}$ Feynman rule we obtain

$$\mathcal{M}^{abc}_{\alpha\beta\gamma}(p,q,r) = gc^{abc}\left[g_{\gamma\alpha}(p-r)_\beta + g_{\gamma\beta}(r-q)_\alpha + g_{\alpha\beta}(q-p)_\gamma\right] \,, \tag{515}$$

where the α, β, γ are to be contracted with the $\epsilon^\alpha_s(p), \epsilon^\beta_{s'}(q), \epsilon^\gamma_{s''}(r)$ for the polarization vectors of the helicity states of the outgoing gluons (indicated by the s, s', s'' subscripts on the ϵ's), or used internally with the α, β, γ indices connected to matching Lorentz indices of internal propagators.

An aside on conventions and comparisons between these notes and various books:

Some authors (like Ryder and Peskin) define a $\mathcal{M}'$ that is related to our $\mathcal{M}$ by $i\mathcal{M}' = \mathcal{M}$. In Mandl-Shaw box normalization,

$$\langle \ldots |S| \ldots \rangle = 1 + \prod \left(\frac{1}{\sqrt{2VE}}\right)^{1/2} (2\pi)^4 \delta^4(\sum_i p_i) i\mathcal{M}' \,. \tag{516}$$

In this convention, our Feynman rules are for $i\mathcal{M}'$. Ryder, Eq. (6.175), and Peskin, Eq. (4.73), use continuum/covariant normalization, for

which

$$\langle \ldots |S| \ldots \rangle = 1 + (2\pi)^4 \delta^4 (\sum_i p_i) i \mathcal{M}' . \qquad (517)$$

After computing using continuum/covariant normalization, the cross section expressions in terms of $|\mathcal{M}'|^2$ are the same as we have obtained in terms of $|\mathcal{M}|^2$. Still, even accounting for these normalization differences, Ryder's Eqs. (7.57) and (7.58) have an inconsistent relative phase, and the factor of 2 in (7.57) is certainly wrong.

My Feynman rules agree with those of Bailin-Love, pp. 127-129, except that they have chosen the opposite convention for the sign of g — see their Eq. (9.29) vs. my Eq. (352) — and there is a typo of a repeated term in their Eq. (10.74).

2. The reduction formalism

Here, I will justify the claim as to how to relate

$$\langle 0|T\{A^{\alpha\, a}(y)A^{\beta\, b}(z)A^{\gamma\, c}(w)\}|0\rangle \tag{518}$$

to

$$\langle p, a, s; q, b, s'; r, c, s''|0\rangle\,, \tag{519}$$

where I have given explicit helicity choices s, s', s'' to the outgoing gauge bosons.

For this, we need to recall some results from last quarter. We write, with the appropriate trivial generalization to include color,

$$A^{\nu\, a}(x) = A^{\nu\, a+}(x) + A^{\nu\, a-}(x) \tag{520}$$

with

$$A^{\nu\, a+} \;=\; \sum_{r\vec{k}} \left(\frac{1}{2V\omega_{\vec{k}}}\right)^{1/2} \epsilon^{\nu}_{r}(\vec{k})a^{a}_{r}(\vec{k})e^{-ik\cdot x}\,, \tag{521}$$

$$A^{\nu\, a-} \;=\; \sum_{r\vec{k}} \left(\frac{1}{2V\omega_{\vec{k}}}\right)^{1/2} \epsilon_r^{\nu}(\vec{k}) a_r^{\dagger\, a}(\vec{k}) e^{+ik\cdot x} \tag{522}$$

where $k^0 = \omega_{\vec{k}} = |\vec{k}|$ and $r = 0, 1, 2, 3$ to describe 4 polarization states. Recall that the ϵ_r^{ν} are real and obey

$$\epsilon_r(\vec{k})\cdot\epsilon_s(\vec{k}) = -\zeta_r \delta_{rs}\,, \quad r, s = 0, 1, 2, 3\,, \quad \sum_r \zeta_r \epsilon_r^{\mu}(\vec{k}) \epsilon_r^{\nu}(\vec{k}) = -g^{\mu\nu}\,, \tag{523}$$

where $\zeta_0 = -1$, $\zeta_{1,2,3} = +1$ and that only the $r = 1, 2$ states are physical.

Note: The above applies only in the Feynman version of the Lorentz gauge for which $\partial_\mu A^\mu = 0$, which is our $\alpha = 1$ choice. Life gets more complicated in proving the reduction result in a more general gauge. (But, of course, it still works.) In this gauge, the equation of motion is simple: $\partial^2 A_\mu = 0$ (for free fields) and we have for the momentum space propagator the result

$$D_{\mu\nu}^{ab}(p) = \delta_{ab} \frac{-g_{\mu\nu}}{p^2}\,. \tag{524}$$

From the above forms of $A^{\nu\, a\, \pm}$ and polarization orthogonalities, it is easy to verify that

$$
\begin{aligned}
a_r^a(\vec{k}) &= -\epsilon_r^\nu(\vec{k}) \int \frac{d^3\vec{x}}{\sqrt{2V\omega_{\vec{k}}}} e^{ik\cdot x} i\overleftrightarrow{\partial}_{x_0} A_\nu^a(x) \\
a_r^{a\dagger}(\vec{k}) &= -\epsilon_r^\nu(\vec{k}) \int \frac{d^3\vec{x}}{\sqrt{2V\omega_{\vec{k}}}} e^{-ik\cdot x} i\overleftrightarrow{\partial}_{x_0} A_\nu^a(x)\,,
\end{aligned}
\tag{525}
$$

where $A\overleftrightarrow{\partial}_{x_0}B \equiv A(\partial_{x_0}B) - (\partial_{x_0}A)B$ and the explicit $-$ sign is because of the $\epsilon_r(\vec{k})\cdot\epsilon_s(\vec{k}) = -\delta_{rs}$ for physical (transverse) polarizations. Armed with the above identities, consider the vertex, including helicity assignments for the outgoing gluons, and a notation $^{\rm out}$ to remind us that the annihilation operators appearing are those for outgoing gluons:

$$
\langle p, s, a; q, s', b; r, s'', c|0\rangle = \langle 0|[a_s^a(p)]^{\rm out}[a_{s'}^b(q)]^{\rm out}[a_{s''}^c(r)]^{\rm out}|0\rangle
\tag{526}
$$

and focus on just one of the a's for the moment, say $a_{s''}^c(r)$. We write

$$\begin{aligned}\langle p,s,a;q,s',b;r,s'',c|0\rangle &= \langle p,s,a;q,s',b|[a^c_{s''}(r)]^{\rm out}|0\rangle \\ &= \lim_{w_0\to+\infty e^{-i\delta}}\langle p,s,a;q,s',b|(-)\epsilon^{\gamma}_{s''}(\vec r)\int\frac{d^3\vec w}{\sqrt{2V\omega_{\vec r}}}e^{ir\cdot w}i\overleftrightarrow{\partial}_{w_0}A^c_{\gamma}(w)|0\rangle\,,\end{aligned} \tag{527}$$

where the $\overleftrightarrow{\partial}_{w_0}$ is for w_0 and where $w_0\to+\infty e^{-i\delta}$ is the appropriate thing since we have stated all the gluons are in the outgoing state. The imaginary component is the same one required to isolate the vacuum at a very late time.

Related to the above time limits, you should be asking yourself why

$$[a^c_{s''}(r)]^{\rm out}|0\rangle\neq 0\,? \tag{528}$$

Well, it is because I have been imprecise and should really be using

$$|0\rangle\to|0\rangle_{\rm in}\,,\quad \textbf{and}\quad \langle 0|\to_{\rm out}\langle 0| \tag{529}$$

and it is only $a^{\rm in}|0\rangle_{\rm in}$ that is 0.

We now use the simple identity that

$$\lim_{t\to+\infty e^{-i\delta}} f(t) = \lim_{t\to-\infty e^{-i\delta}} f(t) + \int_{-\infty e^{-i\delta}}^{+\infty e^{-i\delta}} \frac{df}{dt} \tag{530}$$

to rewrite (including all the appropriate in **and** out **subscripts)**

$$\begin{aligned}
& \lim_{w_0\to+\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a;q,s',b|(-)\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^3\vec w}{\sqrt{2V\omega_{\vec r}}} e^{ir\cdot w} i\overleftrightarrow{\partial}_{w_0} A^c_\gamma(w)|0\rangle_{\rm in} \\
= & \lim_{w_0\to-\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a;q,s',b|(-)\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^3\vec w}{\sqrt{2V\omega_{\vec r}}} e^{ir\cdot w} i\overleftrightarrow{\partial}_{w_0} A^c_\gamma(w)|0\rangle_{\rm in} \\
& + {}_{\rm out}\langle p,s,a;q,s',b|(-)\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^4 w}{\sqrt{2V\omega_{\vec r}}} \frac{d}{dw_0}\left[e^{ir\cdot w} i\overleftrightarrow{\partial}_{w_0} A^c_\gamma(w)\right]|0\rangle_{\rm in} \\
= & {}_{\rm out}\langle p,s,a;q,s',b|[a^c_{s''}(r)]^{\rm in}|0\rangle_{\rm in} \\
& + {}_{\rm out}\langle p,s,a;q,s',b|(-)\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^4 w}{\sqrt{2V\omega_{\vec r}}} i\left[(-\partial^2_{w_0} e^{ir\cdot w})A^c_\gamma(w) + e^{ir\cdot w}(\partial^2_{w_0} A^c_\gamma(w))\right]|0\rangle_{\rm in} \\
= & \; 0 + {}_{\rm out}\langle p,s,a;q,s',b|\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^4 w}{\sqrt{2V\omega_{\vec r}}} i\left[(\vec\nabla^2_w e^{ir\cdot w})A^c_\gamma(w) - e^{ir\cdot w}(\partial^2_{w_0} A^c_\gamma(w))\right]|0\rangle_{\rm in} \\
= & + {}_{\rm out}\langle p,s,a;q,s',b|\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^4 w}{\sqrt{2V\omega_{\vec r}}} \left[(-i)e^{ir\cdot w}[\partial^2_{w_0} - \vec\nabla^2_w]A^c_\gamma(w)\right]|0\rangle_{\rm in} \\
= & -i\epsilon^{\gamma}_{s''}(\vec r)\int \frac{d^4 w}{\sqrt{2V\omega_{\vec r}}} e^{ir\cdot w}\partial^2_w \,{}_{\rm out}\langle p,s,a;q,s',b|A^c_\gamma(w)|0\rangle_{\rm in}\,,
\end{aligned} \tag{531}$$

where we used $\partial^2_{w_0} e^{ir\cdot w} = \vec{\nabla}^2_w e^{ir\cdot w}$ and then did double partial integration on $\vec{\nabla}^2_w$.
Now let us "reduce" in a second final state annihilation operator.

$$
\begin{aligned}
& {}_{\rm out}\langle p,s,a;q,s',b|A^c_\gamma(w)|0\rangle_{\rm in} \\
= \quad & {}_{\rm out}\langle p,s,a|[a^b_{s'}(q)]^{\rm out} A^c_\gamma(w)|0\rangle_{\rm in} \\
= \quad & \lim_{z_0\to+\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^3\vec z}{\sqrt{2V\omega_{\vec q}}} e^{iq\cdot z} i\overleftrightarrow{\partial}_{z_0} A^b_\beta(z)A^c_\gamma(w)|0\rangle_{\rm in} \\
= \quad & \lim_{z_0\to+\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^3\vec z}{\sqrt{2V\omega_{\vec q}}} e^{iq\cdot z} i\overleftrightarrow{\partial}_{z_0} T\{A^b_\beta(z)A^c_\gamma(w)\}|0\rangle_{\rm in} \\
= \quad & \lim_{z_0\to-\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^3\vec z}{\sqrt{2V\omega_{\vec q}}} e^{iq\cdot z} i\overleftrightarrow{\partial}_{z_0} T\{A^b_\beta(z)A^c_\gamma(w)\}|0\rangle_{\rm in} \\
& + {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^4 z}{\sqrt{2V\omega_{\vec q}}}\frac{d}{dz_0}\left[e^{iq\cdot z} i\overleftrightarrow{\partial}_{z_0} T\{A^b_\beta(z)A^c_\gamma(w)\}\right]|0\rangle_{\rm in} \\
= \quad & \lim_{z_0\to-\infty e^{-i\delta}} {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^3\vec z}{\sqrt{2V\omega_{\vec q}}} e^{iq\cdot z} i\overleftrightarrow{\partial}_{z_0} A^c_\gamma(w)A^b_\beta(z)|0\rangle_{\rm in} \\
& + {}_{\rm out}\langle p,s,a|(-)\epsilon^\beta_{s'}(\vec q)\int \frac{d^4 z}{\sqrt{2V\omega_{\vec q}}} i\left[(-\partial^2_{z_0}e^{iq\cdot z})T\{A^b_\beta(z)A^c_\gamma(w)\} + e^{iq\cdot z}\left(\partial^2_{z_0}T\{A^b_\beta(z)A^c_\gamma(w)\}\right)\right]|0\rangle_{\rm in} \\
= \quad & {}_{\rm out}\langle p,s,a|A^c_\gamma(w)[a^b_{s'}(q)]^{\rm in}|0\rangle_{\rm in}
\end{aligned}
$$

$$+ {}_{\rm out}\langle p,s,a|\epsilon^{\beta}_{s'}(\vec q)\int \frac{d^4z}{\sqrt{2V\omega_{\vec q}}} i\left[(\vec\nabla_z^2 e^{iq\cdot z})T\{A^b_\beta(z)A^c_\gamma(w)\} - e^{iq\cdot z}\left(\partial^2_{z_0}T\{A^b_\beta(z)A^c_\gamma(w)\}\right)\right]|0\rangle_{\rm in}$$

$$= \quad 0 + {}_{\rm out}\langle p,s,a|\epsilon^{\beta}_{s'}(\vec q)\int \frac{d^4z}{\sqrt{2V\omega_{\vec q}}} i\left[(-)e^{iq\cdot z}\left(\partial^2_{z_0} - \vec\nabla_z^2\right)T\{A^b_\beta(z)A^c_\gamma(w)\}\right]|0\rangle_{\rm in}$$

$$= \quad -i\epsilon^{\beta}_{s'}(\vec q)\int \frac{d^4z}{\sqrt{2V\omega_{\vec q}}} e^{iq\cdot z}\partial^2_z \, {}_{\rm out}\langle p,s,a|T\{A^b_\beta(z)A^c_\gamma(w)\}|0\rangle_{\rm in}\,. \tag{532}$$

The most important new manipulation above was the insertion of the time-ordered product. This was inserted so that when we converted from $z_0 \to +\infty$ to $z_0 \to -\infty$ the $A^b_\beta(z)$ field moved all the way over to being next to $|0\rangle_{\rm in}$ so that in the next step we had $[a^b_{s'}(q)]^{\rm in}$ sitting next to $|0\rangle_{\rm in}$ giving 0. Without the time ordering instruction the $[a^b_{s'}(q)]^{\rm in}$ operator would have been operating on $A^c_\gamma(w)|0\rangle_{\rm in}$, and we could not say that it was annihilating the incoming vacuum.

At this point, we would continue in similar vein to reduce in the $[a^a_s(p)]^{\rm out}$ operator. Following the same procedure, we would get the analogous result involving the appropriate projection operator operating on

$${}_{\rm out}\langle 0|T\{A^a_\alpha(y)A^b_\beta(z)A^c_\gamma(w)\}|0\rangle_{\rm in}\,. \tag{533}$$

Altogether, we get

$${}_{\rm out}\langle p,a,s;q,b,s';r,c,s''|0\rangle_{\rm in}$$

$$= \left[-i\epsilon^{\gamma}_{s''}(\vec{r})\int \frac{d^4w}{\sqrt{2V\omega_{\vec{r}}}} e^{ir\cdot w}\partial^2_w\right]\left[-i\epsilon^{\beta}_{s'}(\vec{q})\int \frac{d^4z}{\sqrt{2V\omega_{\vec{q}}}} e^{iq\cdot z}\partial^2_z\right]$$

$$\left[-i\epsilon^{\alpha}_{s}(\vec{p})\int \frac{d^4y}{\sqrt{2V\omega_{\vec{p}}}} e^{ip\cdot y}\partial^2_y\right] \times \quad {}_{\rm out}\langle 0|T\{A^a_\alpha(y)A^b_\beta(z)A^c_\gamma(w)\}|0\rangle_{\rm in} \,. \tag{534}$$

We now insert

$$\langle 0|T\{A^{\alpha\, a}(y)A^{\beta\, b}(z)A^{\gamma\, c}(w)\}|0\rangle$$
$$= \int \frac{d^4p'}{(2\pi)^4}e^{-ip'\cdot y}\int \frac{d^4q'}{(2\pi)^4}e^{-iq'\cdot z}\int \frac{d^4r'}{(2\pi)^4}e^{-ir'\cdot w}[iD^{\alpha\alpha'}_{ad}(p')][iD^{\beta\beta'}_{be}(q')][iD^{\gamma\gamma'}_{cf}(r')]\Gamma^{def}_{\alpha'\beta'\gamma'}(p',q',r')\,, \tag{535}$$

where we shifted notation to dummy integration variables, p', q', r'. Let's look at one of the projection operations. Using $iD^{\alpha'}_{\alpha\, ad}(p') = -ig^{\alpha'}_{\alpha}\delta_{ad}/p'^2$, one finds

$$\left[-i\epsilon^{\alpha}_{s}(\vec{p})\int \frac{d^4y}{\sqrt{2V\omega_{\vec{p}}}} e^{ip\cdot y}\partial^2_y\right]\int \frac{d^4p'}{(2\pi)^4}e^{-ip'\cdot y}[iD^{\alpha'}_{\alpha\, ad}(p')]$$

$$= \left[-i\epsilon^{\alpha}_{s}(\vec{p})\int \frac{d^4y}{\sqrt{2V\omega_{\vec{p}}}} e^{ip\cdot y}\right]\int \frac{d^4p'}{(2\pi)^4}(-p'^2)e^{-ip'\cdot y}\frac{-ig^{\alpha'}_{\alpha}\delta_{ad}}{p'^2}$$

$$= \quad \frac{1}{\sqrt{2V\omega_{\vec{p}}}}\epsilon_s^{\alpha}(\vec{p})\int \frac{d^4p'}{(2\pi)^4}(2\pi)^4\delta^4(p-p')g_{\alpha}^{\alpha'}\delta_{ad}$$

$$= \quad \frac{1}{\sqrt{2V\omega_{\vec{p}}}}\epsilon_s^{\alpha}(\vec{p})g_{\alpha}^{\alpha'}\delta_{ad}\,. \tag{536}$$

Notice the cancellation of the propagator p'^2 from the result of the ∂_y^2 differentiation of $e^{-ip'\cdot y}$. Analogous results apply for the other two projections, yielding

$$_{\rm out}\langle p,a,s;q,b,s';r,c,s''|0\rangle_{\rm in} \equiv \frac{1}{\sqrt{2V\omega_{\vec{p}}}\sqrt{2V\omega_{\vec{q}}}\sqrt{2V\omega_{\vec{r}}}}(2\pi)^4\delta^4(p+q+r)\mathcal{M}$$

$$= \quad \left[\frac{1}{\sqrt{2V\omega_{\vec{p}}}}\epsilon_s^{\alpha}(\vec{p})g_{\alpha}^{\alpha'}\delta_{ad}\right]\left[\frac{1}{\sqrt{2V\omega_{\vec{q}}}}\epsilon_{s'}^{\beta}(\vec{q})g_{\beta}^{\beta'}\delta_{be}\right]$$

$$\left[\frac{1}{\sqrt{2V\omega_{\vec{r}}}}\epsilon_{s''}^{\gamma}(\vec{r})g_{\gamma}^{\gamma'}\delta_{cf}\right](2\pi)^4\delta^4(p+q+r)gc^{def}[g_{\gamma'\alpha'}(p-r)_{\beta'}+\ldots]$$

$$= \quad \frac{1}{\sqrt{2V\omega_{\vec{p}}}\sqrt{2V\omega_{\vec{q}}}\sqrt{2V\omega_{\vec{r}}}}\epsilon_s^{\alpha}(\vec{p})\epsilon_{s'}^{\beta}(\vec{q})\epsilon_{s''}^{\gamma}(\vec{r})(2\pi)^4\delta^4(p+q+r)gc^{abc}[g_{\gamma\alpha}(p-r)_{\beta}+\ldots] \tag{537}$$

from which we immediately obtain the previously given Feynman rule for the vertex part of $\mathcal{M}$ to be multiplied by the three ϵ's:

$$\mathcal{M}_{\alpha\beta\gamma}^{abc}(p,q,r) = gc^{abc}\left[g_{\gamma\alpha}(p-r)_{\beta}+g_{\gamma\beta}(r-q)_{\alpha}+g_{\alpha\beta}(q-p)_{\gamma}\right]\,. \tag{538}$$

There is one more thing that I have hidden in the above discussion: that is a whole bunch of $\sqrt{Z}$ factors. I mentioned these last quarter but we did not dwell on them. For tree-level computations we can take $Z = 1$ since the higher powers of g in the the expansion $Z = 1 + \mathcal{O}(g^2) + \ldots$ lead to higher order corrections to the tree-level amplitudes.
The $\sqrt{Z}$ factors arise as follows. We should more correctly write at the first stage of the reduction process

$$
\begin{aligned}
[a_r^a(\vec{k})]^{\rm out} &= -\epsilon_r^\nu(\vec{k}) \int \frac{d^3\vec{x}}{\sqrt{2V\omega_{\vec{k}}}} e^{ik\cdot x} i\overleftrightarrow{\partial}_{x_0} [A_\nu^a(x)]^{\rm out} \\
&= -\epsilon_r^\nu(\vec{k}) \lim_{x_0\to\infty e^{-i\delta}} \int \frac{d^3\vec{x}}{\sqrt{2V\omega_{\vec{k}}}} e^{ik\cdot x} i\overleftrightarrow{\partial}_{x_0} [A_\nu^a(x)](\sqrt{Z})^{-1} \qquad (539)
\end{aligned}
$$

where $A_\nu^a(x)$ is the fully interacting A field, whereas $[A_\nu^a(x)]^{\rm out}$ is a free particle-like A field *with full one-particle normalization*. The interacting A field in the very late time limit (or very early time limit) does not have full one-particle normalization content and so we must multiply by a factor to boost up its normalization. This relative factor is by convention called $\sqrt{Z}$ with $Z < 1$ (after renormalization or cut off). Another subtlety is that

$$
\lim_{x_0\to\infty e^{-i\delta}} A = \sqrt{Z} A^{\rm out} \qquad (540)
$$

cannot be viewed as a statement about the operators, but only as a statement about the operators when applied to ${}_{\rm out}\langle 0|$.
In addition to the above $(\sqrt{Z})^{-1}$ factor for each external particle, there is an additional Z factor for each external particle, coming from the sum of so-called one-particle reducible diagrams inserted into the external particle leg.

We have no time to get into all this. You will encounter these issues in a full discussion of the reduction formalism.

3. **The gluon-ghost-antighost interaction**

This is the next simplest case. The quantity we want to compute is

$$\begin{aligned}
&\langle 0|T\left\{c^b(w)A^{\alpha\, a}(z)\overline{c}^c(y)\right\}|0\rangle \\
&= \left\{\left[\frac{1}{i}\frac{\delta}{\delta\overline{\xi}^b(w)}\right]\left[\frac{1}{i}\frac{\delta}{\delta J^a_\alpha(z)}\right]\left[\frac{1}{-i}\frac{\delta}{\delta\xi^c(y)}\right]\left(\exp\left[iS_I\left(\frac{\delta}{i\delta J},\frac{\delta}{-i\delta\xi},\frac{\delta}{i\delta\overline{\xi}}\right)\right]Z_0[J]Z_0[\xi,\overline{\xi}]\right)\right\}_{J=0,\xi=0,\overline{\xi}=0}
\end{aligned} \tag{541}$$

Of course, we expand perturbatively and write $\exp\left[iS_I\right]\sim 1+iS_I$ and focus on the iS_I term as being that responsible for the true tree-level interactions.

The only iS_I interaction term contributing at tree-level to the particular gluon-ghost-antighost vertex being considered at the moment is [see Eq. (482)]

$$\begin{aligned}
iS_I\left(A,\overline{c},c\right) &= i\int d^4x\overline{c}^d(x)gc^{def}\partial^\mu(A^f_\mu(x)c^e(x)) \\
&= -i\int d^4x(\partial^\mu\overline{c}^d(x))gc^{def}A^f_\mu(x)c^e(x) \\
&\to -i\int d^4x\left(\partial^\mu\frac{\delta}{-i\delta\xi^d(x)}\right)gc^{def}\frac{\delta}{i\delta J^{f\,\mu}(x)}\frac{\delta}{i\delta\overline{\xi}^e(x)}\,.
\end{aligned} \tag{542}$$

where the form on the 2nd line is obtained by partial integration.

So, as usual, we must consider what terms must be brought down from the S_I derivatives of the exponentials appearing in $Z_0[J]Z_0[\xi,\overline{\xi}]$ in

order that the further derivatives appearing in Eq. (541) give a result that does not vanish when the sources are set to 0.
I hope it is clear that one wants to bring down a new term for each of the three derivatives appearing in iS_I.

This yields

$$\exp\left[iS_I\left(\frac{\delta}{i\delta J},\frac{\delta}{-i\delta\xi},\frac{\delta}{i\delta\overline{\xi}}\right)\right]Z_0[J]Z_0[\xi,\overline{\xi}]$$

$$\begin{aligned}
&= -i\int d^4x\left(\partial^\mu\frac{\delta}{-i\delta\xi^d(x)}\right)gc^{def}\frac{\delta}{i\delta J^{f\,\mu}(x)}\frac{\delta}{i\delta\overline{\xi}^e(x)}\\
&\quad\exp\left\{-\frac{i}{2}\int d^4x'd^4y'J^{a'}_{\mu'}(x')D^{\mu'\nu'}_{a'b'}(x'-y')J^{b'}_{\nu'}(y')\right\}\exp\left[-i\int d^4x'd^4y'\overline{\xi}^{c'}(x')\Delta^{c'd'}(x'-y')\xi^{d'}(y')\right]\\
&\ni -i\int d^4x\left(\partial^\mu\frac{\delta}{-i\delta\xi^d(x)}\right)gc^{def}\frac{\delta}{i\delta J^{f\,\mu}(x)}\left[-\int d^4y_1\Delta^{ed''}(x-y_1)\xi^{d''}(y_1)\right]\\
&\quad\exp\left\{-\frac{i}{2}\int d^4x'd^4y'J^{a'}_{\mu'}(x')D^{\mu'\nu'}_{a'b'}(x'-y')J^{b'}_{\nu'}(y')\right\}\exp\left[-i\int d^4x'd^4y'\overline{\xi}^{c'}(x')\Delta^{c'd'}(x'-y')\xi^{d'}(y')\right]\\
&\ni -i\int d^4x\left(\partial^\mu\frac{\delta}{-i\delta\xi^d(x)}\right)gc^{def}\left\{-\int d^4y_2D^{\mu\nu''}_{fb''}(x-y_2)J^{b''}_{\nu''}(y_2)\right\}\left[-\int d^4y_1\Delta^{ed''}(x-y_1)\xi^{d''}(y_1)\right]\\
&\quad\exp\left\{-\frac{i}{2}\int d^4x'd^4y'J^{a'}_{\mu'}(x')D^{\mu'\nu'}_{a'b'}(x'-y')J^{b'}_{\nu'}(y')\right\}\exp\left[-i\int d^4x'd^4y'\overline{\xi}^{c'}(x')\Delta^{c'd'}(x'-y')\xi^{d'}(y')\right]\\
&\ni -i\int d^4x\left(\partial^\mu\left[-\int d^4x_1\overline{\xi}^{c''}(x_1)\Delta^{c''d}(x_1-x)\right]\right)gc^{def}\left\{-\int d^4y_2D^{\mu\nu''}_{fb''}(x-y_2)J^{b''}_{\nu''}(y_2)\right\}\\
&\quad\left[-\int d^4y_1\Delta^{ed''}(x-y_1)\xi^{d''}(y_1)\right]
\end{aligned}$$

$$\exp\left\{-\frac{i}{2}\int d^4x'd^4y' J^{a'}_{\mu'}(x')D^{\mu'\nu'}_{a'b'}(x'-y')J^{b'}_{\nu'}(y')\right\}\exp\left[-i\int d^4x'd^4y'\overline{\xi}^{c'}(x')\Delta^{c'd'}(x'-y')\xi^{d'}(y')\right] \tag{543}$$

where there was a $(-)^3$ that arose when $\frac{\delta}{-i\delta\xi^d(x)}$ had to go past two Grassmann objects to get to the object it acted on and then the Grassmann quantity brought down from the exponential was moved back to the left past a single Grassmann object.

So, now we want

$$\left[\frac{1}{i}\frac{\delta}{\delta\overline{\xi}^b(w)}\right]\left[\frac{1}{i}\frac{\delta}{\delta J^a_\alpha(z)}\right]\left[\frac{1}{-i}\frac{\delta}{\delta\xi^c(y)}\right] \tag{544}$$

operating on the above and then we set the sources to 0. As we have said, we don't want any more operations up in the exponential in order to get the vertex we want. So, we must compute

$$
\begin{aligned}
&\left[\frac{1}{i}\frac{\delta}{\delta\bar{\xi}^b(w)}\right]\left[\frac{1}{i}\frac{\delta}{\delta J^a_\alpha(z)}\right]\left[\frac{1}{-i}\frac{\delta}{\delta\xi^c(y)}\right]\\
&\left\{-i\int d^4x\left(\partial^\mu_x\left[-\int d^4x_1\bar{\xi}^{c''}(x_1)\Delta^{c''d}(x_1-x)\right]\right)gc^{def}\left\{-\int d^4y_2 D^{\mu\nu''}_{fb''}(x-y_2)J^{b''}_{\nu''}(y_2)\right\}\right.\\
&\left.\left[-\int d^4y_1\Delta^{ed''}(x-y_1)\xi^{d''}(y_1)\right]\right\}\\
=&\left[\frac{1}{i}\frac{\delta}{\delta\bar{\xi}^b(w)}\right]\left[\frac{1}{i}\frac{\delta}{\delta J^a_\alpha(z)}\right]\left\{-i\int d^4x\left(\partial^\mu_x\left[-\int d^4x_1\bar{\xi}^{c''}(x_1)\Delta^{c''d}(x_1-x)\right]\right)\right.\\
&\left.gc^{def}\left\{-\int d^4y_2 D^{\mu\nu''}_{fb''}(x-y_2)J^{b''}_{\nu''}(y_2)\right\}\left[i\Delta^{ec}(x-y)\right]\right\}\\
=&\left[\frac{1}{i}\frac{\delta}{\delta\bar{\xi}^b(w)}\right]\left\{-i\int d^4x\left(\partial^\mu_x\left[-\int d^4x_1\bar{\xi}^{c''}(x_1)\Delta^{c''d}(x_1-x)\right]\right)gc^{def}\left\{iD^{\mu\alpha}_{fa}(x-z)\right\}\left[i\Delta^{ec}(x-y)\right]\right\}\\
=&\left\{-i\int d^4x\left(\partial^\mu_x\left[i\Delta^{bd}(w-x)\right]\right)gc^{def}\left\{iD^{\mu\alpha}_{fa}(x-z)\right\}\left[i\Delta^{ec}(x-y)\right]\right\}\\
=&\left\{-i\int d^4x\left(\partial^\mu_x\left[i\int d^4\tilde{p}e^{ip\cdot(x-w)}\Delta^{bd}(p)\right]\right)gc^{def}\left\{i\int d^4\tilde{k}e^{ik\cdot(x-z)}D^{\mu\alpha}_{fa}(k)\right\}\right.\\
&\left.\left[i\int d^4\tilde{q}e^{-iq\cdot(x-y)}\Delta^{ec}(q)\right]\right\}
\end{aligned}
\tag{545}
$$

where at each stage we have kept careful track of Grassmann $(-)$ signs, and at the last stage I have chosen conventions in the exponentials so that p and k are outgoing momenta and q is an incoming momentum. We now perform the $\int d^4x$ and are left with the vertex with external propagators attached in the form:

$$p^\mu g c^{def}(2\pi)^4\delta^4(p+k-q)\left[i\Delta^{bd}(p)\right]\left[iD^{\mu\alpha}_{fa}(k)\right]\left[i\Delta^{ec}(q)\right]. \tag{546}$$

Extracting the external propagators and relabeling to the original external field indices, gives:

$$\mathcal{M}^{abc}_\alpha(k,p,q)=gc^{bca}p_\alpha=gc^{abc}p_\alpha\,, \tag{547}$$

where we must remember that p is the momentum of the outgoing ghost line.

If you compare closely to Ryder, you will see that I think he has the wrong sign for this vertex. In general, my results agree with those of Bailin and Love (after accounting for the fact that they adopted the opposite convention for the sign of g!).

In fact, as noted earlier, the sign of this vertex and the sign of the ghost propagator are not separately physically relevant, but their relative sign is physically relevant since it is fixed by the relative signs of the kinetic and interaction terms appearing in the response function.

Diagrammatically, there is always one ghost-ghost-gluon vertex for each ghost propagator and so it is only the product of the two signs that is important.

The other very crucially important thing is the $(-)$ sign associated with a closed ghost loop.

Put together with the correct relative sign between ghost-ghost-gluon vertex and ghost propagator, this loop minus sign guarantees that there will be a cancellation of the unphysical degrees of freedom associated with a closed gluon loop by a corresponding ghost loop.

2nd quantization approach check

Perhaps we should try to check this using the 2nd quantization "killing" approach. To establish notation, we assume that we write the ghost fields as

$$
\begin{aligned}
c^e(x) &= \sum_{\vec{l}} \frac{1}{\sqrt{2VE_l}} \left[b^e(\vec{l}) e^{-il\cdot x} + d^{e\dagger}(\vec{l}) e^{il\cdot x} \right] \\
\overline{c}^d(x) &= \sum_{\vec{l}'} \frac{1}{\sqrt{2VE_{\vec{l}'}}} \left[d^d(\vec{l}') e^{-il'\cdot x} + b^{d\dagger}(\vec{l}') e^{il'\cdot x} \right]
\end{aligned}
\tag{548}
$$

where the b^a and d^a obey the fermion-like anticommutation relations. And, of course, we have (in a basis where the ϵ's are real)

$$
A^f_\mu(x) = \sum_{t,\vec{r}} \frac{1}{\sqrt{2V\omega_{\vec{r}}}} \left[\epsilon^t_\mu(\vec{r}) a^f_t(\vec{r}) e^{-ir\cdot x} + \epsilon^t_\mu(\vec{r}) a^{f\dagger}_t(\vec{r}) e^{+ir\cdot x} \right] . \tag{549}
$$

To compute the Feynman vertex, we compute (s is the gluon helicity

with polarization vector $\epsilon_s^\alpha(k)$ and we keep only the relevant terms in the field expansions)

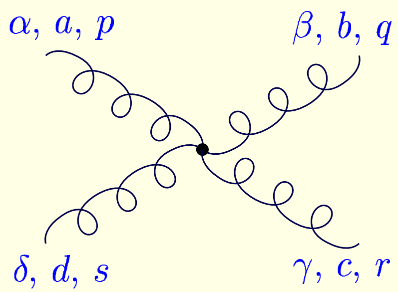
$$
\begin{aligned}
{}_{\rm out}\langle p,b;k,s,a|q,c\rangle_{\rm in} &= \langle p,b;k,s,a|iS_I|q,c\rangle \\
&= \langle 0|a_s^a(\vec k)b^b(\vec p)\left[-i\int d^4x(\partial^\mu \overline{c}^d(x))gc^{def}A_\mu^f(x)c^e(x)\right]b^c(\vec q)^\dagger|0\rangle \\
&= -i\int d^4x\sum_{\vec l,\vec l',\vec r}\sum_t \frac{1}{\sqrt{2VE_{\vec l}}\sqrt{2VE_{\vec l'}}\sqrt{2V\omega_{\vec r}}}e^{-il\cdot x+il'\cdot x+ir\cdot x} \\
&\quad \langle 0|a_s^a(\vec k)b^b(\vec p)\left[(il'^\mu b^{d\dagger}(\vec l')gc^{def}\epsilon_\mu^t(\vec r)a_t^{f\dagger}(\vec r)b^e(\vec l)\right]b^c(\vec q)^\dagger|0\rangle \\
&= \int d^4x\sum_{\vec l,\vec l',\vec r}\sum_t \frac{1}{\sqrt{2VE_{\vec l}}\sqrt{2VE_{\vec l'}}\sqrt{2V\omega_{\vec r}}}e^{-il\cdot x+il'\cdot x+ir\cdot x} \\
&\quad \epsilon_\mu^t(\vec r)\langle 0|\left[\delta_{st}\delta_{af}\delta_{\vec r\vec k}\right]\left[\delta_{bd}\delta_{\vec p\vec l'}l'^\mu\right]\left[\delta_{ce}\delta_{\vec l\vec q}\right]gc^{def}|0\rangle \\
&= \frac{1}{\sqrt{2VE_{\vec q}}\sqrt{2VE_{\vec p}}\sqrt{2V\omega_{\vec k}}}(2\pi)^4\delta^4(p+k-q)\epsilon_\mu^s(\vec k)gc^{bca}p^\mu \\
&\equiv \frac{1}{\sqrt{2VE_{\vec q}}\sqrt{2VE_{\vec p}}\sqrt{2V\omega_{\vec k}}}(2\pi)^4\delta^4(p+k-q)\epsilon_\mu^s(\vec k)\mathcal{M}_{abc}^\mu \qquad (550)
\end{aligned}
$$

from which we obtain (where c belongs to incoming ghost of momentum q, b to outgoing ghost of momentum p, and a to outgoing gluon of momentum k) a result with opposite sign to Ryder (after careful index mapping):

$$\mathcal{M}_{abc}^\mu = gc^{abc}p^\mu\,. \qquad (551)$$

4. **The 4-gluon interaction**

Doing this using functional derivative techniques is the homework assignment. Here, I will do it using the creation/annihilation killing operations approach.



$$\begin{aligned}-ig^2[c^{eab}c^{ecd}&(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\\ +c^{eac}c^{edb}&(g_{\alpha\delta}g_{\gamma\beta}-g_{\alpha\beta}g_{\gamma\delta})\\ +c^{ead}c^{ebc}&(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta})]\end{aligned}$$

Figure 6: The 4-gluon vertex diagram and answer to be obtained.

We wish to compute the S-matrix

$$\langle p,\lambda_1,a;q,\lambda_2,b;r,\lambda_3,c;s,\lambda_4,d|S|0\rangle=\epsilon_\alpha^{\lambda_1\,*}(\vec{p})\epsilon_\beta^{\lambda_2\,*}(\vec{q})\epsilon_\gamma^{\lambda_3\,*}(\vec{r})\epsilon_\delta^{\lambda_4\,*}(\vec{s})V^{\alpha,a;\,\beta,b;\,\gamma,c;\,\delta,d}(p,q,r,s) \tag{552}$$

where V is the Feynman vertex and $S=1+i\int d^4x\mathcal{L}_{\rm int}(x)+\ldots$ implies that we use $i\int d^4x\mathcal{L}_{\rm int}(x)$ for the first non-trivial tree-level vertex, and the relevant term in the interaction Lagrangian is the

quartic term from the quartic interaction:

$$\mathcal{L}_{\rm int}(x) = -\frac{1}{4}F^e_{\mu\nu}F^{\mu\nu\,e} \ni -\frac{1}{4}g^2 c^{efg}c^{ef'g'}A^f_\mu A^g_\nu A^{\mu\,f'}A^{\nu\,g'} \quad (553)$$

with all the A fields evaluated at x. To avoid writing factors that we clearly understand, I will not include the $1/\sqrt{2V\omega}$ factors that are in the Fourier expansions of the A fields. We have seen earlier that these are extracted as multipliers when we define the $\mathcal{M}$ for the process — it is $\mathcal{M}$ that defines our Feynman rules. I will also only write the $a^\dagger$ parts of the fields in $\mathcal{L}$ as we need 4 $a^\dagger$'s to contract (commute) against the a's associated with forming the outgoing state. So, what we have is then

$$\begin{aligned}
&\langle p,\lambda_1,a;q,\lambda_2,b;r,\lambda_3,c;s,\lambda_4,d|S|0\rangle \\
&\ni -\frac{i}{4}g^2c^{efg}c^{ef'g'}\int d^4x\langle 0|a^{\lambda_1}_a(\vec p)a^{\lambda_2}_b(\vec q)a^{\lambda_3}_c(\vec r)a^{\lambda_4}_d(\vec s)\sum_{\vec p',\vec q',\vec r',\vec s'}\ \sum_{\lambda_1',\lambda_2',\lambda_3',\lambda_4'} \\
&\left[a^{\lambda_1'\,\dagger}_f(\vec p')\epsilon^{\lambda_1'\,*}_\mu(\vec p')e^{ix\cdot p'}\right]\left[a^{\lambda_2'\,\dagger}_g(\vec q')\epsilon^{\lambda_2'\,*}_\nu(\vec q')e^{ix\cdot q'}\right]\left[a^{\lambda_3'\,\dagger}_{f'}(\vec r')\epsilon^{\lambda_3'\,\mu\,*}(\vec r')e^{ix\cdot r'}\right] \\
&\left[a^{\lambda_4'\,\dagger}_{g'}(\vec s')\epsilon^{\lambda_4'\,\nu\,*}(\vec s')e^{ix\cdot s'}\right]|0\rangle
\end{aligned} \quad (554)$$

Now, altogether there are $4! = 4\times 6$ ways in which we can get a non-zero contribution using a particular matching up to obtain four $[a, a^\dagger]$ commutators. Of these, only 6 give different algebraic forms and each of these 6 algebraic forms is duplicated 4 times; this latter factor of 4 cancels the $1/4$ out in front. Let us display the term which the notation employed is designed to most easily single out. This is the one that arises from

$$[a_a^{\lambda_1}(p), a_f^{\lambda_1'\,\dagger}(p')] = \delta^{\lambda_1\,\lambda_1'}\delta^{a\,f}\delta_{\vec{p}\vec{p}\,'} \tag{555}$$

$$[a_b^{\lambda_2}(q), a_g^{\lambda_2'\,\dagger}(q')] = \delta^{\lambda_2\,\lambda_2'}\delta^{b\,g}\delta_{\vec{q}\vec{q}\,'} \tag{556}$$

$$[a_c^{\lambda_3}(r), a_{f'}^{\lambda_3'\,\dagger}(r')] = \delta^{\lambda_3\,\lambda_3'}\delta^{c\,f'}\delta_{\vec{r}\vec{r}\,'} \tag{557}$$

$$[a_d^{\lambda_4}(s), a_{g'}^{\lambda_4'\,\dagger}(s')] = \delta^{\lambda_4\,\lambda_4'}\delta^{d\,g'}\delta_{\vec{s}\vec{s}\,'}\,. \tag{558}$$

We first note that after using the $\delta_{\ldots}$'s to perform the p', q', r', s' sums, we develop the integral

$$\int d^4x e^{ix\cdot(p+q+r+s)} = (2\pi)^4\delta^4(p+q+r+s)\,, \tag{559}$$

which is the standard momentum conservation function for the 4 outgoing gluons that we expect to obtain. This same factor emerges for every one of the 'contractions' that we discuss below and so we shall not continue to write it. It is a common factor to all the vertex contributions given below. We will denote the term we have discussed above as the $a \to f, b \to g, c \to f', d \to g'$ contraction, for which we obtain the result given below:

$$
\begin{aligned}
& -\frac{i}{4} g^2 c^{eab} c^{ecd} \epsilon_\mu^{\lambda_1\,*}(\vec{p}) \epsilon^{\lambda_3\,\mu\,*}(\vec{r}) \epsilon_\nu^{\lambda_2\,*}(\vec{q}) \epsilon^{\lambda_4\,\nu\,*}(\vec{s}) \\
& = -\frac{i}{4} g^2 c^{eab} c^{ecd} g^{\alpha\gamma} g^{\beta\delta} \epsilon_\alpha^{\lambda_1\,*}(\vec{p}) \epsilon_\beta^{\lambda_2\,*}(\vec{q}) \epsilon_\gamma^{\lambda_3\,*}(\vec{r}) \epsilon_\delta^{\lambda_4\,*}(\vec{s}) \quad (560)
\end{aligned}
$$

As usual, in the 2nd line, I have restated the Lorentz dot products of the ϵ's using the metric tensor. This is always a necessary step in extracting the vertex function as defined in Eq. (552). Removing the ϵ's, gives the first entry in our list of 'contraction' contributions to the vertex function:

$$
a \to f, b \to g, c \to f', d \to g' : \quad -\frac{i}{4} g^2 c^{eab} c^{ecd} g^{\alpha\gamma} g^{\beta\delta} \quad (561)
$$

So, now where do the other 3 contributions that take exactly the same algebraic form come from. First, if we interchange $(\alpha, a) \leftrightarrow (\beta, b)$ and $(\gamma, c) \leftrightarrow (\delta, d)$, the metric tensors simply switch places while we can use $c^{eba}c^{edc} = (-c^{eab})(-c^{ecd}) = c^{eab}c^{ecd}$ to find that we get the same result:

$$a \to g, b \to f, c \to g', d \to f' : \quad -\frac{i}{4}g^2 c^{eab}c^{ecd}g^{\alpha\gamma}g^{\beta\delta} . \tag{562}$$

Next, we can note that we get another two completely equivalent terms by interchanging the roles of $f, g \leftrightarrow f', g'$ — note that this leaves the $A_\mu A^\mu$ contraction unaltered and the $A_\nu A^\nu$ contraction unaltered in each case:

$$a \to f', b \to g', c \to f, d \to g : \quad -\frac{i}{4}g^2 c^{eab}c^{ecd}g^{\alpha\gamma}g^{\beta\delta} \tag{563}$$

$$a \to g', b \to f', c \to g, d \to f : \quad -\frac{i}{4}g^2 c^{eab}c^{ecd}g^{\alpha\gamma}g^{\beta\delta} . \tag{564}$$

So, we have the required 4 identical terms to cancel the 1/4 factor. We represent these 4 terms by the first sequence of contractions, writing

$$\left(a \to f, b \to g, c \to f', d \to g' + \text{equiv.}\right) : \quad -ig^2 c^{eab}c^{ecd}g^{\alpha\gamma}g^{\beta\delta} . \tag{565}$$

Now let us consider an inequivalent contraction that nonetheless gives the same color factor out front. This is the $b \to f, a \to g, c \to f', d \to g'$ contraction. This interchanges the roles of $(\alpha, a) \leftrightarrow (\beta, b)$ giving us

$$b \to f, a \to g, c \to f', d \to g' : \qquad -\frac{i}{4} g^2 c^{eba} c^{ecd} g^{\beta\gamma} g^{\alpha\delta} . \tag{566}$$

Again, there are 3 more completely equivalent terms so that we have

$$(b \to f, a \to g, c \to f', d \to g' + \text{equiv.}) : \quad -ig^2 c^{eba} c^{ecd} g^{\beta\gamma} g^{\alpha\delta} . \tag{567}$$

Hopefully, the game is clear now and we can write the complete list without further ado. First, we give the forms obtained by the permutations of the $(\alpha, a), (\beta, b), (\gamma, c), (\delta, d)$ directly in the manner that we have illustrated:

$$\left(a \to f, b \to g, c \to f', d \to g' + \text{equiv.}\right) : \quad -ig^2 c^{eab} c^{ecd} g^{\alpha\gamma} g^{\beta\delta} \tag{568}$$

$$(b \to f, a \to g, c \to f', d \to g' + \text{equiv.}) : \quad -ig^2 c^{eba} c^{ecd} g^{\beta\gamma} g^{\alpha\delta} \tag{569}$$

$$\left(a \to f, c \to g, b \to f', d \to g' + \text{equiv.}\right) : \quad -ig^2 c^{eac} c^{ebd} g^{\alpha\beta} g^{\gamma\delta} \tag{570}$$

$$\left(a \to f, c \to g, d \to f', b \to g' + \text{equiv.}\right) : \quad -ig^2 c^{eac} c^{edb} g^{\alpha\delta} g^{\gamma\beta} \tag{571}$$

$$\left(a \to f, d \to g, c \to f', b \to g' + \text{equiv.}\right) : \quad -ig^2 c^{ead} c^{ecb} g^{\alpha\gamma} g^{\beta\delta} \tag{572}$$

$$\left(a \to f, d \to g, b \to f', c \to g' + \text{equiv.}\right) : \quad -ig^2 c^{ead} c^{ebc} g^{\alpha\beta} g^{\gamma\delta} . \tag{573}$$

Now, we have written these terms in the order such that they can combined pair-wise using the antisymmetry of the c structure constants. Thus, for instance, we can combine the 1st and 2nd term by using $c^{eba} = -c^{eab}$ in the 2nd term, and so forth. In this way, we obtain the net result:

$$\begin{aligned}
\mathcal{M}^{abcd}_{\alpha\beta\gamma\delta}(p,q,r,s) &= -ig^2\Big\{ c^{eab}c^{ecd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\
&\quad + c^{eac}c^{edb}\left(g^{\alpha\delta}g^{\gamma\beta} - g^{\alpha\beta}g^{\gamma\delta}\right) \\
&\quad + c^{ead}c^{ebc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right)\Big\}, \qquad (574)
\end{aligned}$$

for the Lorentz-indexed $\mathcal{M}$ that is contracted with external polarizations or with Lorentz indices of internal gluon propagators. The overall momentum conservation factor of $(2\pi)^4\delta^4(p+q+r+s)$ and the $1/\sqrt{2V\omega}$ factors that are present in $V^{\alpha,a;\beta,b;\gamma,c;\delta,d}(p,q,r,s)$ are explicit multipliers of $\mathcal{M}$ and thus not included above.
Of course, this brute force technique can be avoided by simply starting with the very 1st term form derived and using the fact that the result

must be symmetric under interchange of any two gluons. Thus,

$$-ig^2 c^{eab} c^{ecd} g^{\alpha\gamma} g^{\beta\delta} \tag{575}$$

under interchange $(\alpha, a) \leftrightarrow (\beta, b)$ yields

$$-ig^2 c^{eba} c^{ecd} g^{\beta\gamma} g^{\alpha\delta} = +ig^2 c^{eab} c^{ecd} g^{\beta\gamma} g^{\alpha\delta}\,, \tag{576}$$

the second term in Eq. (574). If we interchange $(\alpha, a) \leftrightarrow (\gamma, c)$ we get

$$-ig^2 c^{eab} c^{ecd} g^{\alpha\gamma} g^{\beta\delta} \rightarrow -ig^2 c^{ecb} c^{ead} g^{\gamma\alpha} g^{\beta\delta} = +ig^2 c^{ebc} c^{ead} g^{\gamma\alpha} g^{\beta\delta}\,, \tag{577}$$

the last term in Eq. (574), and so forth.

5. The fermion-antifermion-gluon interaction

For this, we need to introduce the fermion part of the action. Recall that

$$\mathcal{L}_F = \overline{\psi}^A \left[i\gamma^\mu (D_\mu)_{AB} - m\delta_{AB}\right] \psi^B\,,$$
$$\text{with} \quad (D_\mu)_{AB} = \partial_\mu \delta_{AB} - igL^a_{AB} A^a_\mu\,. \tag{578}$$

(Compare to Bailin-Love (9.15) to again check they have opposite sign of g.) **The L^a matrix is the matrix representing the group for the particular fermion representation being considered. Here, A and B denote the "color" indices of the fermion field. Dirac indices are not explicitly shown above, but are implicitly present in the usual way.**

Going through the usual game of first getting the Z_0 we would end up with the obvious close analogue of the QED result:

$$Z_0(\eta, \overline{\eta}) = \exp\left[-i\int d^4x' d^4y' \overline{\eta}^C_\gamma(x')[S_F(x'-y')]_{\gamma\delta}\delta_{CD}\eta^D_\delta(y')\right]. \tag{579}$$

from which we would compute

$$\langle 0|T\{\psi^B_\beta(w),\overline{\psi}^A_\alpha(z)\}|0\rangle = \left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}^B_\beta(w)}\right)\left(\frac{1}{-i}\frac{\delta}{\delta\eta^A_\alpha(z)}\right)Z_0(\eta,\overline{\eta})\bigg|_{\eta=\overline{\eta}=0}$$
$$= [iS_F(w-z)]_{\beta\alpha}\delta_{BA} \quad (580)$$

We next express the remaining part of $\mathcal{L}_F$ using the functional derivative form:

$$iS_I = i\int d^4x'\left[-ig\left(\frac{1}{-i}\frac{\delta}{\delta\eta_{C\,\gamma}(x')}\right)i\gamma^\nu_{\gamma\delta}L^c_{CD}\left(\frac{1}{i}\frac{\delta}{\delta J^{\nu\,c}(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{D\,\delta}(x')}\right)\right]. \quad (581)$$

where I have switched to dummy indices and integration variable. As usual, we imagine expanding $\exp[iS_I]\sim 1+iS_I$ and using the perturbative approach to evaluating the vertex

$$\langle 0|T\left\{\psi_{A\,\alpha}(x)A^{\mu\,a}(y)\overline{\psi}_{B\,\beta}(z)\right\}|0\rangle$$
$$= \left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{A\,\alpha}(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^a_\mu(y)}\right)\left(\frac{1}{-i}\frac{\delta}{\delta\eta_{B\,\beta}(z)}\right)iS_IZ_0[J]Z_0[\xi,\overline{\xi}]Z_0[\eta,\overline{\eta}]. \quad (582)$$

The procedure for getting a non-zero contribution after setting the sources to 0 is now familiar. The derivatives in S_I must each act on the exponentials in the appropriate Z_0 to bring down a pre-multiplier that has a single remaining source of the corresponding type, and then the 3 functional derivatives defining the time-ordered product act on the remaining single sources in the 3 pre-multipliers.

So, we focus first on

$$\begin{aligned} iS_I Z_0[J]Z_0[\eta,\overline{\eta}] &= i\int d^4x' \left[-ig\left(\frac{1}{-i}\frac{\delta}{\delta\eta_{C\,\gamma}(x')}\right) i\gamma^{\nu}_{\gamma\delta}L^c_{CD}\left(\frac{1}{i}\frac{\delta}{\delta J^{\nu\,c}(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{D\,\delta}(x')}\right)\right] \\ &\quad \times \exp\left\{-\frac{i}{2}\int d^4x' d^4y' J^{a'}_{\mu'}(x')D^{\mu'\nu'}_{a'b'}(x'-y')J^{b'}_{\nu'}(y')\right\} \\ &\quad \times \exp\left[-i\int d^4x'' d^4y'' \overline{\eta}^A(x'')S_F(x''-y'')\delta_{AB}\eta^B(y'')\right] \\ &\ni ig\gamma^{\nu}_{\gamma\delta}L^c_{CD}\int d^4x' \left[(-)^3\int d^4x_1 \overline{\eta}^{A'}_{\gamma'}(x_1)[S_F(x_1-x')]_{\gamma'\gamma}\delta_{A'C}\right] \\ &\quad \times \left\{-\int d^4y_2 D^{\nu''}_{\nu\,cb''}(x'-y_2)J^{b''}_{\nu''}(y_2)\right\}\left[-\int d^4y_1[S_F(x'-y_1)]_{\delta\delta'}\delta_{DB'}\eta^{B'}_{\delta'}(y_1)\right] \\ &\quad \times \exp\{\ldots\}\exp\{\ldots\} \end{aligned} \tag{583}$$

where, we will not be needing the exponentials anymore. So the next stage is:

$$\begin{aligned} &\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{A\,\alpha}(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^a_\mu(y)}\right)\left(\frac{1}{-i}\frac{\delta}{\delta\eta_{B\,\beta}(z)}\right) \\ &\times ig\gamma^{\nu}_{\gamma\delta}L^c_{CD}\int d^4x' \left[(-)^3\int d^4x_1 \overline{\eta}^{A'}_{\gamma'}(x_1)[S_F(x_1-x')]_{\gamma'\gamma}\delta_{A'C}\right] \\ &\times \left\{-\int d^4y_2 D^{\nu''}_{\nu\,cb''}(x'-y_2)J^{b''}_{\nu''}(y_2)\right\}\left[-\int d^4y_1[S_F(x'-y_1)]_{\delta\delta'}\delta_{DB'}\eta^{B'}_{\delta'}(y_1)\right] \end{aligned}$$

$$
\begin{aligned}
&= && ig\gamma^{\nu}_{\gamma\delta}L^{c}_{CD}\int d^4x'\left[\frac{(-)^3}{i}\delta_{AA'}\delta_{\alpha\gamma'}[S_F(x-x')]_{\gamma'\gamma}\delta_{A'C}\right] \\
& && \times\left\{-\frac{1}{i}D^{\mu}_{\nu\,ca}(x'-y)\right\}\left[(\frac{-1}{-i})(-)[S_F(x'-z)]_{\delta\delta'}\delta_{DB'}\delta_{B'B}\delta_{\delta'\beta}\right] \\
&= && ig\gamma^{\nu}_{\gamma\delta}L^{c}_{CD}\int d^4x'\left[i\delta_{AC}[S_F(x-x')]_{\alpha\gamma}\right] \\
& && \times\left\{iD^{\mu}_{\nu}(x'-y)\delta_{ca}\right\}\left[i[S_F(x'-z)]_{\delta\beta}\delta_{DB}\right] \\
&= && ig\int d^4x'\left[[iS_F(x-x')]_{\alpha\gamma}\delta_{AC}\right]\gamma^{\nu}_{\gamma\delta}L^{c}_{CD}\left[[iS_F(x'-z)]_{\delta\beta}\delta_{DB}\right][iD^{\mu}_{\nu\,ca}(x'-y)] \\
&= && ig\int d^4x'\left[\int d^4\tilde{p}'e^{-ip'\cdot(x-x')}\left[S_F(p')\right]_{\alpha\gamma}\delta_{AC}\right]\gamma^{\nu}_{\gamma\delta}L^{c}_{CD} \\
& && \times\left[\int d^4\tilde{p}e^{-ip\cdot(x'-z)}\left[S_F(p)\right]_{\delta\beta}\delta_{DB}\right]\left[\int d^4\tilde{k}e^{ik\cdot(x'-y)}D^{\mu}_{\nu\,ca}(k)\right] \\
&\propto && igL^{c}_{CD}\gamma^{\nu}_{\gamma\delta}(2\pi)^4\delta^4(p'+k-p)\times\text{external propagators and trivial plane wave factors} \\
&\rightarrow && igL^{a}_{AB}\gamma^{\mu}_{\alpha\beta}(2\pi)^4\delta^4(p'+k-p)\quad\text{after standard relabeling .}
\end{aligned}
\tag{584}
$$

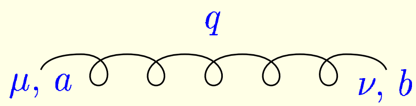
So, our Feynman rule for $\mathcal{M}$ (removing $(2\pi)^4\delta^4(p'+k-p)$ and $1/\sqrt{2VE}$ factors that would appear in the reduction formalism approach) is:

$$
[\mathcal{M}^{\mu\,a}_{AB}]_{\alpha\beta}(p,k,p') = igL^{a}_{AB}\gamma^{\mu}_{\alpha\beta}\,. \tag{585}
$$

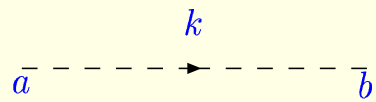
where A,α (B,β) is for the outgoing (incoming) fermion. (Ryder has diagram labeling wrong and wrong sign, I claim.)

- **Summary**

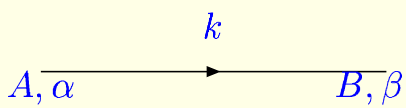
We are now in a position to summarize our results in a graphical manner. We separate our summary into propagators and vertices.



$$\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\delta_{ab}$$

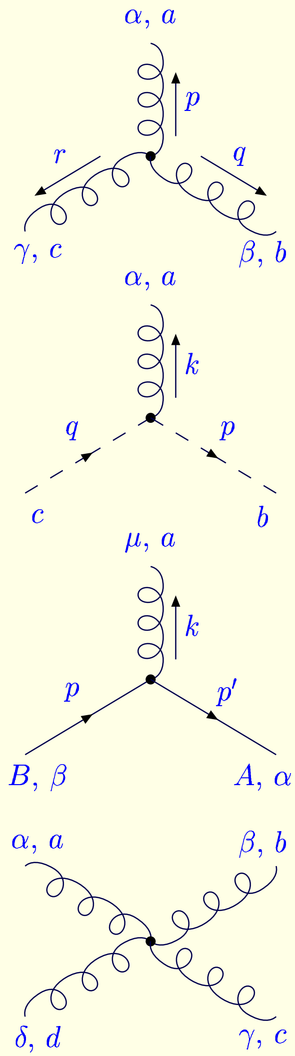


$$\frac{i}{k^2+i\epsilon}\delta_{ab}$$



$$\left[i\frac{\not{k}+m}{k^2-m^2+i\epsilon}\right]_{\beta\alpha}\delta_{BA}$$

Figure 7: **Propagator rules for the gluon, the ghost and a fundamental representation fermion.** $a, b, c = 1, \ldots, (N^2-1)$ **are the gauge group indices (for** $SU(N)$**),** $A, B = 1, \ldots, N$ **are fundamental fermion representation indices. For the gluon propagator, we have written the simple** $\alpha = 1$ **Feynman gauge form: more generally, the propagator contains** $\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}(1-\alpha)\right]$**. In the fermion propagator case,** α, β **are Dirac indices.**



$$gc^{abc}[g_{\gamma\beta}(r-q)_\alpha + g_{\alpha\gamma}(p-r)_\beta + g_{\beta\alpha}(q-p)_\gamma]$$

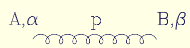
$$gc^{abc}p^\alpha$$

$$igL^a_{AB}\gamma^\mu_{\alpha\beta}$$

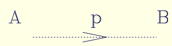
$$\begin{aligned}-ig^2[&c^{eab}c^{ecd}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\\ &+c^{eac}c^{edb}(g_{\alpha\delta}g_{\gamma\beta} - g_{\alpha\beta}g_{\gamma\delta})\\ &+c^{ead}c^{ebc}(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta})]\end{aligned}$$

Figure 8: Feynman vertex rules: 3-gluon, ghost→ghost+gluon, fermion→fermion+gluon and 4-gluon. In the 3rd diagram, α and β are Dirac indices; the line labeled by p' is the outgoing fermion. All vertex rules are for $\mathcal{M}$; they should be implicitly multiplied by a $(2\pi)^4\delta^4$(momentum conservation) when inserted into calculations.

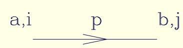
For comparison, I give the Feynman rules in the Bailin-Love conventions, which are also those employed by Keith Ellis (http://theory.fnal.gov/people/ellis/Calctools/tools.html) which, as stated earlier, assume the opposite sign for g. In addition: the 3-gluon vertex is written in terms of *incoming* momenta; their capitalized $A, \ldots$ indices are $a, \ldots$ in my notation and vice versa; they employ the $SU(3)$ specializations of $c^{abc} \to f^{abc}$ and $L^a_{CB} \to t^a_{CB} = \frac{\lambda^a_{CB}}{2}$; their gauge parameter λ corresponds to my α; they use i, j for Dirac indices. After accounting for these convention differences and being very careful with the indices, you can check that the rules below agree with my rules.



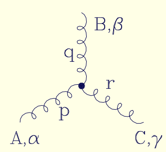
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$



$$\delta^{AB}\frac{i}{(p^2+i\epsilon)}$$

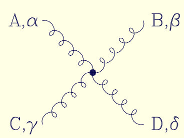


$$\delta^{ab}\frac{i}{(\not p-m+i\epsilon)_{ji}}$$

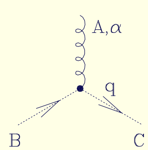


$$-g\ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

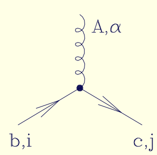
(all momenta incoming, $p+q+r=0$)



$$\begin{aligned}&-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]\\&-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]\\&-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]\end{aligned}$$



$$g\ f^{ABC}q^\alpha$$



$$-ig\ (t^A)_{cb}\ (\gamma^\alpha)_{ji}$$

- **Gauge Invariance Check**

We expect that Feynman amplitudes in QCD should obey Ward identities such as found in QED.

In QED, Ward identities required that the external charged particles (the fermions) be on-shell so that these sources of electromagnetic current interactions corresponded to conserved currents.

In QED, we found that one did not need to have the external photons on-shell, but we noted that this was a special case arising from the fact that the photons themselves do not carry any charge.

In QCD, we expect that we will have to place the external gluons on-shell as well as the external fermions, since in QCD the gluons themselves carry charge.

So, let us verify this in one detailed sample calculation.

We consider gluon-gluon annihilation into a quark and antiquark. There are 3 diagrams: t channel quark exchange; u channel quark exchange; and s channel gluon exchange. These are depicted below, along with the momentum and other labels. All external momenta are defined as flowing out of the diagram. We use a real ϵ basis.

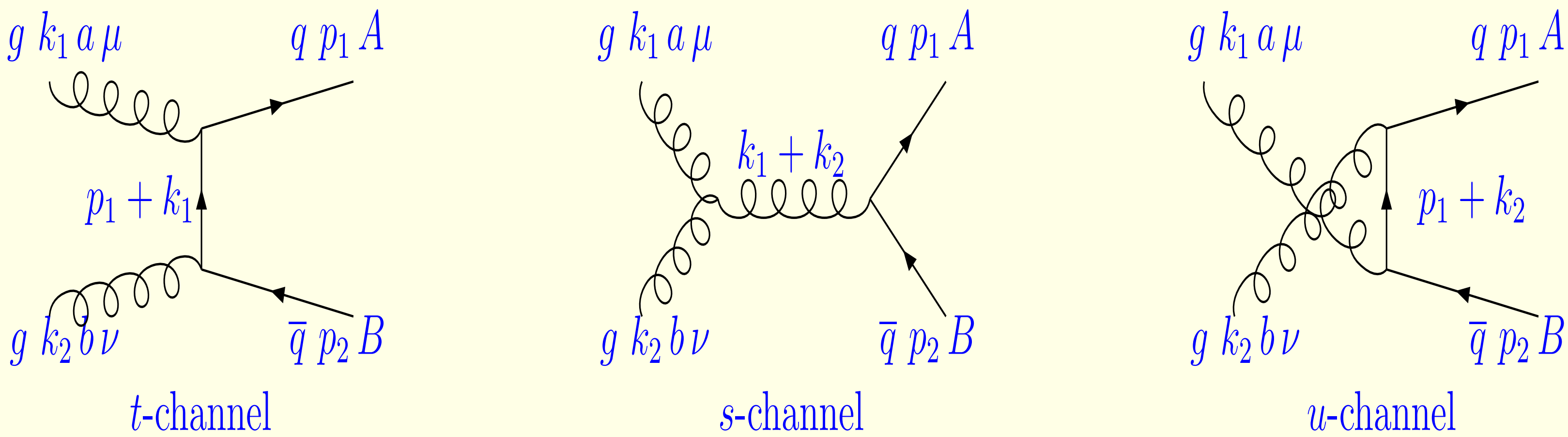


Figure 9: The t, s and u channel Feynman diagrams for $gg \rightarrow q\overline{q}$.

For the two gluons, we take that with momentum k_1 to have color a and Lorentz index μ, and the 2nd to have color b and Lorentz index ν. The outgoing quark has color label A and the outgoing antiquark has color label B. The Feynman amplitudes for the two fermion exchange diagrams are:

$$\begin{aligned} \mathcal{M}^{\mu\nu}_{t+u}\epsilon_\mu(k_1)\epsilon_\nu(k_2) \quad &= \quad (ig)^2\overline{u}_A(p_1)\Bigg(\gamma^\mu L^a_{AC}\frac{i\delta_{CD}}{-\not{p}_2-\not{k}_2-m}\gamma^\nu L^b_{DB} \\ &\quad +\gamma^\nu L^b_{AC}\frac{i\delta_{CD}}{\not{k}_2+\not{p}_1-m}\gamma^\mu L^a_{DB}\Bigg)v_B(p_2)\epsilon_\mu(k_1)\epsilon_\nu(k_2)\,. \end{aligned} \tag{586}$$

where we have hidden the Dirac indices on the spinors. Let us now replace $\epsilon_\nu(k_2)$ by $k_{2\,\nu}$. We obtain (hiding the quark line color indices)

$$
\begin{aligned}
& \mathcal{M}^{\mu\nu}_{t+u}\epsilon_\mu(k_1)k_{2\,\nu} \\
= \quad & (ig)^2\overline{u}(p_1)\left(\gamma^\mu L^a\frac{i}{-\not p_2-\not k_2-m}\not k_2 L^b+\not k_2 L^b\frac{i}{\not k_2+\not p_1-m}\gamma^\mu L^a\right)v(p_2)\epsilon_\mu(k_1) \\
= \quad & (ig)^2\overline{u}(p_1)\left(\gamma^\mu L^a\frac{i}{-\not p_2-\not k_2-m}(\not k_2+\not p_2+m)L^b+(\not k_2+\not p_1-m)L^b\frac{i}{\not k_2+\not p_1-m}\gamma^\mu L^a\right)v(p_2)\epsilon_\mu(k_1) \\
= \quad & (ig)^2\overline{u}(p_1)\left(-i\gamma^\mu[L^a,L^b]\right)v(p_2)\epsilon_{1\,\mu} \\
= \quad & -g^2\overline{u}(p_1)\gamma^\mu L^c v(p_2)\epsilon_{1\,\mu}f^{abc}
\end{aligned}
\tag{587}
$$

where we went from the 2nd line to the 3rd line by using

$$
\overline{u}(p_1)(\not p_1-m)=0\,, \quad \text{and} \quad (\not p_2+m)v(p_2)=0. \tag{588}
$$

The final result is non-zero because the color matrices do not commute, unlike the abelian case where one has (effectively) $[I,I]=0$. It is the gluon exchange diagram that will cancel the above non-zero contribution. It has exactly the right structure as it involves the color structure factor f^{abc} (labeling the exchange gluon with color index c) and also a gluon quark antiquark vertex which will be assigned L^c.

The s-channel diagram takes the form (defining $k_3 = -k_1 - k_2$, also "outgoing" from the 3 gluon vertex)

$$
\begin{aligned}
& \mathcal{M}_s^{\mu\nu}\epsilon_\mu(k_1)\epsilon_\nu(k_2) \\
= \quad & ig\overline{u}(p_1)\gamma^{\rho'}L^c v(p_2)\frac{-ig_{\rho'\rho}}{k_3^2}gf^{acb}\Big[g^{\mu\nu}(k_1-k_2)^\rho + g^{\nu\rho}(k_2-k_3)^\mu \\
& \qquad\qquad +g^{\rho\mu}(k_3-k_1)^\nu\Big]\epsilon_\mu(k_1)\epsilon_\nu(k_2) \\
\stackrel{\epsilon_2\to k_2}{=} \quad & ig\overline{u}(p_1)\gamma_\rho L^c v(p_2)\frac{-i}{k_3^2}gf^{acb}[k_2^\mu(k_1-k_2)^\rho + k_2^\rho(k_2-k_3)^\mu + g^{\rho\mu}(k_3-k_1)\cdot k_2]\epsilon_\mu(k_1) \\
\stackrel{k_2\to -k_3-k_1}{=} \quad & ig\overline{u}(p_1)\gamma_\rho L^c v(p_2)\frac{-i}{k_3^2}gf^{acb}[-g^{\rho\mu}k_3^2 + g^{\rho\mu}k_1^2 + k_3^\rho k_3^\mu + k_1^\rho k_1^\mu]\epsilon_\mu(k_1) \\
\stackrel{k_1\cdot\epsilon_1=0,\ k_1^2=0}{=} \quad & ig\overline{u}(p_1)\gamma_\rho L^c v(p_2)\frac{-i}{k_3^2}gf^{acb}[-g^{\rho\mu}k_3^2 + k_3^\rho k_3^\mu]\epsilon_\mu(k_1) \\
\stackrel{\overline{u}(p_1)\not{k}_3 v(p_2)=0}{=} \quad & ig\overline{u}(p_1)\gamma^\mu L^c v(p_2)(-ig)f^{acb}(-)\epsilon_\mu(k_1) \quad \text{(remember: } k_3 = p_1+p_2\text{)} \\
= \quad & g^2\overline{u}(p_1)\gamma^\mu L^c v(p_2)f^{abc}\epsilon_{1\,\mu}
\end{aligned}
\tag{589}
$$

where we had to be careful to note our clockwise convention for the 3-gluon vertex to write f^{acb} and only in the last line converted to $f^{abc} = -f^{acb}$. Obviously, this result cancels the result from the $t+u$ channel diagrams and we have the appropriate Ward identity. However, you will note that this cancellation depended on:

1. The coupling constant for the quark-antiquark-gluon interaction had to be the same as that for the $\mathbf{3}$-gluon interaction.
   This, of course, came directly from the requirement of invariance under gauge transformations of the 2nd kind which correlates these two couplings according to the minimal substitution rule.
2. We had to use $\epsilon_{1\,\mu}k_1^\mu = 0$ and $k_1^2 = 0$, i.e. we had to keep all the "charged" particles on-shell and physical, that is having transverse polarization.

- In a similar way, we could consider the process $gg \to gg$. There are s, t, and u channel g-exchange diagrams, and *in addition*, a diagram with all four external gluons coming together at a single point (the quartic interaction vertex term).

  If we tried to check for the Ward identity without the "contact" quartic interaction diagram, we would have failed. By including the contact interaction diagram, the Ward identity is satisfied, *provided the g appearing there is the same as the g in the $\mathbf{3}$-gluon interaction vertices*.

  Thus, once again the full gauge invariance under gauge transformations of the 2nd kind is required for a sensible theory with Ward identities, which, in turn, are crucial for a renormalizable theory.

- **The role of ghosts**

In proving the Ward identity above, we had to assume that the 2nd gauge boson was transverse ($k_1 \cdot \epsilon_1 = 0$).

Should we have expected that this would come out of the argument rather than having to be an input? In QED, the Feynman diagrams predict that the photons produced by e^+e^- annihilation are transverse. Amplitudes for producing the two other polarization states cancelled one another. Recall:

If we go to a Lorentz frame where $k^\mu = (k, 0, 0, k)$ has only a direction 3 vector component, then, in the Lorentz gauge we know that $\epsilon_1(\vec{k}) = (0, 1, 0, 0)$ and $\epsilon_2(\vec{k}) = (0, 0, 1, 0)$, *i.e.* $\epsilon_1^1 = 1$ and $\epsilon_2^2 = 1$, all others zero. So,

$$X = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 \,, \tag{590}$$

where the subscripts are the Lorentz indices. The Ward identity reduces to

$$k(\mathcal{M}_0 + \mathcal{M}_3) = 0 \,, \quad \Rightarrow \quad \mathcal{M}_0 = -\mathcal{M}_3 \,. \tag{591}$$

This is the cancellation of which I speak. It meant that we could equally well write

$$X = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2 - |\mathcal{M}_0|^2 = -g^{\mu\nu}\mathcal{M}_\mu \mathcal{M}_\nu^* = -\mathcal{M}^\nu \mathcal{M}_\nu^* . \tag{592}$$

So, effectively, we could use the replacement

$$\sum_{r=1,2} \epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k}) \to -g^{\mu\nu} . \tag{593}$$

All of this fails in the case of QCD!

This is related to the question of how the optical theorem works in QCD. The optical theorem says that, for example, if you have $q\overline{q} \to q\overline{q}$ at one loop with intermediate gg states mediating the loop, then by taking the imaginary part of this diagram you should get something proportional to the amplitude-squared for $q\overline{q} \to gg$ at tree-level. Without including the ghost-ghost intermediate states, taking the imaginary part gives you something that includes contributions to $q\overline{q} \to q\overline{q}$ corresponding to unphysical intermediate gg polarization states. The role of the ghost-loop intermediaries is to cancel away these unphysical polarization states so that the optical theorem (a statement of probability conservation) is

obeyed.

General Derivation of Optical Theorem

Let us write the S matrix in the form $S = 1 + iT$, in which case unitarity $S^\dagger S = 1$ takes the form

$$-i(T - T^\dagger) = T^\dagger T\,. \tag{594}$$

Take the matrix element between two-particle states, $\langle p_1 p_2|$ and $|k_1 k_2\rangle$, and use the definition of $\mathcal{M}$ that has an extra i compared to what we have been employing (which is more convenient for the moment):

$$\langle p_1 p_2 | iT | k_1 k_2 \rangle = \frac{1}{\sqrt{2VE_{\vec{p}_1}}}\frac{1}{\sqrt{2VE_{\vec{p}_2}}}\frac{1}{\sqrt{2VE_{\vec{k}_1}}}\frac{1}{\sqrt{2VE_{\vec{k}_2}}}(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)\, i\widetilde{\mathcal{M}}(k_1 k_2 \to p_1 p_2)\,. \tag{595}$$

The left hand side of Eq. (594) becomes

$$\frac{1}{\sqrt{2VE_{\vec{p}_1}}}\frac{1}{\sqrt{2VE_{\vec{p}_2}}}\frac{1}{\sqrt{2VE_{\vec{k}_1}}}\frac{1}{\sqrt{2VE_{\vec{k}_2}}}(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)$$
$$\times \left[-i\widetilde{\mathcal{M}}(k_1 k_2 \to p_1 p_2) + i\widetilde{\mathcal{M}}^*(p_1 p_2 \to k_1 k_2)\right]\,. \tag{596}$$

Meanwhile, on the right hand side, we insert a complete set of states labeled by momentum states $\{q_i\}$ and find

$$\begin{aligned}
\langle p_1 p_2|T^\dagger T|k_1 k_2\rangle &= \sum_n \left(\prod_{i=1}^n \int V \frac{d^3 q_i}{(2\pi)^3}\right) \langle p_1 p_2|T^\dagger|\{q_i\}\rangle\langle\{q_i\}|T|k_1 k_2\rangle \\
&= (2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum_n \left(\prod_{i=1}^n \int V \frac{d^3 q_i}{(2\pi)^3}\right) \\
&\quad \times \frac{1}{\sqrt{2VE_{\vec{p}_1}}}\frac{1}{\sqrt{2VE_{\vec{p}_2}}}\frac{1}{\sqrt{2VE_{\vec{k}_1}}}\frac{1}{\sqrt{2VE_{\vec{k}_2}}}\left(\prod_{i=1}^n \frac{1}{\sqrt{2VE_{\vec{q}_i}}}\right)^2 \\
&\quad \times \widetilde{\mathcal{M}}^*(p_1 p_2 \to \{q_i\})\widetilde{\mathcal{M}}(k_1 k_2 \to \{q_i\})(2\pi)^4\delta^4(k_1+k_2-\sum_i q_i) \qquad (597)
\end{aligned}$$

where we used the relationship between T and $\widetilde{\mathcal{M}}$. Equating the two sides of our starting equation and removing the overall $(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$ gives

$$\begin{aligned}
&\left[-i\widetilde{\mathcal{M}}(k_1 k_2 \to p_1 p_2) + i\widetilde{\mathcal{M}}^*(p_1 p_2 \to k_1 k_2)\right] \\
&= \sum_n \left(\prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3}\frac{1}{2E_i}\right) \\
&\qquad \times \widetilde{\mathcal{M}}^*(p_1 p_2 \to \{q_i\})\widetilde{\mathcal{M}}(k_1 k_2 \to \{q_i\})(2\pi)^4\delta^4(k_1+k_2-\sum_i q_i)\,.
\end{aligned}$$

$$(598)$$

Let us now take the forward case of $p_1 = k_1$ and $p_2 = k_2$. In this case, the above relation can be abbreviated as

$$
\begin{aligned}
2\mathrm{Im}\widetilde{\mathcal{M}}(k_1k_2 \to k_1k_2) &= \sum_f \int d\Pi_f \widetilde{\mathcal{M}}^*(k_1k_2 \to f)\widetilde{\mathcal{M}}(k_1k_2 \to f) \\
&= 4E_{cm}p_{cm}\sigma_{\rm tot}(k_1k_2 \to \text{anything}) \\
&\xrightarrow{\text{small masses}} 2s\sigma_{\rm tot}(k_1k_2 \to \text{anything})\,. \qquad (599)
\end{aligned}
$$

Back to the role of ghosts

We can elucidate some of the words describing the role of ghosts in satisfying unitarity constraints by doing a little calculation. Consider $k^\mu = (k^0, \vec{k})$ with $k^2 = 0$. There are two purely spatial vectors orthogonal to $\vec{k}$ which define the transverse polarizations for a vector boson of momentum k. Our normal approach to completing the basis set is to include the longitudinal polarization state, with polarization parallel to $\vec{k}$, and the timelike polarization state. Instead, let us employ two lightlike linear combinations of these latter states with polarization vectors parallel to the vectors k^μ and $\widetilde{k}^\mu = (k^0, -\vec{k})$:

$$\epsilon^+_\mu(k) = \left(\frac{k^0}{\sqrt{2}|\vec{k}|}, \frac{\vec{k}}{\sqrt{2}|\vec{k}|} \right) \,, \quad \epsilon^-_\mu(k) = \left(\frac{k^0}{\sqrt{2}|\vec{k}|}, -\frac{\vec{k}}{\sqrt{2}|\vec{k}|} \right) . \tag{600}$$

We call these the forward and backward lightlike polarization vectors. Together with the transverse states $\epsilon^T_\mu(k)$, $i = 1, 2$, we have a complete basis with

$$\epsilon^T_i \cdot \epsilon^{*\,T}_j = -\delta_{ij}, \quad \epsilon^+ \cdot \epsilon^T_i = \epsilon^- \cdot \epsilon^T_i = 0, \quad \epsilon^+ \cdot \epsilon^+ = \epsilon^- \cdot \epsilon^- = 0, \quad \epsilon^+ \cdot \epsilon^- = 1 . \tag{601}$$

They also satisfy the completeness relation

$$g_{\mu\nu} = \epsilon^-_\mu \epsilon^{+\,*}_\nu + \epsilon^+_\mu \epsilon^{-\,*}_\nu - \sum_{i=1,2} \epsilon^T_{i\,\mu} \epsilon^{T\,*}_{i\,\nu} . \tag{602}$$

Let us now convert our earlier $gg \to q\overline{q}$ calculation to $q\overline{q} \to gg$ (which just means we need to replace $\overline{u}$ by $\overline{v}$ and v by u with appropriate momentum arguments, p_1 for the incoming $\overline{q}$ and p_2 for the incoming q) and compute the amplitude for outgoing gluons, which we shall call $i\widetilde{\mathcal{M}}'$ (the amplitude for incoming gluons being in our new notation $i\widetilde{\mathcal{M}}$).

Let us in particular consider the case where the outgoing gluons have the polarizations $\epsilon^-(k_1)$ and $\epsilon^+(k_2)$. Since $\epsilon^{+\,*}_{\nu}(k_2) = \frac{k_{2\,\nu}}{\sqrt{2}|\vec{k}_2|}$, we will get the same formulae (aside from above spinor replacements and an extra factor of $\frac{1}{\sqrt{2}|\vec{k}_2|}$) as obtained after the gauge invariance game for the $t+u$ channel diagrams as before and it is only $i\widetilde{\mathcal{M}}'^{\mu\nu}_s \epsilon^{-\,*}_{\mu}(k_1)\epsilon^{+\,*}_{\nu}(k_2)$ that we must reexamine. For this we obtain the same piece as before (with extra $\frac{1}{\sqrt{2}|\vec{k}_2|}$) that cancels the $t+u$ diagrams, plus a piece that for transverse polarizations vanished, but now no longer vanishes. This part arises since $\epsilon^{-\,*}(k_1)\cdot k_1 \neq 0$. It is obtained from Eq. (589) as shown below:

$$
\begin{aligned}
i\widetilde{\mathcal{M}}'^{\mu\nu}_s \epsilon^{-\,*}_{\mu}(k_1)\epsilon^{+\,*}_{\nu}(k_2) \quad &\ni \quad ig\overline{v}(p_1)\gamma_\rho L^c u(p_2)\frac{-i}{k_3^2} g f^{acb} \frac{1}{\sqrt{2}|\vec{k}_2|}[k_1^\rho k_1^\mu]\epsilon^{-\,*}_{\mu}(k_1) \\
&= \quad ig\overline{v}(p_1)\gamma_\rho L^c u(p_2)\frac{-i}{k_3^2} g f^{acb} k_1^\rho \frac{|\vec{k}_1|}{|\vec{k}_2|}\,. \qquad (603)
\end{aligned}
$$

In the center of mass, we can set $|\vec{k}_1| = |\vec{k}_2|$. I will implicitly use this frame in what follows. (I have not quite figured out how the ensuing calculations work in some other frame.)

So, it appears that there is a non-zero amplitude for the unphysical process of making two gluons with these lightlike polarizations.

Of course, for tree-level processes we can just simply say that the only external physical gluon states we allow are those with transverse polarizations.

But, these unphysical polarizations have the potential for getting us into trouble when we go to loop diagrams and try to apply the optical theorem. Why is this?

According to the optical theorem, two times the imaginary part of $\widetilde{\mathcal{M}}(q\overline{q} \to (1 - loop) \to q\overline{q})$ coming from the indicated diagrams and computed in the forward momentum configuration (i.e. initial momenta equal to final momenta) is equal to the absolute square of the $q\overline{q} \to gg$ amplitude integrated over the gg final state phase space and keeping only physical intermediate 2-particle states — i.e. transversely polarized gluons. The proof appeared above. This does not work if we consider only the gluon loop!

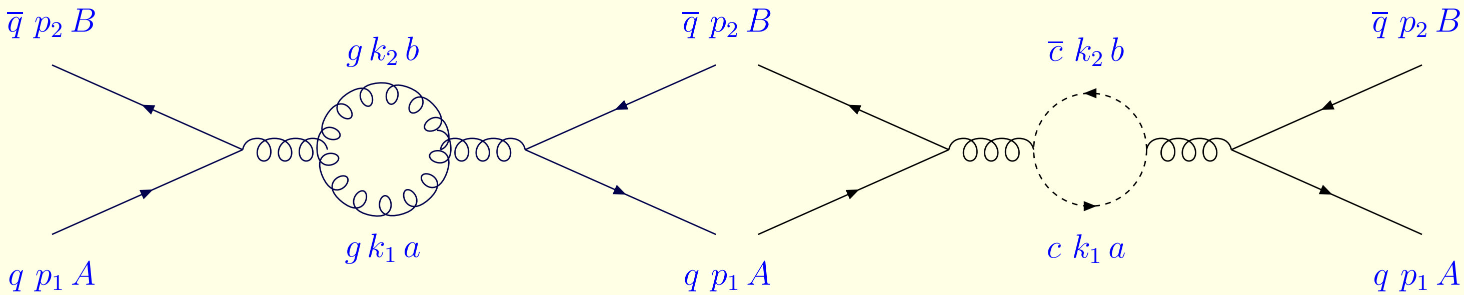


Figure 10: The gluon-loop with two s-channel exchanges and the ghost loop. In the gluon-loop case, where we show an s-channel exchange, one should actually sum over the s, t and u-channel diagrams.

For the gluon loop, if we replace the gluon propagator $g_{\mu\nu}$'s by the sum over four polarizations, Eq. (602), it appears that all the polarizations, including the unphysical ones, should be included for the gauge bosons in the sum over the $q\overline{q} \to gg$ amplitudes absolute squared. It is the $q\overline{q} \to (ghost - loop) \to q\overline{q}$ diagram that must be added to the one-loop $q\overline{q} \to (gluon - loop) \to q\overline{q}$ amplitude computation. The ghost loop cancels off the loop contributions coming from the unphysical polarizations hidden in the $g_{\mu\nu}$'s and only transverse polarizations for the final state g's need be included in summing over the $q\overline{q} \to gg$ squared amplitudes.

So, let us show this in detail. We write the amplitude for $q\overline{q} \to gg$ in the form

$$i\widetilde{\mathcal{M}}'^{\,\mu\nu}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,. \tag{604}$$

The corresponding amplitude for $gg \to q\overline{q}$ is that which we have already written down and is of the form

$$i\widetilde{\mathcal{M}}^{\mu\nu}\epsilon_\mu(k_1)\epsilon_\nu(k_2)\,, \tag{605}$$

where, of course, the k_1 and k_2 are the same since we are examining the situation where the initial state of the $gg \to q\overline{q}$ process is the same as the final state of the $q\overline{q} \to gg$ process.

The Cutkosky rules state that[4] $\mathrm{Disc}\widetilde{\mathcal{M}} = 2i\mathrm{Im}\,\widetilde{\mathcal{M}}$ for the $q\overline{q} \to (gluon-loop) \to q\overline{q}$ diagram is obtained by replacing the "cut" gauge boson propagators with momentum k_i by

$$-ig_{\mu\nu}(-2\pi i)\delta^+(k_i^2)\,, \tag{606}$$

where the δ^+ means "keep only the positive energy solution" (i.e. $\theta(k_i^0)$)

[4]Here, you should think of $\widetilde{\mathcal{M}}$ as an analytic function with a branch cut across which there is a discontinuity. The branch cut begins when the incoming cm energy is sufficient for the process to take place. A sketchy proof of the Cutkosky rules will be given when renormalization is discussed.

and where the $-ig_{\mu\nu}$ is simply the numerator part of the propagator. I will not keep writing the $^+$, but you should assume it is implicitly there. The $(-2\pi i)\delta(k_i^2)$ can sort of be thought of as the discontinuity associated with the propagator obtained by replacing $+i\epsilon$ by $-i\epsilon$. We make this replacement for both the k_1 and k_2 gluon propagators, thereby turning the two 4-dimensional loop integrals into two 3-dimensional integrals as below:

$$
\begin{aligned}
& \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^4(P-k_1-k_2)(-2\pi i)^2\delta(k_1^2)\delta(k_2^2) \\
= \quad & (-i)^2\int \frac{d^3\vec{k}_1}{(2\pi)^3 2|k_1|}\frac{d^3\vec{k}_2}{(2\pi)^3 2|k_2|}(2\pi)^4\delta^4(P-k_1-k_2) \\
= \quad & (-i)^2\int d\Pi \qquad (607)
\end{aligned}
$$

which is, aside from the $(-i)^2$, exactly the form of the final state phase space for the gg final state of $q\overline{q}\to gg$.

So, the net result for $2i\mathrm{Im}\widetilde{\mathcal{M}}(q\overline{q}\to(gluon-loop)\to q\overline{q})$ coming from the diagram is

$$\int d\Pi \frac{1}{2}(i\widetilde{\mathcal{M}}'^{\,\mu\nu}) g_{\mu\rho} g_{\nu\sigma} (i\widetilde{\mathcal{M}}^{\rho\sigma}) , \tag{608}$$

where the $\frac{1}{2}$ is the appropriate "symmetry" factor for the gluon-loop.

Now we introduce the representation Eq. (602) for the two metric tensors.

1. The pieces that have only transverse polarizations correspond to the expected imaginary part associated with $q\bar{q} \to gg$ with transverse polarizations for the gluons.
2. The cross terms between a transverse substitution for one metric tensor and a lightlike substitution for the other vanish. Just follow the Ward identity proof noting that one of the gluons just has transverse polarizations — we showed precisely that (again, we are using $i\widetilde{\mathcal{M}}$ in place of $\mathcal{M}$ for this material):

$$i\widetilde{\mathcal{M}}^{\mu\nu} \epsilon_\mu^{T\,*}(k_1) \epsilon_\nu^{+\,*}(k_2) = 0 \tag{609}$$

since $\epsilon_\nu^{+\,*}(k_2) \propto k_{2\,\nu}$. Precisely the same result applies also for $i\widetilde{\mathcal{M}}'$.[5]

[5]This material is similar to that in Peskin's Chapt. 16, but at this point I start disagreeing with him. In particular, it is always $\epsilon_\mu^+(k)$ that is proportional to k_μ, regardless of whether we are talking about k_1 or k_2.

Similarly, if we keep the k_2 gluon transverse and use $\epsilon_\nu^{+\,*}(k_1) \propto k_{1\,\nu}$, we will get zero. Since the non-transverse part of the $g_{\mu\nu}$ replacement always includes at least one $\epsilon^+(k_1)$ or one $\epsilon^+(k_2)$, we will get zero for all such cross terms.

3. If we make lightlike substitutions for both metric tensors then four different types of contributions arise. We must be careful. We write (keeping only the lightlike parts and dropping the star notation since actually everything is real)

$$g_{\mu\rho}g_{\nu\sigma} \ni (\epsilon_\mu^-(k_1)\epsilon_\rho^+(k_1) + \epsilon_\mu^+(k_1)\epsilon_\rho^-(k_1))(\epsilon_\nu^-(k_2)\epsilon_\sigma^+(k_2) + \epsilon_\nu^+(k_2)\epsilon_\sigma^-(k_2)) \quad (610)$$

We saw that if we have $\epsilon_\mu^-(k_1)\epsilon_\nu^+(k_2)$ then we have a non-zero residual. The same applies if we have $\epsilon_\mu^+(k_1)\epsilon_\nu^-(k_2)$. And also for the similar structures in the $\rho\sigma$ indices: $\epsilon_\rho^-(k_1)\epsilon_\sigma^+(k_2)$ and $\epsilon_\rho^+(k_1)\epsilon_\sigma^-(k_2)$ will give non-zero residuals.

If we have $\epsilon_\mu^+(k_1)\epsilon_\nu^+(k_2)$, then the residual will be proportional to $k_1^2 = 0$, so such a term vanishes. Similarly, $\epsilon_\mu^-(k_1)\epsilon_\nu^-(k_2)$ will be proportional to $k_2^2 = 0$.

Thus, the only cross terms that give a (dangerous) non-zero result are:

$$\epsilon_\mu^-(k_1)\epsilon_\nu^+(k_2)\epsilon_\rho^-(k_1)\epsilon_\sigma^+(k_2) + \epsilon_\mu^+(k_1)\epsilon_\nu^-(k_2)\epsilon_\rho^+(k_1)\epsilon_\sigma^-(k_2) \quad (611)$$

4. The structure that arises from the above can be read off of Eq. (603). We get

$$\frac{1}{2}\left(ig\overline{v}(p_2)\gamma_\rho L^c u(p_1)\frac{i}{k_3^2}gf^{abc}k_1^\rho\right)\left(ig\overline{u}(p_1)\gamma_{\rho'} L^{c'} v(p_2)\frac{i}{k_3^2}gf^{abc'}k_1^{\rho'}\right)+(k_1\leftrightarrow k_2)\,. \tag{612}$$

Now, the two terms above are actually equal since

$$\overline{v}(p_2)\gamma_\rho(k_1+k_2)^\rho u(p_1)=\overline{v}(p_2)\gamma_\rho(p_1+p_2)^\rho u(p_1)=0\,, \tag{613}$$

where we used the Dirac equations for the spinors.

5. So, what is it that cancels this stuff? It is, of course, the diagram corresponding to $q\overline{q}\to(ghost-loop)\to q\overline{q}$. We take the imaginary part of this in the same way so that each propagator is replaced by $i(-2\pi i)\delta(k_i^2)$, and convert to the phase space form. Using the ghost-ghost-gluon vertex, we find the result for $q\overline{q}\to c(k_1)+\overline{c}(k_2)$:

$$i\widetilde{\mathcal{M}}'_{ghost}=ig\overline{v}(p_2)\gamma_\rho L^c u(p_1)\frac{i}{k_3^2}gf^{abc}k_1^\rho \tag{614}$$

for the $q\overline{q} \to gg$ direction and

$$i\widetilde{\mathcal{M}}_{ghost} = ig\overline{u}(p_1)\gamma_{\rho'}L^{c'}v(p_2)\frac{i}{k_3^2}gf^{ac'b}k_2^{\rho'} \tag{615}$$

for the $gg \to q\overline{q}$ direction. In the latter term, we replace $k_2^{\rho'} \to (p_1 + p_2 - k_1)^{\rho'}$, use the fact that the $(p_1 + p_2)^{\rho'}$ portion gives zero by virtue of the Dirac equations for $v(p_2)$ and $\overline{u}(p_1)$ and use $f^{ac'b} = -f^{abc'}$. The product $(i\widetilde{\mathcal{M}}'_{ghost})(i\widetilde{\mathcal{M}}_{ghost})$ is then of exactly the same form as the first of the two gluon-loop terms in Eq. (612); the factor of $\frac{1}{2}$ in the latter is compensated by the fact that the 2nd product term simply doubles the first product term. It would seem that the ghost loop simply doubles the net result from the gluon loop? Is there some reason for which they cancel? The answer is **YES**. They cancel because we still must include the $(-)$ sign for the ghost loop.

So, we have verified that the Faddeev-Popov ghosts serve to cancel the timelike and longitudinal polarization states of the gauge bosons in the context of situations (such as unitarity) where the gauge bosons of internal loops are placed on shell.

- **Separating Color Algebra from Dirac/Feynman Algebra**

We will now return to $\mathcal{M}$ in the following.

It will be convenient to separate off some additional i's into the color algebra (as opposed to the momentum, Dirac, Lorentz algebra) by writing

$$\begin{aligned} gc^{abc} &= (-ig)(ic^{abc}) \quad \text{for the 3-g and gh-gh-}g\text{ vertex} \\ -ig^2c^{eab}c^{ecd} &= (ig^2)(ic^{eab})(ic^{ecd}) \quad \text{for the 4-g vertex}\,. \end{aligned}$$

Our Feynman rules for $\mathcal{M}$ will then take the factorized form:

1. $(-)$ sign for closed fermion or ghost loops;
2. $\int \frac{d^4k_i}{(2\pi)^4}$ for each *independent* loop i;
3. Other internal loop momenta are determined by momentum conservation at each vertex;
4. Symmetry factors, such as a factor of $1/2$ for gluon loop insertion into gluon propagator;
5. Times, the following graphical rules.

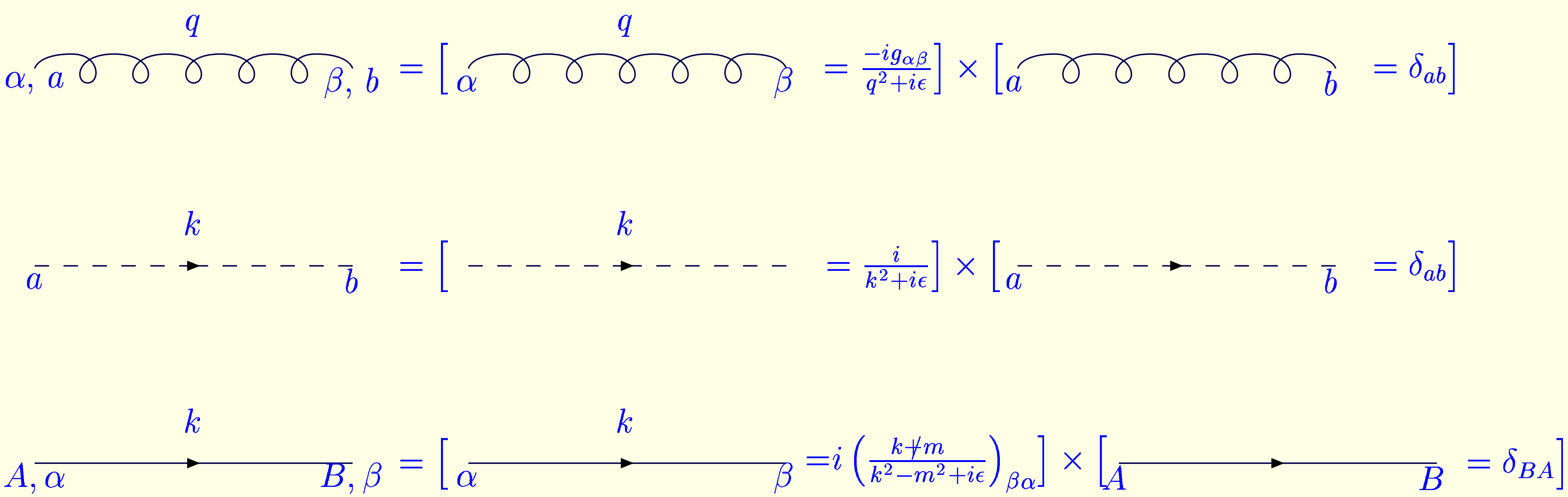
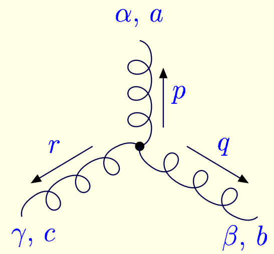
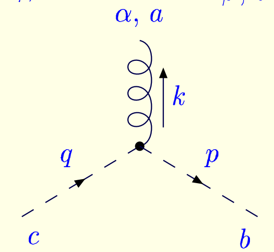


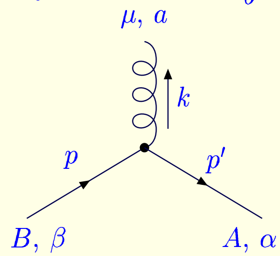
Figure 11: Propagator rules for the gluon, the ghost and a fundamental representation fermion in the Feynman/color factorized form.



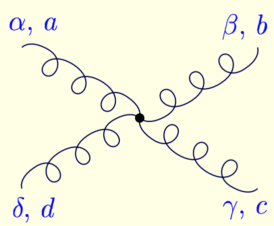
$$-ig[g_{\gamma\beta}(r-q)_\alpha + g_{\alpha\gamma}(p-r)_\beta + g_{\beta\alpha}(q-p)_\gamma] \times (ic^{abc})$$



$$-igp^\alpha \times (ic^{abc})$$

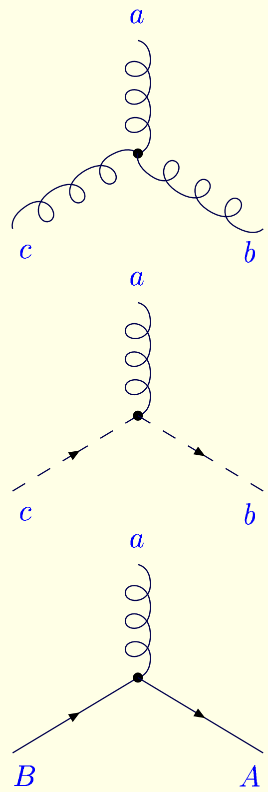


$$ig\gamma^\mu_{\alpha\beta} \times (L^a_{AB})$$



$$\begin{aligned} ig^2[&(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \times (ic^{abe})(ic^{ecd}) \\ &+(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\gamma\beta}) \times (ic^{ace})(ic^{ebd}) \\ &+(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\beta}g_{\gamma\delta}) \times (ic^{ade})(ic^{ecb})] \end{aligned}$$

Figure 12: **Factorized forms for Feynman vertex rules: 3-gluon, ghost→ghost+gluon, fermion→fermion+gluon and 4-gluon. We will draw a separate Feynman/momentum diagram times a color diagram for each of the expressions given; in particular, the 4-gluon vertex will be reduced to the sum of three pieces, each represented in this factorized form. Note that I have set up a certain clock-wise convention for the ic^{abc} factors; in order to do this for the three 4-gluon terms, I changed the 2nd and 3rd metric tensor combination signs.**

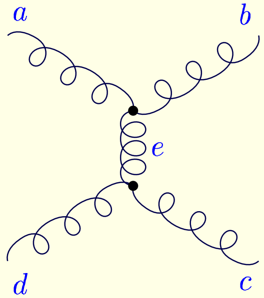


(ic^{abc})

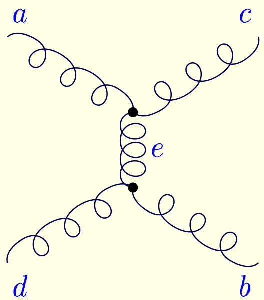
(ic^{abc})

(L^a_{AB})

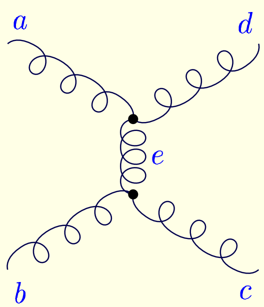
Figure 13: The color diagrams for the 3-gluon, ghost$\rightarrow$ghost+gluon, fermion$\rightarrow$fermion+gluon diagrams.



$$(ic^{abe})(ic^{ecd})$$



$$(ic^{ace})(ic^{ebd})$$



$$(ic^{ade})(ic^{ecb})$$

Figure 14: The color diagrams for the three subdiagrams coming from the 4-gluon interaction. They have an s-, u- and t-channel type topology, respectively, from the perspective of the original diagram.

Color Algebra Techniques

- We now need to learn a bit better how to deal with color algebra. There are many different techniques. Here, I will discuss a technique developed by Cvitanovic in PRD14, p1536.

  It is a diagrammatically based technique, that once mastered allows extremely rapid computation of color factors. We will focus on the application to $SU(N)$ groups, and in particular our notation will be adapted for QCD $SU(3)$.

  Cvitanovic uses slightly different conventions regarding index names than those I have employed above. In his notation $a \to i$ for the $L^i = \frac{\lambda^i}{2}$, and $A \to a$ for the fundamental fermion representation color index. I will try to stick to our earlier notation.

  So, we have

$$[L^a, L^b] = i \sum_{c'} c^{abc'} L^{c'} \tag{616}$$

$$\mathrm{Tr}[L^a] = 0 \tag{617}$$

$$\mathrm{Tr}[L^a L^b] = \frac{1}{2}\delta^{ab} \tag{618}$$

$$ic^{abc} = 2\mathrm{Tr}[L^c[L^a, L^b]] = 2\mathrm{Tr}[L^a L^b L^c - L^c L^b L^a] \tag{619}$$

$$2\sum_a (L^a)_{BA}(L^a)_{DC} = \delta_{DA}\delta_{BC} - \frac{1}{N}\delta_{BA}\delta_{DC}\,, \tag{620}$$

where the latter applies in the conventional representations of the L^a's, [e.g. the $\vec{\tau}/2$ matrices for $SU(2)$ or the $\vec{\lambda}/2$ matrices for SU(3), as you can check] and agrees with the trace condition since we have from (618) that $\mathrm{Tr}[L^a L^a] = \frac{1}{2}\delta^{aa} = \frac{N^2-1}{2}$ whereas (620) implies

$$2\mathrm{Tr}[L^a L^a] = \delta_{AA}\delta_{BB} - \frac{1}{N}\delta_{AA} = N^2 - 1\,. \tag{621}$$

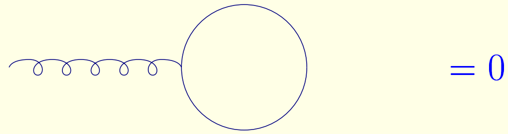
The $-\frac{1}{N}$ term in (620) is simply keeping track of the traceless requirement for the generators of $SU(N)$. Finally, we note that

$$L^c_{BA} = -L^c_{AB} \tag{622}$$

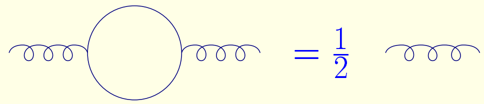
in a real representation basis (fundamental representations are not real in general).

- **We now convert the above identities into diagrammatic equivalent forms using our previously established notation.**

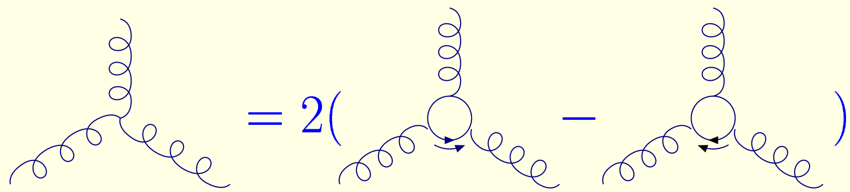
Eq. (617) becomes



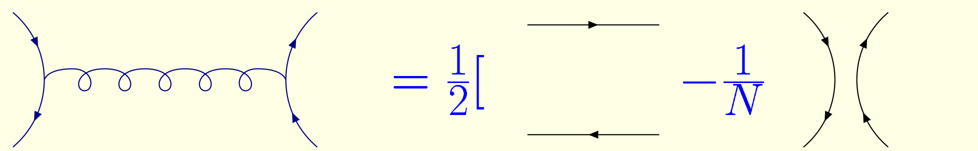
Eq. (618) becomes



Eq. (619) becomes (note arrows on loops)

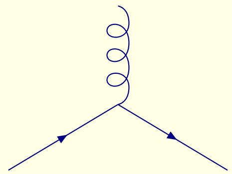


Eq. (620) becomes (note arrows)

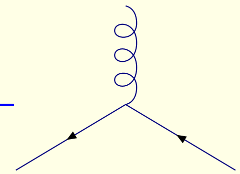


In words, this last result says (in QCD) that a gluon is almost a quark-antiquark combination except that the singlet quark-antiquark combination is subtracted. Finally,

Eq. (622) becomes (note arrow reversal)



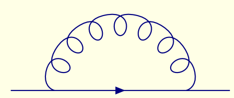
$= -$



Let us give a simple, yet important application of these graphical rules: the computation of the fundamental representation Casimir, C_F. This is defined algebraically by

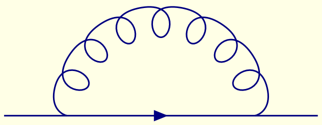
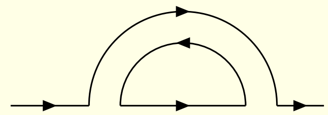
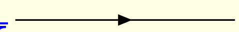
$$\sum_d L^d_{BC} L^d_{CA} = C_F \delta_{BA} \tag{623}$$

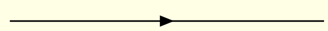
Graphically, this is represented by:



$= \quad C_F \times$

The computation is a simple application of Eq. (620). Graphically we have:

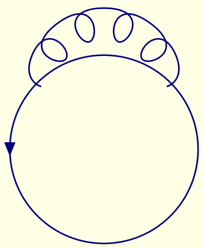
 $= \frac{1}{2}\Big[$  $- \frac{1}{N}$  $\Big]$

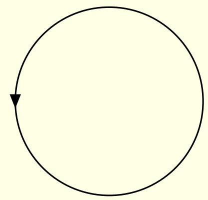
$= \frac{1}{2}\left[N - \frac{1}{N}\right]$ 

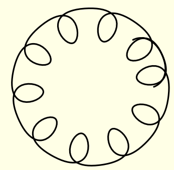
This implies that

$$C_F = \frac{N^2-1}{2N} = \frac{4}{3} \quad \text{for } N = 3 \text{ (QCD).} \tag{624}$$

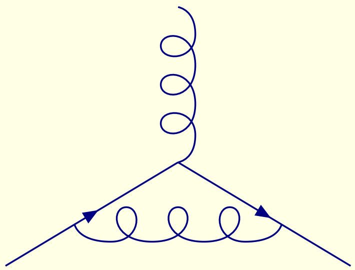
This same result can be obtained by a different graphical procedure that makes use of Eq. (618) as shown below.



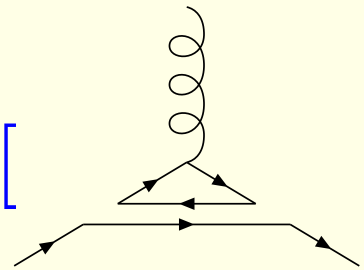
$\equiv C_F$  $= C_F N$

$= \frac{1}{2}$  $= \frac{1}{2}(N^2 - 1)$

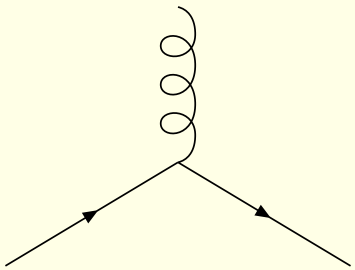
Another little example that illustrates how we will use these color computations when it comes to analyzing the effects of loops is the following result.



$$= \frac{1}{2}\Big[$$

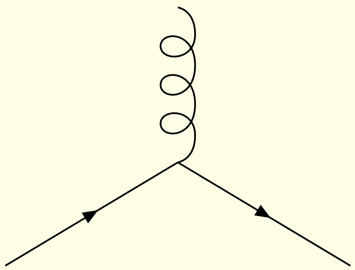


$$- \frac{1}{N}$$



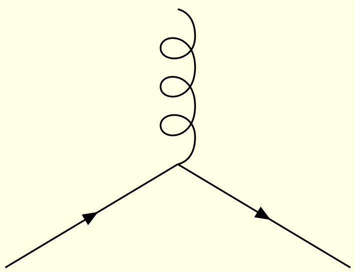
$$\Big]$$

$$= \frac{1}{2}\Big[0 - \frac{1}{N}$$



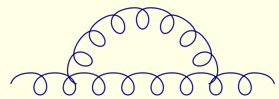
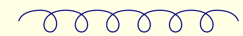
$$\Big]$$

$$= -\frac{1}{2N}$$

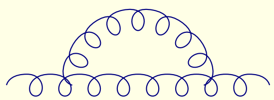
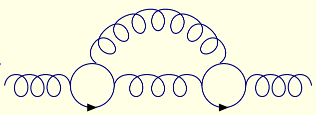
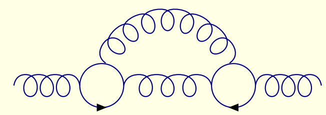


The other, and perhaps most important, Casimir operator is that for the adjoint representation, i.e. the gluons themselves: $(ic^{ade})(ic^{dbe}) = C_A\delta^{ba}$.

Graphically, this is represented by:

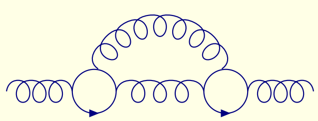
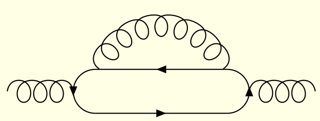
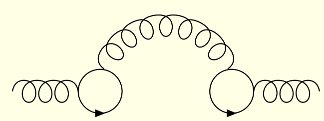
 $= \quad C_A\times$ 

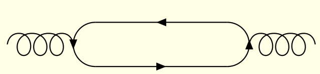
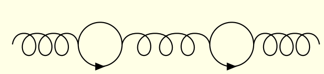
The computation is an application of Eq. (619). Graphically we have:

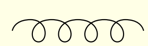
 $= (2^2)(2)[$  $-$  $]$

where the $(2)^2$ **comes from using Eq. (619) twice, and the single** (2) **comes from the fact that doing so actually yields** 4 **terms containing two pairs of identical terms. Note: the arrows on the loops are in the same direction for the 1st term and are in opposite directions for the 2nd term.**

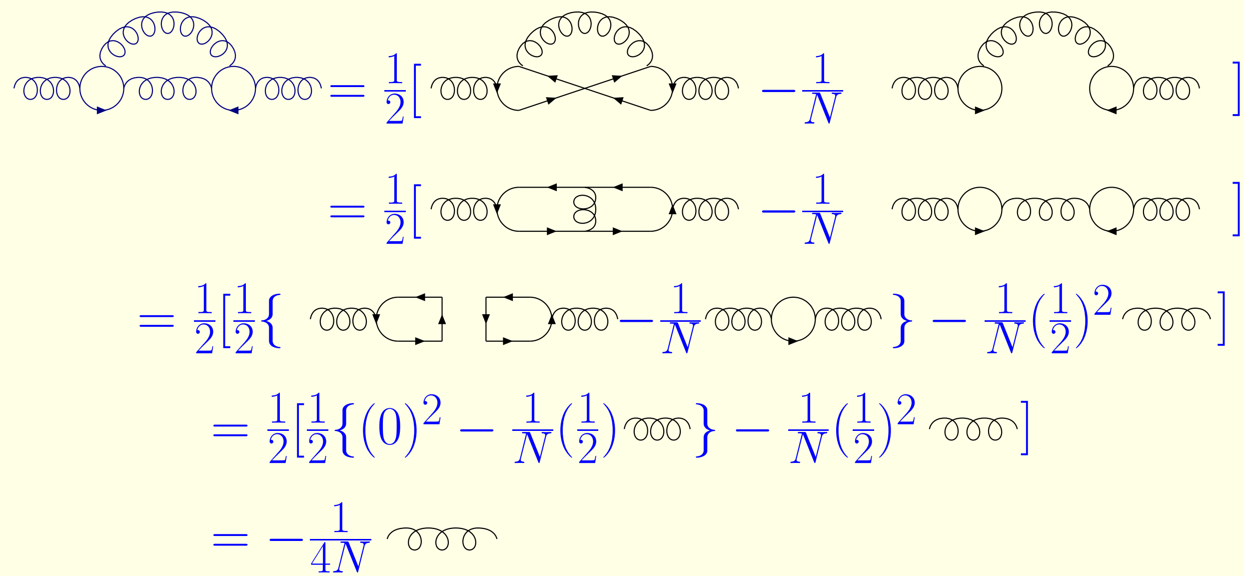
The first term inside the brackets is, using Eq. (620) (in graphical form), the definition of C_F **and then Eq. (618) (in graphical form) for the successive equalities:**

 $= \frac{1}{2}[$  $-\frac{1}{N}$  $]$

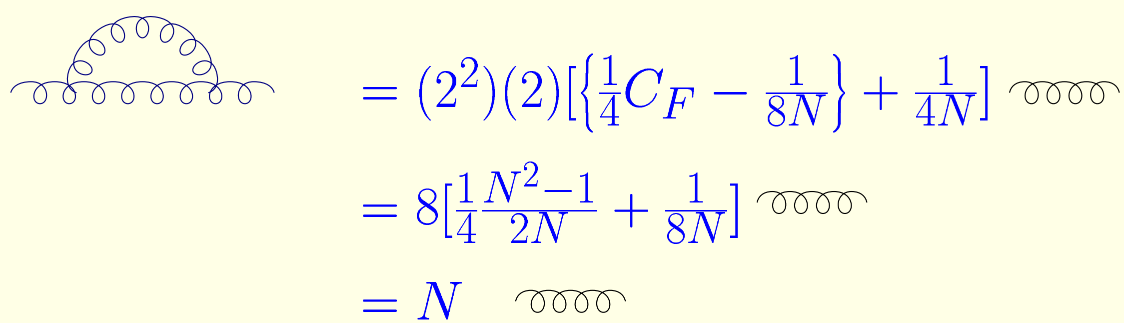
$= \frac{1}{2}[\,C_F$  $-\frac{1}{N}$  $]$

$= \frac{1}{2}[\,C_F\frac{1}{2} - \frac{1}{N}(\frac{1}{2})^2]$ 

The second term inside the brackets is reduced using the graphical forms of Eqs. (620) and (618) as follows:



The net result is then:



implying that $C_A = N$.

It is useful to know that the values of C_F and C_A have powerful experimental implications. For example, a quark moving along at high momentum will radiate soft gluons (which then turn into hadrons) with probability C_F, whereas a gluon moving along at high momentum will radiate additional soft gluons with probability C_A. As a result the average hadron multiplicity associated with a gluon jet will be larger than the average multiplicity associated with a quark jet by the factor C_A/C_F. This famous prediction was made in

Hadron Multiplicity in Color Gauge Theory Models. Stanley J. Brodsky (SLAC) , J.F. Gunion (UC, Davis) . SLAC-PUB-1749, UCD-76-5, May 1976. 13pp. Published in Phys.Rev.Lett.37:402-405,1976.

and is now checked experimentally with quite good precision. For example, in

Tests of Quantum Chromo Dynamics at e+ e- Colliders. Stefan Kluth (Munich, Max Planck Inst.) . MPP-2006-19, Mar 2006. 91pp. Published in Rept.Prog.Phys.69:1771-1846,2006. e-Print: hep-ex/0603011

The results from combined LEP data are

$$C_A = 2.89 \pm 0.21\,, \quad C_F = 1.30 \pm 0.09\,. \tag{625}$$

and, for the ratio (in which some uncertainties cancel) is

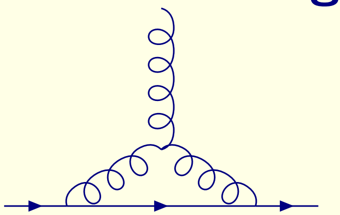
$$\frac{C_A}{C_F} = 2.23 \pm 0.01(statistical) \pm 0.14(systematic)\,. \tag{626}$$

Extracting these numbers from actual data is a bit of a tricky business as there are finite phase space effects, mass effects, leading particle effects and the need to precisely define a jet in a meaningful way using a cone or other algorithm. You can look at the above thesis to see how much work is actually involved.

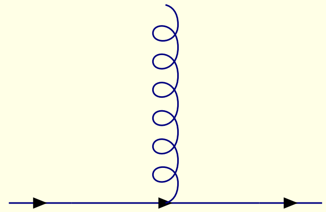
Also, one must include higher order corrections to the naive multiplicity ratio. These are significant.

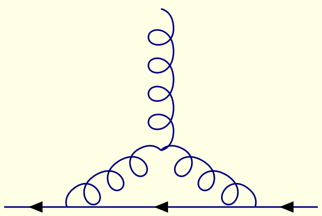
Still, after all is said and done, the bottom line is very much as predicted.

Finally, some additional triangle diagram type identities useful for loop calculations are the following (perhaps assigned as homework). Note that the directions of the arrows are reversed in the upper diagram compared to the middle diagram.

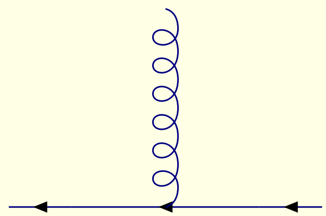


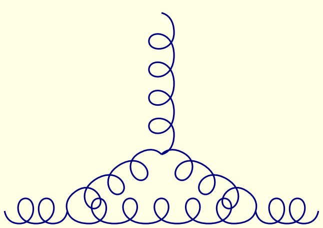
$$= -\frac{C_A}{2}$$



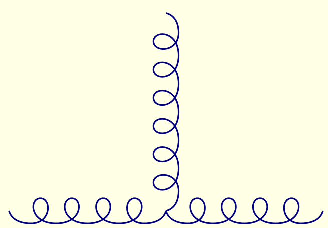


$$= +\frac{C_A}{2}$$





$$= +\frac{C_A}{2}$$



These vertex corrections control the evolution of α_s (after renormalization) with energy and, in particular, lead to the fact that α_s decreases with increasing energy (asymptotic freedom). The non-abelian structure turns out to be critical for this kind of decrease.

Spinor Algebra Techniques

It is useful to learn some clever calculational techniques that work extremely well when the masses of the particles can be neglected. For some particles, like the electron, photon, gluon, ..., this is a good approximation at essentially all energies. For other particles, it is a good approximation when the energies and momentum transfers all become large.

The approach I will introduce is based on the paper by Gunion and Kunszt (GK) from which the spinor techniques originally derived: ("IMPROVED ANALYTIC TECHNIQUES FOR TREE GRAPH CALCULATIONS AND THE G G Q ANTI-Q LEPTON ANTI-LEPTON SUBPROCESS", J.F. Gunion, Z. Kunszt, Phys.Lett.B161:333,1985). Some aspects of the material given below appear in Peskin problems 3.3, 5.3 and 5.6, Please let me know if you spot any typos.

- One begins by defining some standardized spinors, and their inner products. I will deviate somewhat from GK and begin ala Peskin by first writing down the left-handed particle Dirac spinor for momentum $k_0 = (E, 0, 0, -E)$. Peskin Eq. (3.49), states that particle spinors are given by

$$u = \left(\begin{array}{c} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \overline{\sigma}}\xi \end{array} \right) , \tag{627}$$

where σ is the 4-vector-like collection of 2×2 matrices $(1, \sigma^1, \sigma^2, \sigma^3) \equiv (1, \vec{\sigma})$ and $\overline{\sigma} = (1, -\vec{\sigma})$ and where

$$\xi(\text{spin } +\hat{z}) = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) , \quad \xi(\text{spin } -\hat{z}) = \left(\begin{array}{c} 0 \\ 1 \end{array} \right) . \tag{628}$$

To construct a left-handed spinor for k_0 (which is in the negative z direction) we should employ $+\hat{z}$ spin direction, *i.e.* $\xi = \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$. From this result one finds that the full 4-component spinor takes the form (in the Weyl representation of the Dirac matrices)

$$u_{L\,0} = \sqrt{2E} \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right) . \tag{629}$$

Note that $u_{L\,0}$ is an eigenstate of $P_L = (1 - \gamma_5)/2$ which in the Weyl representation is

$$P_L = \begin{pmatrix} \mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix}, \tag{630}$$

as expected in the massless limit. We also need a standard right handed spinor that will be an eigenstate of $P_R = (1 + \gamma_5)/2$ which, in the Weyl representation takes the form

$$P_R = \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \end{pmatrix}. \tag{631}$$

As noted in Peskin, a convenient way to get such a state is to employ $k_1 = (0, 1, 0, 0)$ and define $u_{R\,0} = \not{k}_1 u_{L\,0} = -\gamma^1 u_{L\,0}$. That this is an eigenstate of P_R is obvious from

$$P_R u_{R\,0} = P_R \not{k}_1 u_{L\,0} = \not{k}_1 P_L u_{L\,0} = \not{k}_1 u_{L\,0} = u_{R\,0}. \tag{632}$$

Computing explicitly, we find

$$u_{R\,0} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{633}$$

which is indeed the spinor for momentum k_0 and spin in the $-\hat{z}$ direction ($i.e.$ in the same spatial direction as the 3-momentum of the k_0 4-vector).

- Continuing to follow Peskin, the k_0 spinors $u_{L\,0}$ and $u_{R\,0}$ are next used to construct spinors $u_L(p)$ and $u_R(p)$ that are eigenstates of P_L and P_R, respectively, and obey the Dirac equation $\not p u_{L,R} = 0$. One writes

$$u_L(p) = \frac{\not p u_{R\,0}}{\sqrt{2p\cdot k_0}} = \frac{1}{\sqrt{p^+}}\not p \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{p^+}} \begin{pmatrix} -p^1 + ip^2 \\ p^+ \\ 0 \\ 0 \end{pmatrix} \tag{634}$$

where $p^+ \equiv p^0 + p^3$ and the final form is obtained by explicitly writing out

$$\not p = \begin{pmatrix} 0_{2\times 2} & p\cdot\sigma \\ p\cdot\overline{\sigma} & 0_{2\times 2} \end{pmatrix}. \tag{635}$$

That $\not p u_L(p) = 0$ is evident from $\not p \not p = p^2 = 0$ (for a massless particle). That $P_L u_L(p) = u_L(p)$ is evident from $P_L \not p u_{R\,0} = \not p P_R u_{R\,0} = \not p u_{R\,0}$.

In a similar fashion, we construct

$$u_R(p) = \frac{\not{p} u_{L\,0}}{\sqrt{2p\cdot k_0}} = \frac{1}{\sqrt{p^+}} \begin{pmatrix} 0 \\ 0 \\ p^+ \\ p^1 + ip^2 \end{pmatrix} . \tag{636}$$

However one arrives at these spinors (one could have just constructed $u_R(p)$ and $u_L(p)$ explicitly by rotating $u_{R\,0}$ and $u_{L\,0}$ from the negative $\hat{z}$ momentum direction to the general $\hat{p}$ direction), they are the right- and left-handed spinors for a massless particle with momentum p as written in the Weyl representation. The only difference at this point with GK is that GK use the Dirac representation for these same spinors.

- Using these spinors we now construct the fundamental "inner products". To construct these, we also need

$$\overline{u}_R(p) = u_R^\dagger(p)\gamma^0 = \frac{1}{\sqrt{p^+}}\left(p^+, p^1 - ip^2, 0, 0\right) \tag{637}$$

$$\overline{u}_L(p) = u_L^\dagger(p)\gamma^0 = \frac{1}{\sqrt{p^+}}\left(0, 0, -(p^1 + ip^2), p^+\right) \tag{638}$$

where the right-most expressions are obtained by explicitly taking the $u^\dagger_{R,L}$ forms and multiplying by

$$\gamma^0 = \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \tag{639}$$

where we have given the Weyl form of γ^0. The spinor "inner products" are defined by

$$\overline{u}_R(k)u_L(p) \equiv \langle k+|p-\rangle = \sqrt{\frac{p^+}{k^+}}(k^1 - ik^2) - \sqrt{\frac{k^+}{p^+}}(p^1 - ip^2) \tag{640}$$

$$\overline{u}_L(k)u_R(p) \equiv \langle k-|p+\rangle = -\sqrt{\frac{p^+}{k^+}}(k^1 + ik^2) + \sqrt{\frac{k^+}{p^+}}(p^1 + ip^2), \tag{641}$$

where the explicit expressions are simply those obtained by explicit computation. In the bra-ket notation, the $+$ or $-$ denotes the R or L, respectively, of the spinors. We also have, by explicit computation, the results

$$\overline{u}_R(k)u_R(p) = \overline{u}_L(k)u_L(p) = 0\,, \quad i.e.\ \langle k+|p+\rangle = \langle k-|p-\rangle = 0\,. \tag{642}$$

This same result can be proved as follows. First, we note that

$$\overline{u}_R(k) = u^\dagger_R(k)\gamma^0 = [P_R u_R(k)]^\dagger\gamma^0 = u^\dagger_R(k)P_R\gamma^0 = u^\dagger_R(k)\gamma^0 P_L = \overline{u}_R(k)P_L \tag{643}$$

where we used the fact that $\{\gamma^5, \gamma^0\} = 0$ for the 4th equality. We may then write, for example,

$$\overline{u}_R(k) u_R(p) = \overline{u}_R(k) P_L u_R(p) = 0 \tag{644}$$

where the last equality is simply the statement that u_R is a right-handed helicity spinor. More generally, we may prove using this latter kind of technique that

$$\langle k \pm |(\text{odd number of } \gamma \text{ matrices})|p\mp\rangle = 0 \tag{645}$$

$$\langle k \pm |(\text{even number of } \gamma \text{ matrices})|p\pm\rangle = 0 \tag{646}$$

- Various properties of these spinor inner products are immediately apparent from their explicit expressions. First, we have

$$\langle k + |p-\rangle = -\langle p + |k-\rangle \,, \quad \langle k - |p+\rangle = -\langle p - |k+\rangle \,. \tag{647}$$

Next, we note that

$$\langle k + | p - \rangle^* = \langle p - | k + \rangle \,, \tag{648}$$

as suggested by the bra-ket notation. Next, we may take the explicit forms and compute

$$\begin{aligned}
& |\langle k + | p - \rangle|^2 \\
= \quad & \frac{k^+}{p^+}[(p^1)^2 + (p^2)^2] + \frac{p^+}{k^+}[(k^1)^2 + (k^2)^2] - (p^1 - ip^2)(k^1 + ik^2) - (p^1 + ip^2)(k^1 - ik^2) \\
= \quad & \frac{k^+}{p^+}[(p^0)^2 - (p^3)^2] + \frac{p^+}{k^+}[(k^0)^2 - (k^3)^2] - 2p^1k^1 - 2p^2k^2 \\
= \quad & (k^0 + k^3)(p^0 - p^3) + (p^0 + p^3)(k^0 - k^3) - 2p^1k^1 - 2p^2k^2 \\
= \quad & 2(k^0p^0 - k^1p^1 - k^2p^2 - k^3p^3) \\
= \quad & 2p \cdot k \,,
\end{aligned} \tag{649}$$

with the same result for $|\langle k - | p + \rangle|^2$, implying that the inner products can be thought of as the square roots of twice the dot product. Another way of arriving at this same result illustrates some further manipulation techniques:

$$\begin{aligned}
& |\langle k + | p - \rangle|^2 \\
= \quad & \langle k + | p - \rangle \langle k + | p - \rangle^* = \langle k + | p - \rangle \langle p - | k + \rangle = \langle k + | (| p - \rangle \langle p - | + | p + \rangle \langle p + |) | k + \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle k+|\not{p}|k+\rangle = \langle k+|\not{p}P_R|k+\rangle = \mathrm{Tr}[\not{p}P_R(|k+\rangle\langle k+|+|k-\rangle\langle k-|)] \\
&= \mathrm{Tr}[\not{p}\frac{1+\gamma^5}{2}\not{k}] = 2p\cdot k\,,
\end{aligned} \tag{650}
$$

where for the 3rd equality we used Eq. (642) and for the 6th equality we used the fact that $P_R|k-\rangle = 0$, as is obvious from the definition of negative helicity and also the explicit expression for $|k-\rangle \equiv u_L(k)$.

- Of course, we will also need the helicity states for antiparticles in computing some processes. These are obtained starting with the Peskin form

$$
v = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\overline{\sigma}}\,\eta \end{pmatrix} \tag{651}
$$

where

$$
\eta(+\hat{z}\ \text{spin}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \quad \eta(-\hat{z}\ \text{spin}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{652}
$$

One then finds that

$$
v_{L\,0} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = u_{R\,0}\,, \quad v_{R\,0} = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = u_{L\,0}\,, \tag{653}
$$

with the result that for general momentum

$$v_+(p) \equiv v_R(p) = u_L(p) \equiv u_-(p) \equiv |p-\rangle \tag{654}$$

$$v_-(p) \equiv v_L(p) = u_R(p) \equiv u_+(p) \equiv |p+\rangle\,. \tag{655}$$

- Another very important identity is

$$\langle k+|\gamma^\mu|p+\rangle = \langle p-|\gamma^\mu|k-\rangle\,. \tag{656}$$

To prove this identity, we must basically use the charge conjugation operator as defined in Peskin's chapter 3. In particular, it may be verified explicitly, using the forms given in Eqs. (634), (636) and (638), that

$$u_\pm(k) = C[\overline{u}_\mp(k)]^T \tag{657}$$

where

$$C = -i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{658}$$

and, as always, the $\pm$ subscripts are shorthand for R and L, respectively. Accepting this result we can now proceed to prove Eq. (656). We have

$$
\begin{aligned}
\langle k+|\gamma^\mu|p+\rangle &\equiv \overline{u}_+(k)\gamma^\mu u_+(p) = u_+^\dagger(k)\gamma^0\gamma^\mu u_+(p) \\
&= \left\{C[\overline{u}_-(k)]^T\right\}^\dagger \gamma^0\gamma^\mu C[\overline{u}_-(p)]^T \\
&= \overline{u}_-(p)C^T[\gamma^\mu]^T[\gamma^0]^T \left\{C[\overline{u}_-(k)]^T\right\}^* \\
&= \overline{u}_-(p)C^T[\gamma^\mu]^T C C^{-1}[\gamma^0]^T C^* \left\{[u_-^\dagger(k)\gamma^0]^T\right\}^* \\
&= \overline{u}_-(p)\left[C[\gamma^\mu]^T C\right]\left[C[\gamma^0]^T C\right][\gamma^0]^\dagger u_-(k) \\
&= \overline{u}_-(p)\gamma^\mu\gamma^0\gamma^0 u_-(k) \\
&= \overline{u}_-(p)\gamma^\mu u_-(k) \\
&\equiv \langle p-|\gamma^\mu|k-\rangle\,, \qquad (659)
\end{aligned}
$$

where at various points we have used the facts that $C = -C^T = -C^{-1} = C^*$, that $[\gamma^0]^\dagger = \gamma^0$ and that $C[\gamma^\mu]^T C = \gamma^\mu$. This latter is the crux of the charge conjugation symmetry and may be explicitly verified for each choice of μ using the explicit form of C and the explicit forms of the γ^μ, all in the Weyl representation of course.

- **Polarization vectors for gluons (or photons) can be constructed as follows:**

$$[\epsilon^{\mu\pm}(k,p)]^* = \frac{\pm\langle k\pm|\gamma^\mu|p\pm\rangle}{\sqrt{2}\langle p\mp|k\pm\rangle}\,. \tag{660}$$

Here, p is any light-like momentum, $p^2=0$. Note that $k_\mu[\epsilon^{\mu\pm}(k,p)]^* = 0$ by virtue of $\langle k\pm|\,\not{k} \equiv \overline{u}_\pm(k)\not{k} = 0$.

In particular, it is not hard to explicitly evaluate these expressions for the case of $k=(E,0,0,E)$. One finds

$$[\epsilon^{\mu+}(k,p)]^* = -\frac{1}{\sqrt{2}}(0,1,-i,0) - \sqrt{2}\frac{p^+}{k^+(p^1+ip^2)}k^\mu = \textbf{standard} + \beta k^\mu\,, \tag{661}$$

and

$$[\epsilon^{\mu-}(k,p)]^* = \frac{1}{\sqrt{2}}(0,1,i,0) - \sqrt{2}\frac{p^+}{k^+(p^1-ip^2)}k^\mu = \textbf{standard} + \beta' k^\mu\,, \tag{662}$$

where by "standard" we mean we are getting the results equivalent to

the usual photon helicity circular polarizations defined by

$$\epsilon^{\mu}(\pm) = \mp\frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \tag{663}$$

($i.e.$ without any *).

Note that the the pieces proportional to k^μ in Eqs. (661) and (662) will have no effect in the context of computing a gauge-invariant quantity so long as our choices of polarizations correspond to some kind of gauge choice.

Correspondingly, if we change the reference light-like momentum from p to l let's say, one finds, for example,

$$\begin{aligned}
[\epsilon^{\mu +}(k,p)]^* - [\epsilon^{\mu +}(k,l)]^* &= \frac{1}{\sqrt{2}}\left[\frac{\langle k+|\gamma^\mu|p+\rangle}{\langle p-|k+\rangle} - \frac{\langle k+|\gamma^\mu|l+\rangle}{\langle l-|k+\rangle}\right] \\
&= \frac{1}{\sqrt{2}}\frac{\langle l-|k+\rangle\langle k+|\gamma^\mu|p+\rangle - \langle p-|k+\rangle\langle k+|\gamma^\mu|l+\rangle}{\langle p-|k+\rangle\langle l-|k+\rangle} \\
&= \frac{1}{\sqrt{2}}\frac{\langle l-|\not{k}\gamma^\mu|p+\rangle - \langle p-|\not{k}\gamma^\mu|l+\rangle}{\langle p-|k+\rangle\langle l-|k+\rangle} \\
&= \frac{1}{\sqrt{2}}\frac{\langle l-|\not{k}\gamma^\mu|p+\rangle + \langle l-|\gamma^\mu\not{k}|p+\rangle}{\langle p-|k+\rangle\langle l-|k+\rangle} \quad \text{using the } C \text{ game} \\
&= \sqrt{2}k^\mu\frac{\langle l-|p+\rangle}{\langle p-|k+\rangle\langle l-|k+\rangle}
\end{aligned} \tag{664}$$

This shows that if we change the light-like reference momentum of one gluon from p to l, keeping reference momentum p for all other external gluons, then the Ward identity (applied for the altered gluon) guarantees that the result for the net amplitude (after summing all Feynman diagrams of course) will remain unchanged.

It is easy to see what gauge choice we have made, by noting that $p_\mu \epsilon^{\mu\,\pm\,*}(k,p) = 0$ by virtue of $\not{p}|p\pm\rangle = 0$ (the massless Dirac equation). This means that we are in a light-like axial gauge. This may also be verified by computing the completeness sum:

$$\begin{aligned}
\sum_{\lambda=\pm} \epsilon^\mu(\lambda)[\epsilon^\nu(\lambda)]^* &= \frac{\langle k+|\gamma^\mu|p+\rangle^*}{\sqrt{2}\langle p-|k+\rangle^*}\frac{\langle k+|\gamma^\nu|p+\rangle}{\sqrt{2}\langle p-|k+\rangle} + \frac{-\langle k-|\gamma^\mu|p-\rangle^*}{\sqrt{2}\langle p+|k-\rangle^*}\frac{-\langle k-|\gamma^\nu|p-\rangle}{\sqrt{2}\langle p+|k-\rangle} \\
&= \frac{\langle p+|\gamma^\mu|k+\rangle\langle k+|\gamma^\nu|p+\rangle}{2(2p\cdot k)} + \frac{\langle p-|\gamma^\mu|k-\rangle\langle k-|\gamma^\nu|p-\rangle}{2(2p\cdot k)} \\
&= \frac{\mathrm{Tr}[\not{p}\gamma^\mu P_R \not{k}\gamma^\nu]}{4p\cdot k} + \frac{\mathrm{Tr}[\not{p}\gamma^\mu P_L \not{k}\gamma^\nu]}{4p\cdot k} \\
&= \frac{\mathrm{Tr}[\not{p}\gamma^\mu \not{k}\gamma^\nu]}{4p\cdot k} \\
&= -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{p\cdot k}\,, \qquad (665)
\end{aligned}$$

which is precisely the expression for the gluon propagator in the so-called “light-like axial” gauge with the light-like axial vector n chosen to be p.

- **Now come the very important Fierz identities.**

1. **The fundamental identity, that is obtained as part of a Peskin problem (5.3, see also problem 3.6) is:**

$$(\gamma_\mu P_R)_{\alpha\beta}(\gamma^\mu P_L)_{\gamma\delta} = 2(P_L)_{\alpha\delta}(P_R)_{\gamma\beta}\,, \tag{666}$$

where the Greek indices are Dirac indices. A proof following the Peskin 5.3 route appears in a short section following this section. If we now sandwich the $\alpha\beta$ between the two $+$ spinors $\overline{u}_+(a)_\alpha \ldots u_+(b)_\beta$, we obtain

$$\overline{u}_+(a)_\alpha(\gamma_\mu P_R)_{\alpha\beta}u_+(b)_\beta(\gamma^\mu P_L)_{\gamma\delta} = 2(P_R)_{\gamma\beta}u_+(b)_\beta\,\overline{u}_+(a)_\alpha(P_L)_{\alpha\delta} = 2u_+(b)_\gamma\,\overline{u}_+(a)_\delta\,. \tag{667}$$

In our bra-ket notation, this is written as

$$\langle a+|\gamma_\mu P_R|b+\rangle(\gamma^\mu P_L)_{\gamma\delta} = 2|b+\rangle_\gamma\langle a+|_\delta \tag{668}$$

or, since $P_R|b+\rangle = |b+\rangle$, simply

$$\langle a+|\gamma_\mu|b+\rangle(\gamma^\mu P_L)_{\gamma\delta} = 2|b+\rangle_\gamma\langle a+|_\delta\,, \tag{669}$$

or, dropping the explicit Dirac indices,

$$\langle a+|\gamma_\mu|b+\rangle(\gamma^\mu P_L) = 2|b+\rangle\langle a+|\,. \tag{670}$$

Note that this result also implies, using Eq. (656), that

$$\langle b-|\gamma_\mu|a-\rangle(\gamma^\mu P_L) = 2|b+\rangle\langle a+|\,. \tag{671}$$

We can get related identities by sandwiching the $\gamma\delta$ Dirac index items with $\overline{u}_-(b)_\gamma \ldots u_-(a)_\delta$. We obtain

$$\langle b-|\gamma_\mu|a-\rangle(\gamma^\mu P_R) = 2|a-\rangle\langle b-| \tag{672}$$

and, after using Eq. (656) again,

$$\langle a+|\gamma_\mu|b+\rangle(\gamma^\mu P_R) = 2|a-\rangle\langle b-|\,. \tag{673}$$

2. A very important use of these identities is the evaluation of the typical sum that appears in Feynman diagram calculations associated with an

internally exchanged gluon:

$$\begin{aligned}
\langle a+|\gamma_\mu|b+\rangle\langle c-|\gamma^\mu|d-\rangle &= \langle a+|\gamma_\mu|b+\rangle\langle c-|\gamma^\mu P_L|d-\rangle \\
&= \langle c-|\left[\langle a+|\gamma_\mu|b+\rangle\gamma^\mu P_L\right]|d-\rangle \\
&= \langle c-|\left[2|b+\rangle\langle a+|\right]|d-\rangle \\
&= 2\langle c-|b+\rangle\langle a+|d-\rangle\,. \qquad (674)
\end{aligned}$$

Using Eq. (656), we also find

$$\begin{aligned}
\langle a+|\gamma_\mu|b+\rangle\langle c+|\gamma^\mu|d+\rangle &= \langle a+|\gamma_\mu|b+\rangle\langle d-|\gamma^\mu|c-\rangle \\
&= 2\langle a+|c-\rangle\langle d-|b+\rangle \\
&= 2[-\langle c+|a-\rangle][-\langle b-|d+\rangle] \\
&= 2\langle c+|a-\rangle\langle b-|d+\rangle \\
&= \ldots \qquad (675)
\end{aligned}$$

3. **Finally, from the Fierz identities Eqs. (670)-(673), we find the important identity needed when an external polarization for a gluon is Lorentz**

contracted with a γ matrix at some QCD vertex on a fermion line. The result of such a contraction is to produce a $\not{\epsilon}$ or $\not{\epsilon}^*$, where ϵ is the gluon polarization. Using the explicit forms given earlier for $\epsilon^*_\pm(k)$, and the Fierz identities, we find

$$\begin{aligned}\not{\epsilon}^*_\pm(k) &= \frac{\pm\langle k\pm|\gamma^\mu|p\pm\rangle}{\sqrt{2}\langle p\mp|k\pm\rangle}\gamma_\mu(P_L+P_R) \\ &= \pm\sqrt{2}\left\{\frac{|k\mp\rangle\langle p\mp| + |p\pm\rangle\langle k\pm|}{\langle p\mp|k\pm\rangle}\right\}.\end{aligned} \tag{676}$$

The specific forms of the Fierz identities (670)-(673) employed in obtaining this result are:

$$\begin{aligned}\langle k+|\gamma^\mu|p+\rangle\gamma_\mu P_L &= 2|p+\rangle\langle k+| \\ \langle k+|\gamma^\mu|p+\rangle\gamma_\mu P_R &= 2|k-\rangle\langle p-| \\ \langle k-|\gamma^\mu|p-\rangle\gamma_\mu P_L &= 2|k+\rangle\langle p+| \\ \langle k-|\gamma^\mu|p-\rangle\gamma_\mu P_R &= 2|p-\rangle\langle k-|.\end{aligned} \tag{677}$$

4. There is one final Fierz-like identity that is useful in more complex $2 \to 4$ calculations than those we will be working on. However, I give it for completeness. It reads in bra-ket notation

$$|b+\rangle\langle a-| - |a+\rangle\langle b-| = \langle a-|b+\rangle P_R\,. \tag{678}$$

If we sandwich this between $\langle c-|\ldots|d+\rangle$, we obtain the identity:

$$\langle c-|b+\rangle\langle a-|d+\rangle - \langle c-|a+\rangle\langle b-|d+\rangle = \langle a-|b+\rangle\langle c-|d+\rangle \tag{679}$$

as well as related identities obtained by using the antisymmetry of the spinor inner product at various locations in the above identity.

- So, now just a few words about using this formalism — in particular for the gluons.

  First, consider external gluons. The idea is to choose reference momentum p in $\epsilon(k,p)$ that will simplify your calculation. Typically, the simplification will occur if p is chosen equal to one of the external fermion momenta. In this way, one or more diagrams can often be zeroed. Of course, once you make a choice of p for computing one diagram, you must continue to make this same (gauge) choice for all other diagrams. The sum of diagrams will be independent of this choice of course, but the calculation will be vastly simpler if you make a choice of p that zeroes as many diagrams as possible. Now, in many simple tree-level computations involving external fermions you can actually choose p differently for different external gluons. This is because the difference between the $\epsilon(k,p)$ and $\epsilon(k,p')$ introduced thereby is simply proportional to k which will be connecting to a conserved current. This trick can be employed for the cases we have considered.

For internal gluons, you should in principle use the full axial gauge propagator. Again, for the calculations we are doing, this internal propagator is attaching to at least one conserved fermionic current, and so one can drop the non-$g_{\mu\nu}$ part of the gluon propagator. However, in general one must be careful.

- **The Fierz Identity**

Here, we prove the identity

$$\langle k-|\gamma^\mu|p-\rangle\gamma_\mu P_L = 2|k+\rangle\langle p+| \tag{680}$$

from which all the others follow trivially. Let us rewrite it in terms of the old spinor notation:

$$\overline{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma_\mu \frac{1}{2}(1-\gamma^5)]_{\alpha\beta} = 2\,[u_R(p_1)\overline{u}_R(p_2)]_{\alpha\beta}\,. \tag{681}$$

Clearly, the right hand side of the above equation must just be some 4×4 matrix M in Dirac space and so it can be written as a linear combination of the standard 16 Γ matrices (1, γ^5, γ_μ, $\gamma_\mu\gamma^5$, $\frac{1}{2}[\gamma_\mu, \gamma_\nu]$). The only question is which of these actually appear on the left hand side.

Now, from the rhs we see explicitly that $\gamma^5 M = -M\gamma^5$ since $\gamma^5 u_R = +u_R$ and $\overline{u}_R\gamma^5 = u_R^\dagger\gamma^0\gamma^5 = -u_R^\dagger\gamma^5\gamma^0 = -u_R^\dagger\gamma^0 = -\overline{u}_R$.

Of the above listed 16 matrices, only γ_μ and $\gamma_\mu\gamma^5$ have this property. Thus, we must have

$$M = \gamma_\mu \frac{1}{2}(1-\gamma^5)V^\mu + \gamma_\mu \frac{1}{2}(1+\gamma^5)W^\mu\,, \tag{682}$$

where V^μ and W^μ are 4-vectors. These 4 vectors can be computed using trace technology. For example, we project out V^μ by taking

$$V^\nu = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{2}(1-\gamma^5)\gamma^\nu M\right]. \tag{683}$$

Thus, we would compute

$$\begin{aligned}
V^\nu &= \frac{1}{2}\mathrm{Tr}\left[\frac{1}{2}(1-\gamma^5)\gamma^\nu(2u_R(p_1)\overline{u}_R(p_2))\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\nu\frac{1}{2}(1+\gamma^5)(2u_R(p_1)\overline{u}_R(p_2))\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\nu 2u_R(p_1)\overline{u}_R(p_2))\right] \\
&= \overline{u}_R(p_2)\gamma^\nu u_R(p_1) \\
&= \overline{u}_L(p_1)\gamma^\nu u_L(p_2)\,,
\end{aligned} \tag{684}$$

where in the last step we used Eq. (656). Meanwhile, we would compute W^ν as

$$W^\nu = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{2}(1+\gamma^5)\gamma^\nu(2u_R(p_1)\overline{u}_R(p_2))\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\nu \frac{1}{2}(1-\gamma^5)(2u_R(p_1)\overline{u}_R(p_2))\right] \\
&= 0\,,
\end{aligned}
\tag{685}
$$

since $(1-\gamma^5)u_R = 0$. Thus, we verify the identity we were aiming at. The P_R version of this identity was

$$
\overline{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma_\mu \frac{1}{2}(1+\gamma^5)]_{\alpha\beta} = 2\left[u_L(p_2)\overline{u}_L(p_1)\right]_{\alpha\beta}\,. \tag{686}
$$

We would prove this in just the same manner as above except that it would be W^ν that would survive and we would compute

$$
\begin{aligned}
W^\nu &= \frac{1}{2}\mathrm{Tr}\left[\frac{1}{2}(1+\gamma^5)\gamma^\nu(2u_L(p_2)\overline{u}_L(p_1))\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\nu\frac{1}{2}(1-\gamma^5)(2u_L(p_2)\overline{u}_L(p_1))\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\nu 2u_L(p_2)\overline{u}_L(p_1))\right] \\
&= \overline{u}_L(p_1)\gamma^\nu u_L(p_2)\,,
\end{aligned}
\tag{687}
$$

The other two Fierz identities of Eq. (677) are obtained from the two derived by just using Eq. (656).

- $\boxed{\text{Applications}}$

- Let us first consider the computation requested in Peskin 5.3(d), namely

$$\frac{d\sigma}{d\Omega}(e^-_R(k)e^+_L(p) \to \mu^-_R(k')\mu^+_L(p')) = \frac{\alpha^2}{4E^2_{\rm cm}}(1+\cos\theta)^2 . \qquad (688)$$

The Feynman graph was given earlier

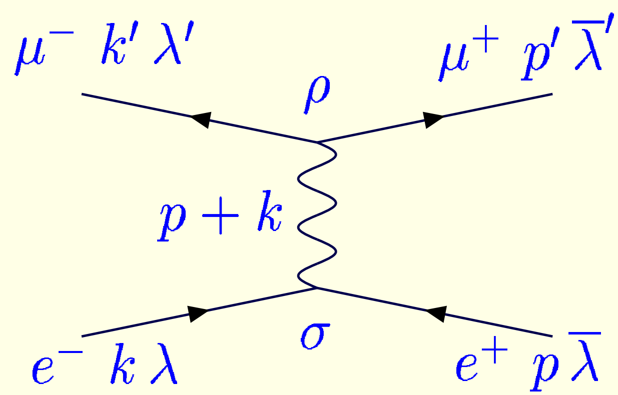


Figure 15: The one Feynman diagram contributing to $e^+e^- \to \mu^+\mu^-$.

The amplitude desired is (remembering that $v_-(p) = |p+\rangle$, etc.)

$$\begin{aligned}
\mathcal{M} &= \overline{u}_+(k')ie\gamma_\rho v_-(p')i\frac{-g^{\rho\sigma}}{(p+k)^2}\overline{v}_-(p)ie\gamma_\sigma u_+(k) \\
&= i\frac{e^2}{s}\langle k'+|\gamma_\rho|p'+\rangle\langle p+|\gamma^\rho|k+\rangle \\
&= i\frac{e^2}{s}2\langle p+|k'-\rangle\langle p'-|k+\rangle\,, \qquad (689)
\end{aligned}$$

where we used one of the Fierz identities, Eq. (675), for the last equality above. From this, we obtain (using $|\langle a+|b-\rangle|^2 = 2a\cdot b$)

$$|\mathcal{M}|^2 = 4\frac{e^4}{s^2}(2p\cdot k')(2p'\cdot k)\,. \qquad (690)$$

Now, we need to recall our kinematics

$$\begin{aligned}
k &= (E,0,0,E)\,, \quad p = (E,0,0,-E) \\
k' &= (E, E\sin\theta, 0, E\cos\theta) \\
p' &= (E, -E\sin\theta, 0, -E\cos\theta)\,, \qquad (691)
\end{aligned}$$

where $E = E_{\rm cm}/2 = \sqrt{s}/2$. From this we have

$$2p \cdot k' = 2p' \cdot k = 2E^2(1 + \cos\theta) = \frac{1}{2}E_{\rm cm}^2(1 + \cos\theta) \qquad (692)$$

so that

$$|\mathcal{M}|^2 = 4\frac{e^4}{E_{\rm cm}^4}\tfrac{1}{4}E_{\rm cm}^4(1 + \cos\theta)^2 = e^4(1 + \cos\theta)^2\,. \qquad (693)$$

So now, remembering that we are not spin averaging, but rather computing the cross section for a given spin configuration, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{\rm cm}^2}\frac{|\vec{k}|}{16\pi^2 E_{\rm cm}}\,|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2 E_{\rm cm}^2}\,|\mathcal{M}|^2 \quad \text{for } E \gg m_\mu,\ |\vec{k}| = E_{\rm cm}/2 \\
&= \frac{e^4}{64\pi^2 E_{\rm cm}^2}(1 + \cos\theta)^2
\end{aligned}$$

$$= \frac{\alpha^2}{4E_{\rm cm}^2}(1+\cos\theta)^2\,, \quad \text{using } e^2 = 4\pi\alpha\,. \tag{694}$$

At this point, you should find it useful to improve your understanding of the spinor techniques to check the other results found in Peskin's Eqs. (5.23) and (5.24) for the 3 other non-zero helicity configurations. You can also easily check that helicity configurations not shown there are zero (in the small mass limit). For instance,

$$\frac{d\sigma}{d\Omega}(e^-_R(k)e^+_R(p) \to \mu^-_R(k')\mu^+_L(p')) \tag{695}$$

requires the amplitude

$$\begin{aligned} \mathcal{M} &= \overline{u}_+(k')ie\gamma_\rho v_-(p')i\frac{-g^{\rho\sigma}}{(p+k)^2}\overline{v}_+(p)ie\gamma_\sigma u_+(k) \\ &= i\frac{e^2}{s}\langle k'+|\gamma_\rho|p'+\rangle\langle p-|\gamma^\rho|k+\rangle \\ &= 0 \end{aligned} \tag{696}$$

by virtue of the fact that $\langle a-|\gamma^\rho|b+\rangle = 0$ always as a result of helicity conservation.

- So, now let us attempt a real QCD calculation. The process I will discuss is $gg \to q\overline{q}$. We wish to obtain the form of the cross section

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\frac{1}{n_{\rm spin} n_{\rm color}} \sum_{\rm colors, spins} |\mathcal{M}|^2 \tag{697}$$

where, as indicated, we must average over initial spins *and over initial colors*.

The result we will obtain when the quark masses can be neglected is

$$\frac{d\sigma}{dt} = \frac{\pi\alpha_s^2}{s^2}\left(\frac{1}{6}\frac{u^2+t^2}{ut} - \frac{3}{8}\frac{u^2+t^2}{s^2}\right) . \tag{698}$$

As you will see, even using the tricks of the massless limit, the calculation is not easy.

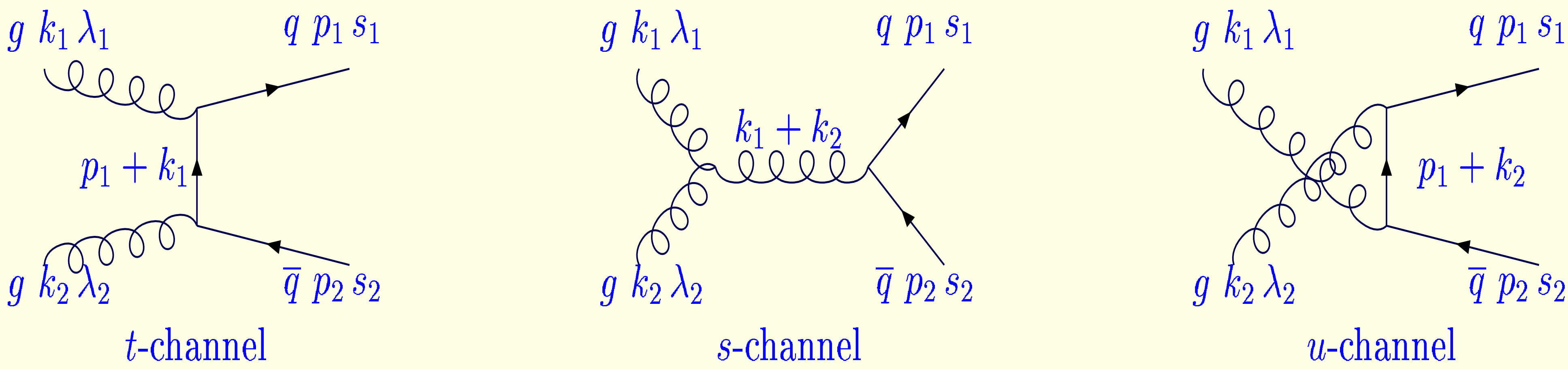
• 

Figure 16: The t, s and u channel Feynman diagrams for $gg \to q\overline{q}$. I will use a convention in which *all* **the momenta are outgoing. For this computation, unlike the unitarity check, I will use a convention in which** *all* **the particles and momenta are outgoing. This will mean that in our computation we will be using ϵ^*'s for the gluons.**

There are three contributing diagrams as given in Fig. 16, where I employ a convention where all particles are outgoing. The diagrams will be labeled by the channel of the virtual exchange, t, s or u. So, for a given set of external particle helicity choices, we will write

$$\mathcal{M} = \mathcal{A}_t C_t + \mathcal{A}_s C_s + \mathcal{A}_u C_u \tag{699}$$

where $\mathcal{A}$ represents the Feynman part of the amplitude (momenta, vertices etc.) and C represents the color amplitude associated with the diagram.

First, let us take care of the color algebra. We have two initial gluons. Each of these initial gluons can have $N^2 - 1 = 8$ different colors. Thus, $n_{\rm color} = (N^2 - 1)^2 = 64$.

1. The square of the t channel color factor produces, after summing over the colors of the initial gluons and of the final quarks, a closed quark loop with two gluon C_F type attachments leading to $|C_t|^2 = C_F^2 N = N\left(\frac{N^2-1}{2N}\right)^2 = \frac{(N^2-1)^2}{4N} = \frac{16}{3}$.

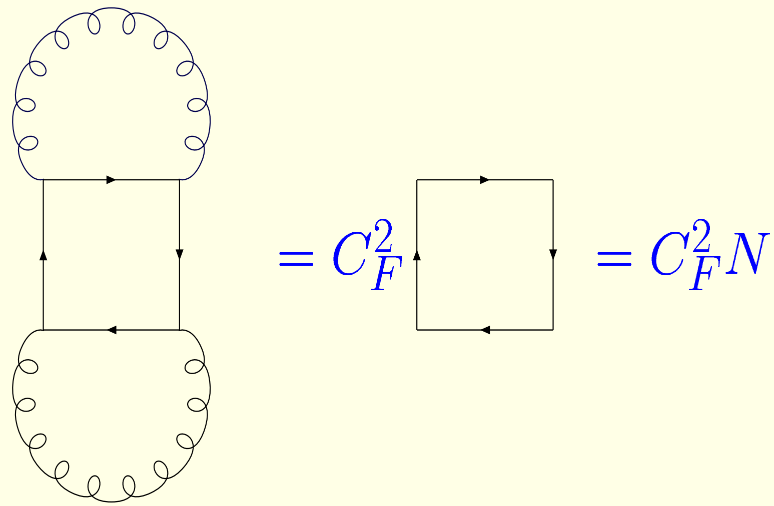


Figure 17: The color factor computation for C_t^2.

To understand how this diagram arises in more detail, let us call the color of gluon 1 a, that of gluon 2 b, that of quark 1 C and that of anti-quark 2 D. Ignoring the internal structure of the color diagram and calling the color amplitude $\langle CD|\mathcal{M}|ab\rangle$, what we want is

$$\sum_{a,b,C,D} |\langle CD|\mathcal{M}|ab\rangle|^2 = \langle CD|\mathcal{M}|ab\rangle\langle a'b'|\mathcal{M}|C'D'\rangle\delta_{aa'}\delta_{bb'}\delta_{CC'}\delta_{DD'}\,, \tag{700}$$

where in the 2nd form we have assumed the repeated index summation convention. Diagrammatically:

(a) $\langle CD|\mathcal{M}|ab\rangle$ is represented by a diagram (in the present case the t-channel diagram) with 2 gluons in and quark and antiquark out.

(b) $\langle a'b'|\mathcal{M}|C'D'\rangle$ is represented by a diagram with quark and antiquark in and 2 gluons out.

(c) $\delta_{aa'}$ is represented by a gluon color propagator line joining the in gluon a to the out gluon a'.

(d) $\delta_{bb'}$ is represented by a gluon color propagator line joining the in gluon b to the out gluon b'.

(e) $\delta_{CC'}$ is represented by a quark color propagator joining the out quark C to the in quark C'.

(f) $\delta_{DD'}$ is represented by a quark color propagator joining the out antiquark D to the in antiquark D', except that since this is an antiquark the direction of the color (vs. anticolor) flow is opposite.

Putting these diagram elements together, we get the diagram drawn above in Fig. 17.

2. The square of the u channel color factor gives the same: $|C_u|^2 = \frac{(N^2-1)^2}{4N} = \frac{16}{3}$.

3. The square of the s channel color factor gives a diagram in which there are two closed gluon loops (i.e. 3 gluon lines with two 3-gluon vertices) with a closed quark loop inserted in one of them. The latter gives a

factor of $1/2$, leaving a diagram equivalent to a gluon loop correction to a closed gluon loop, which equals C_A times a single closed gluon loop, which in turn is simply counting the number of gluons $N^2 - 1$. The result is thus $|C_s|^2 = \frac{1}{2}C_A(N^2 - 1) = \frac{1}{2}N(N^2 - 1) = 12$.

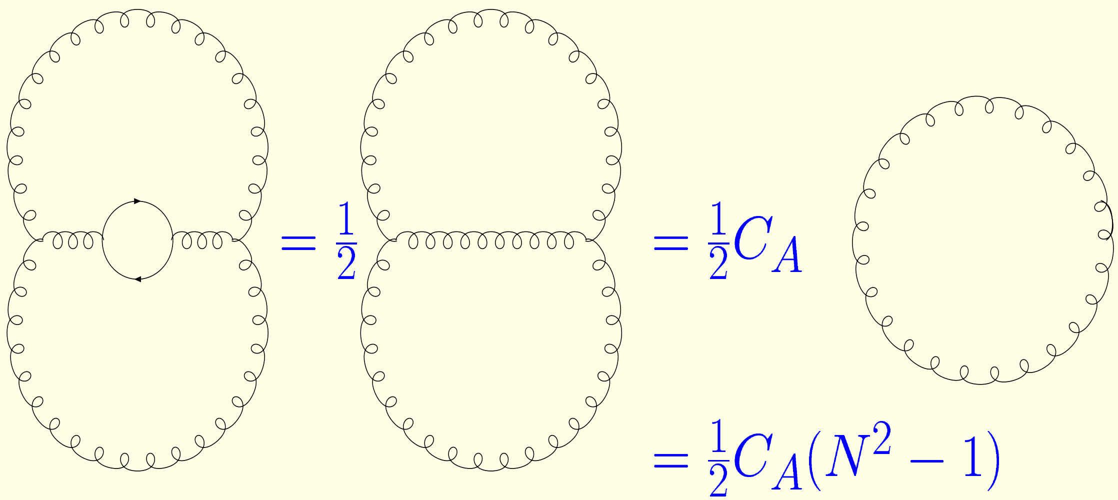


Figure 18: The color factor computation for C_s^2.

4. The interference term C_tC_u gives a color diagram equivalent to a closed quark loop with one horizontal gluon attached to opposite sides of the loop and one vertical gluon attached to the top and bottom of the loop. One can then "reduce" out the vertical gluon (for example), with associated numerical factor of $1/2$, leaving a "double-tadpole" diagram (which is 0) and a $-\frac{1}{N}$ times a closed quark loop with just the horizontal gluon, which topology is equivalent to C_F times a bare

closed quark loop (which corresponds to a factor of N). The result is $C_t C_u = \frac{1}{2}\left[0 - \frac{1}{N} C_F N\right] = -\frac{N^2-1}{4N} = -\frac{2}{3}$.

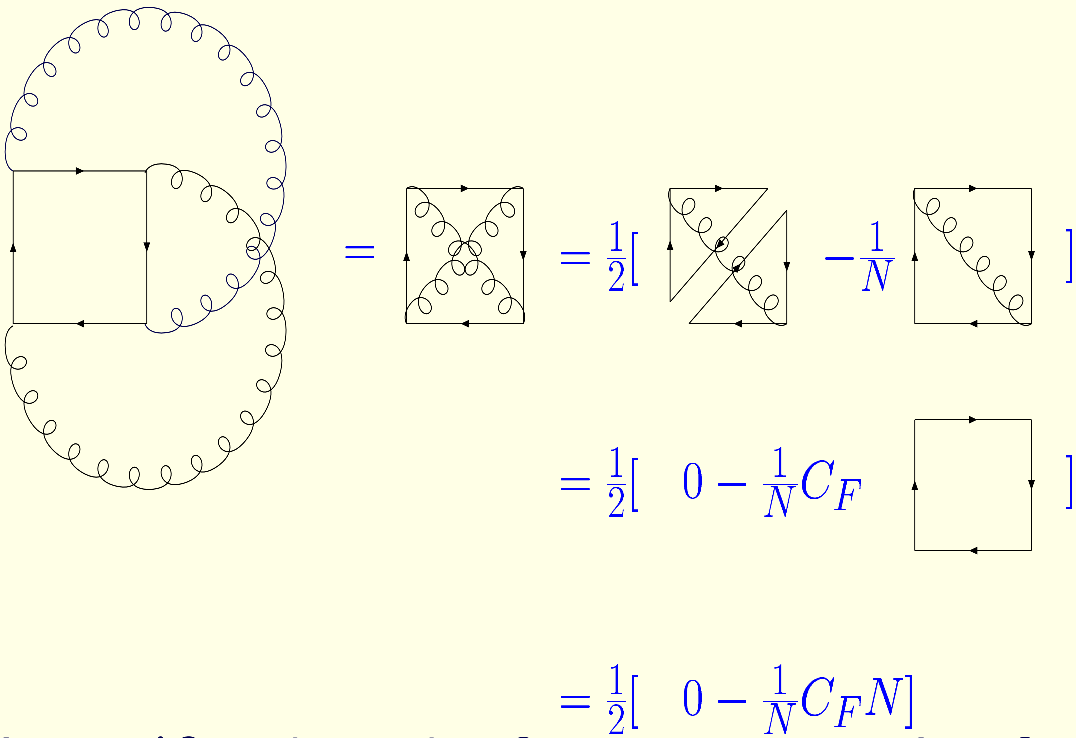
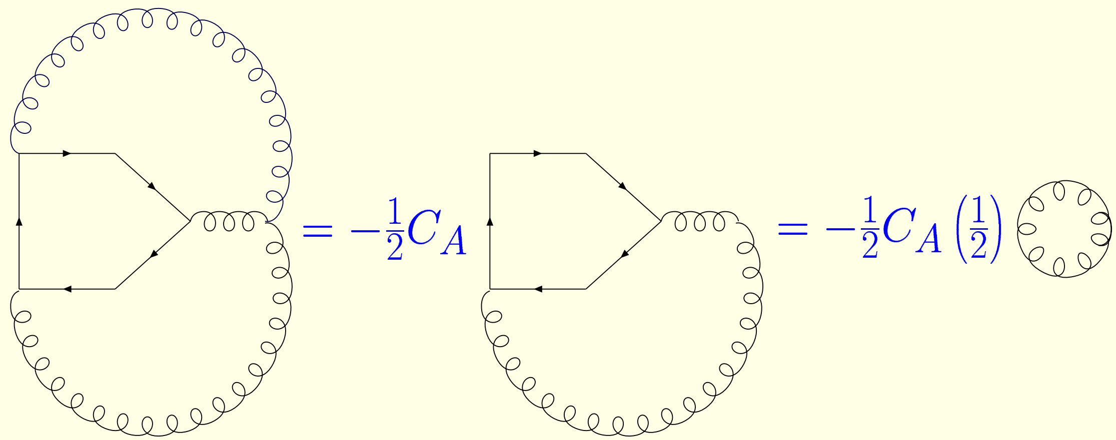


Figure 19: The color factor computation for $C_t C_u$.

5. Finally, we have the color calculation for $C_t C_s = -C_u C_s$, where the latter $-$ sign comes from the fact that the gluons are crossed (in their attachment to a closed quark loop) in the $C_u C_s$ calculation compared to the $C_t C_s$ calculation.
   The $C_t C_s$ diagram has a closed quark loop to which 3 gluons attach. Two of the gluons circle from left to right to attach to a 3-gluon vertex

into which enters the 3rd gluon emanating from the quark loop. Thus, a substructure that appears is the gluon-gluon-quark loop correction to a gluon quark vertex, which yields $-\frac{C_A}{2}$ times the bare quark-gluon vertex. After this replacement we are left with a gluon loop correction to a closed quark loop, giving $C_F N = \frac{1}{2}(N^2 - 1)$. The net result is thus $C_t C_s = -\frac{C_A}{2} C_F N = -\frac{N}{2}\frac{N^2-1}{2N} N = -6$



$$= -\frac{1}{2}C_A\left(\frac{1}{2}\right)(N^2 - 1)$$

$$= -\frac{1}{2}N\left(\frac{1}{2}\right)(N^2 - 1)$$

Figure 20: The color factor computation for $C_t C_s$.

- **So, now we turn to the Feynman helicity amplitudes, $\mathcal{A}_{s,t,u}(\lambda_1, \lambda_2, s_1, s_2)$.**

1. Consider first the t-channel amplitude:

$$\mathcal{A}_t(\lambda_1, \lambda_2, +, +) = \langle p_1 + | \not{\epsilon}_1(\lambda_1)^* (\not{p}_1 + \not{k}_1) \not{\epsilon}_2(\lambda_2)^* | p_2 - \rangle = 0 \quad (701)$$

where the final $|p_2-\rangle$ comes from remembering that $v_+(p_2) = u_-(p_2)$, and the final result of 0 arises because we need to have an even number of γ matrices between a bra and ket of opposite helicities.
A similar argument applies to the s and u channel diagrams for the $s_1 = s_2 = +$ helicity configurations, as well as to the $s_1 = s_2 = -$ helicity configurations.

2. Next consider $s_1 = +, s_2 = -$. Here we must work case by case in λ_1, λ_2.

   Consider first $\lambda_1 = \lambda_2 = +$.

   The t-channel diagram gives a numerator algebra:

$$\mathcal{A}_t(\lambda_1, \lambda_2, +, -) = \langle p_1 + | \not{\epsilon}_1(\lambda_1)^* (\not{p}_1 + \not{k}_1) \not{\epsilon}_2(\lambda_2)^* | p_2 + \rangle \,. \quad (702)$$

For $\epsilon_2(+)$ choose reference momentum p_2. Then,

$$\not{\epsilon}_2^+(k_2,p_2)^*|p_2+\rangle = \sqrt{2}\left[\frac{|k_2-\rangle\langle p_2-| + |p_2+\rangle\langle k_2+|}{\langle p_2-|k_2+\rangle}\right]|p_2+\rangle = 0 \tag{703}$$

by virtue of the facts that $\langle p_2-|p_2+\rangle = 0$ (antisymmetry of inner product) and $\langle k_2+|p_2+\rangle = 0$ (helicity "conservation").
The u-channel diagram gives a numerator algebra expression of the form:

$$\mathcal{A}_u(\lambda_1,\lambda_2,+,-) = \langle p_1+|\not{\epsilon}_2(\lambda_2)^*(\not{p}_1+\not{k}_2)\not{\epsilon}_1(\lambda_1)^*|p_2+\rangle\,. \tag{704}$$

This will be 0 if we choose reference momentum p_2 for $\epsilon_1(+)$, just as we did for ϵ_2.
The s-channel numerator algebra expression takes the form

$$\mathcal{A}_s(\lambda_1,\lambda_2,+,-) = \langle p_1+|\gamma^\alpha|p_2+\rangle V_\alpha(\epsilon_1,\epsilon_2)\,, \tag{705}$$

where

$$V_\alpha \propto \epsilon_1^\gamma(\lambda_1)^*\epsilon_2^\beta(\lambda_2)^*\left[(k_1-k_2)_\alpha g_{\gamma\beta} + (2k_2+k_1)_\gamma g_{\alpha\beta} - (2k_1+k_2)_\beta g_{\alpha\gamma}\right]. \tag{706}$$

The latter two terms insert $\not\epsilon_2(+)$ or $\not\epsilon_1(+)$, respectively, into the structure $\langle p_1 + | \ldots | p_2 + \rangle$, and both insertions give 0 for the above reference momentum choices. As regards the first term in V_α, it has the structure (remembering that p_2 is chosen as the reference momentum for both ϵ_1 and ϵ_2)

$$\begin{aligned} \propto \epsilon_1^* \cdot \epsilon_2^* \ldots \quad & \propto \quad \langle k_1 + | \gamma^\mu | p_2 + \rangle \langle k_2 + | \gamma_\mu | p_2 + \rangle \\ & \propto \quad \langle k_1 + | \gamma^\mu | p_2 + \rangle \langle p_2 - | \gamma_\mu | k_2 - \rangle \\ & \propto \quad 2 \langle k_1 + | k_2 - \rangle \langle p_2 - | p_2 + \rangle \\ & = \quad 0 \end{aligned} \tag{707}$$

because the 2nd inner product is 0.
The net result is that $\mathcal{A}(+,+,+,-) = 0$.

$\boxed{\textbf{What about } \mathcal{A}_{s,t,u}(-,-,+,-)\textbf{?}}$

Starting with (as before)

$$\mathcal{A}_t(\lambda_1, \lambda_2, +, -) = \langle p_1 + | \not\epsilon_1(\lambda_1)^* (\not p_1 + \not k_1) \not\epsilon_2(\lambda_2)^* | p_2 + \rangle \,, \tag{708}$$

we choose p_1 for the $\epsilon_1(-)$ reference momentum and then note that

$$\langle p_1 + | \not\epsilon_1^-(k_1, p_1)^* = \langle p_1 + | \left(-\sqrt{2} \left[\frac{|k_1+\rangle\langle p_1 + | + |p_1-\rangle\langle k_1 - |}{\ldots} \right] \right) \tag{709}$$

which gives 0.

Similarly, for $\mathcal{A}_u$ we will get zero if we choose p_1 as the reference momentum for $\epsilon_2(-)$.

As in the previous $\lambda_1 = \lambda_2 = +$ case, we will also find that the s-channel diagram is zero in the present $\lambda_1 = \lambda_2 = -$ case when both $\epsilon_1(-)$ and $\epsilon_2(-)$ are referenced to p_1.

Note: In case you have not noticed, we have been very careful to choose the same reference momentum (same gauge) for all diagrams contributing to a given helicity configuration. We are free to change the gauge when computing the diagrams for a different helicity configuration. *This is allowed, since each helicity configuration is an independent physical observable.*

Thus, by a somewhat clever choice of gauge, we have been able to show that all helicity amplitudes other than $\mathcal{A}(+,-,+,-)$ and

$\mathcal{A}(-,+,+,-)$ (and the parity flips thereof) are zero.

$\boxed{\text{So, now let us work on } \mathcal{A}_{s,t,u}(+,-,+,-).}$

We will choose reference momentum p_1 for ϵ_2 and p_2 for ϵ_1. This is ok for this and similar processes with some external fermions. Of course, we must keep these choices for all three diagrams.

The required ϵ-related forms are then

$$
\begin{aligned}
\not{\epsilon}_2^-(k_2,p_1)^* &= -\sqrt{2}\frac{|k_2+\rangle\langle p_1+| + |p_1-\rangle\langle k_2-|}{\langle p_1+|k_2-\rangle} \\
\not{\epsilon}_1^+(k_1,p_2)^* &= +\sqrt{2}\frac{|k_1-\rangle\langle p_2-| + |p_2+\rangle\langle k_1+|}{\langle p_2-|k_1+\rangle} \\
\epsilon_2^-(k_2,p_1)^*_\beta &= -\frac{\langle k_2-|\gamma_\beta|p_1-\rangle}{\sqrt{2}\langle p_1+|k_2-\rangle} \\
\epsilon_1^+(k_1,p_2)^*_\gamma &= +\frac{\langle k_1+|\gamma_\gamma|p_2+\rangle}{\sqrt{2}\langle p_2-|k_1+\rangle}
\end{aligned} \tag{710}
$$

We start by noting that $\mathcal{A}_u = 0$ by virtue of the fact that

$$\langle p_1+| \not\epsilon_2^{-\,*}(k_2,p_1) = \langle p_1+| \left(-\sqrt{2}\frac{|k_2+\rangle\langle p_1+| + |p_1-\rangle\langle k_2-|}{\langle p_1+|k_2-\rangle} \right) = 0 \tag{711}$$

using $\langle p_1+|p_1-\rangle = 0$.
Next we consider $\mathcal{A}_t$. For this non-zero amplitude we write out all the details. We will use repeatedly the fact that $\langle a\pm|b\pm\rangle = 0$.

$$\begin{aligned}
\mathcal{A}_t(+,-,+,-) &= \langle p_1+|\not\epsilon_1^{+}(k_1,p_2)^*(ig_s)\frac{i(\not p_1+\not k_1)}{(p_1+k_1)^2}\not\epsilon_2^{-}(k_2,p_1)^*(ig_s)|p_2+\rangle \\
&= -ig_s^2(+\sqrt{2})(-\sqrt{2})\frac{\langle p_1+|k_1-\rangle}{\langle p_2-|k_1+\rangle}\frac{\langle p_2-|(\not p_1+\not k_1)|p_1-\rangle}{2p_1\cdot k_1}\frac{\langle k_2-|p_2+\rangle}{\langle p_1+|k_2-\rangle} \\
&= 2ig_s^2\frac{\langle p_1+|k_1-\rangle}{\langle p_2-|k_1+\rangle}\frac{\langle p_2-|(0+|k_1+\rangle\langle k_1+|)|p_1-\rangle}{\langle p_1+|k_1-\rangle\langle k_1-|p_1+\rangle}\frac{\langle k_2-|p_2+\rangle}{\langle p_1+|k_2-\rangle} \\
&= 2ig_s^2\frac{\langle k_1+|p_1-\rangle\langle k_2-|p_2+\rangle}{\langle k_1-|p_1+\rangle\langle p_1+|k_2-\rangle}
\end{aligned} \tag{712}$$

The s-channel amplitude takes the form, now including all factors carefully:

$$\begin{aligned}
\mathcal{A}_s &= ig_s\langle p_1+|\gamma_\alpha|p_2+\rangle\epsilon_2^{-}(k_2,p_1)^*_\beta\epsilon_1^{+}(k_1,p_2)^*_\gamma\left[\frac{-ig_{\alpha\alpha'}}{(k_1+k_2)^2}\right]\times \\
&\qquad (-ig_s)\left[g_{\gamma\alpha'}(-2k_1-k_2)_\beta + g_{\gamma\beta}(k_1-k_2)_{\alpha'} + (2k_1+k_2)_\gamma g_{\alpha'\beta}\right]
\end{aligned}$$

$$= \quad \frac{-ig_s^2}{(k_1+k_2)^2}\langle p_1+|(\not{k}_1-\not{k}_2)|p_2+\rangle\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\frac{2\langle k_2-|p_2+\rangle\langle k_1+|p_1-\rangle}{\langle p_1+|k_2-\rangle\langle p_2-|k_1+\rangle} \quad (713)$$

where, in the above, we used the explicit ϵ forms given for this case earlier and noted that

$$\langle p_1+|\not{\epsilon}_2^{-\,*}(k_2,p_1)|p_2+\rangle = \langle p_1+|\not{\epsilon}_1^{-\,*}(k_1,p_2)|p_2+\rangle = 0 \quad (714)$$

for the reference momentum choices made, implying that we only had to keep the $g_{\gamma\beta}$ term in the vertex component in the above reduction. Now, we use $k_2 = -(p_1+p_2+k_1)$ and $\langle p_1+|\not{p}_1 = \not{p}_2|p_2+\rangle = 0$ to rewrite

$$\begin{aligned}\langle p_1+|(\not{k}_1-\not{k}_2)|p_2+\rangle &= \langle p_1+|(2\not{k}_1)|p_2+\rangle \\ &= 2\langle p_1+|k_1-\rangle\langle k_1-|p_2+\rangle \\ &= -2\langle p_1+|k_1-\rangle\langle p_2-|k_1+\rangle\,. \quad (715)\end{aligned}$$

Inserting this into Eq. (713), and canceling a common $\langle p_2-|k_1+\rangle$ in numerator and denominator, we obtain

$$\mathcal{A}_s(+,-,+,-) = -\frac{2ig_s^2}{2k_1\cdot k_2}\frac{\langle p_1+|k_1-\rangle\langle k_2-|p_2+\rangle\langle k_1+|p_1-\rangle}{\langle p_1+|k_2-\rangle} \quad (716)$$

$\boxed{\text{Finally we have } \mathcal{A}_{s,t,u}(-,+,+,-)}$

This can be obtained from the $\mathcal{A}_{s,t,u}(+,-,+,-)$ amplitudes by noting that the amplitudes differ by the interchange of gluon-1 and gluon-2, implying that

$$\mathcal{A}_t(-,+,+,-)(k_1,k_2) = \mathcal{A}_u(+,-,+,-)(k_2,k_1) = 0 \quad (717)$$

and

$$\begin{aligned}\mathcal{A}_u(-,+,+,-)(k_1,k_2) &= \mathcal{A}_t(+,-,+,-)(k_2,k_1)\\ &= 2ig_s^2\frac{\langle k_2+|p_1-\rangle\langle k_1-|p_2+\rangle}{\langle k_2-|p_1+\rangle\langle p_1+|k_1-\rangle} \quad (718)\end{aligned}$$

and (remembering that the non-color part of the 3-gluon vertex is

antisymmetric under such an interchange)

$$\begin{aligned}\mathcal{A}_s(-,+,+,-)(k_1,k_2) &= -\mathcal{A}_s(+,-,+,-)(k_2,k_1)\\ &= +\frac{2ig_s^2}{2k_1\cdot k_2}\frac{\langle p_1+|k_2-\rangle\langle k_1-|p_2+\rangle\langle k_2+|p_1-\rangle}{\langle p_1+|k_1-\rangle}\end{aligned} \tag{719}$$

Thus, we have computed all the momentum space structures by using tricks to get many zeroes and only, in the end, had to perform 2 non-trivial computations, both of which reduced to relatively simple forms in terms of the inner products.

Note that these inner products could be easily computed numerically for given choices of the momenta generated inside some Monte Carlo program.

Here, however, we want to continue so as to get the final simple analytic form for the cross section. This involves combining the color stuff with these momentum space amplitudes. The process is a bit tricky and fairly revealing.

Let us simplify our notation somewhat before proceeding. Our non-zero

helicity amplitudes are

$$\mathcal{A}_t(+-+-;k_1,k_2) = \mathcal{A}_u(-++-;k_2,k_1) \equiv \mathcal{A}_t(1,2)$$
$$\mathcal{A}_s(+-+-;k_1,k_2) = -\mathcal{A}_s(-++-,k_2,k_1) \equiv \mathcal{A}_s(1,2)\,. \tag{720}$$

Our two helicity amplitudes are then, bringing back in the color factors,

$$\mathcal{M}(+-+-)(k_1,k_2) = \mathcal{A}_t(1,2)C_t + \mathcal{A}_s(1,2)C_s$$
$$\mathcal{M}(-++-)(k_1,k_2) = \mathcal{A}_t(2,1)C_u - \mathcal{A}_s(2,1)C_s\,. \tag{721}$$

We wish to sum the absolute squares of these two physically distinct helicity amplitudes. We have

$$\begin{aligned} |\mathcal{M}(+-+-)|^2 &= |\mathcal{A}_t(1,2)|^2 C_t^2 + |\mathcal{A}_s(1,2)|^2 C_s^2 + 2\mathrm{Re}\mathcal{A}_t(1,2)\mathcal{A}_s^*(1,2)C_tC_s \\ |\mathcal{M}(-++-)|^2 &= |\mathcal{A}_t(2,1)|^2 C_u^2 + |\mathcal{A}_s(2,1)|^2 C_s^2 - 2\mathrm{Re}\mathcal{A}_t(2,1)\mathcal{A}_s^*(2,1)C_uC_s\,. \end{aligned} \tag{722}$$

We now recall our color factors. Including the $1/64$ for initial gluon color averaging, we have

$$C_t^2 = C_u^2 = \frac{1}{12}, \quad C_s^2 = \frac{3}{16}, \quad C_t C_s = -C_u C_s = -\frac{3}{32}. \tag{723}$$

Putting this into the previous equation gives

$$\begin{aligned} |\mathcal{M}(+-+-)|^2 + |\mathcal{M}(-++-)|^2 &= \frac{1}{12}\left(|\mathcal{A}_t(1,2)|^2 + |\mathcal{A}_t(2,1)|^2\right) \\ &+ \frac{3}{16}\left(|\mathcal{A}_s(1,2)|^2 + |\mathcal{A}_s(2,1)|^2 - \mathrm{Re}\mathcal{A}_t(1,2)\mathcal{A}_s^*(1,2) - \mathrm{Re}\mathcal{A}_t(2,1)\mathcal{A}_s^*(2,1)\right) \end{aligned} \tag{724}$$

which exhibits the expected final symmetry under interchange of the two initial gluons. (Note how the $-$ sign of $C_u C_s$ made up for the extra $-$ sign in the momentum-space calculation of $\mathcal{A}_s(-+-+)$.)

The final task is then to compute explicitly the various $\mathcal{A}$ products. This is easy using $|\langle a+|b-\rangle|^2 = 2a\cdot b$. We have

$$|\mathcal{A}_t(1,2)|^2 = 4g_s^4 \frac{2k_1\cdot p_1 2k_2\cdot p_2}{2k_1\cdot p_1 2p_1\cdot k_2} = 4g_s^4\left(\frac{t}{u}\right), \tag{725}$$

$$|\mathcal{A}_t(2,1)|^2 = 4g_s^4\left(\frac{u}{t}\right) \quad \text{since } u \leftrightarrow t \text{ under } k_1 \leftrightarrow k_2\,, \tag{726}$$

$$|\mathcal{A}_s(1,2)|^2 = \frac{4g_s^4}{s^2}\frac{2k_2\cdot p_2(2k_1\cdot p_1)^2}{2p_1\cdot k_2} = \frac{4g_s^4}{s^2}\left(\frac{t^3}{u}\right)\,, \tag{727}$$

$$|\mathcal{A}_s(2,1)|^2 = \frac{4g_s^4}{s^2}\left(\frac{u^3}{t}\right)\,, \tag{728}$$

and, finally, the only tricky case,

$$\begin{aligned}
\mathcal{A}_t^*(1,2)\mathcal{A}_s(1,2) &= -\frac{4g_s^4}{s}\frac{\langle p_1-|k_1+\rangle\langle p_2+|k_2-\rangle}{\langle p_1+|k_1-\rangle\langle k_2-|p_1+\rangle}\frac{\langle p_1+|k_1-\rangle\langle k_2-|p_2+\rangle\langle k_1+|p_1-\rangle}{\langle p_1+|k_2-\rangle} \\
&= -\frac{4g_s^4}{s}\frac{2p_1\cdot k_1 2p_2\cdot k_2}{2p_1\cdot k_2} \\
&= -\frac{4g_s^4}{s}\frac{t^2}{u}\,,
\end{aligned} \tag{729}$$

where we canceled a common factor of $\langle p_1+|k_1-\rangle$ in numerator and denominator and then used 3 versions of

$$|\langle a-|b+\rangle|^2 = \langle a-|b+\rangle\langle b+|a-\rangle = 2a\cdot b. \tag{730}$$

Further,

$$\mathcal{A}_t(1,2)\mathcal{A}_s^*(1,2) = \mathcal{A}_t^*(1,2)\mathcal{A}_s(1,2)\,. \tag{731}$$

Inserting these results into Eq. (724), we obtain

$$|\mathcal{M}(+-+-)|^2+|\mathcal{M}(-++-)|^2 = 4g_s^4\left[\frac{1}{12}\left(\frac{t}{u}+\frac{u}{t}\right)+\frac{3}{16}\left(\frac{u^3}{s^2t}+\frac{t^3}{s^2u}+\frac{t^2}{su}+\frac{u^2}{st}\right)\right]. \quad (732)$$

This can be further simplified by noting that

$$\frac{t^3}{s^2u}+\frac{t^2}{su} = \frac{1}{s^2u}t^2(t+s)=\frac{1}{s^2u}t^2(-u)=-\frac{t^2}{s^2}, \quad (733)$$

and, similarly,

$$\frac{u^3}{s^2t}+\frac{u^2}{st}=-\frac{u^2}{s^2}, \quad (734)$$

resulting in

$$\frac{1}{4}\sum_{\lambda_1,\lambda_2,s_1,s_2}|\mathcal{M}(\lambda_1,\lambda_2,s_2,s_2)|^2 = \frac{1}{4}2\left(|\mathcal{M}(+,-,+,-)|^2+|\mathcal{M}(-,+,+,-)|^2\right)$$
$$= g_s^4\left[\frac{1}{6}\left(\frac{t}{u}+\frac{u}{t}\right)-\frac{3}{8}\frac{u^2+t^2}{s^2}\right], \quad (735)$$

where the extra factor of 2 comes from including also the $(-,+,-,+)$ and $(+,-,-,+)$ squared amplitudes which are equal to the $(+,-,+,-)$

and $(-,+,+,-)$ squared amplitudes, respectively, (due to parity invariance)[6], and the $1/4$ is for averaging over the initial gluon helicities. The final result for the helicity-averaged and color-averaged cross section is then:

$$\begin{aligned}
\frac{d\sigma}{dt} &= \frac{1}{16\pi s^2}\frac{1}{4}\sum_{\lambda_1,\lambda_2,s_1,s_2}\left|\mathcal{M}_(\lambda_1,\lambda_2,s_2,s_2)\right|^2 \\
&= \frac{(4\pi\alpha_s)^2}{16\pi s^2}\left[\frac{1}{6}\left(\frac{t}{u}+\frac{u}{t}\right)-\frac{3}{8}\frac{u^2+t^2}{s^2}\right] \\
&= \frac{\pi\alpha_s^2}{s^2}\left[\frac{1}{6}\left(\frac{t}{u}+\frac{u}{t}\right)-\frac{3}{8}\frac{u^2+t^2}{s^2}\right] \\
&= \frac{\pi\alpha_s^2}{s^2}\left[\frac{1}{6}\frac{t^2+u^2}{tu}-\frac{3}{8}\frac{u^2+t^2}{s^2}\right],
\end{aligned} \tag{736}$$

which is the promised result.

- **From this result, we can also get $\frac{d\sigma}{dt}$ for other processes related by crossing. For example, consider $gq \to gq$. If we denote with primes the**

[6] Parity changes $\vec{p} \to -\vec{p}$ but leaves the spin (like angular momentum) unchanged. As a result, helicity is changed to minus helicity for each particle. QCD is parity invariant since there are no γ_5's floating around. Thus, amplitudes related to one another by a parity transformation should be equal (up to a possible phase which we don't care about).

s', t', u' variables for the $gq \to gq$ process, we have

$$s' = t\,, \quad u' = u\,, \quad t' = s\,. \tag{737}$$

We must also account for the change in color averaging. The color averaging factor of $1/(N^2-1)^2 = 1/64$ for $gg \to q\overline{q}$ must be replaced by $1/[N(N^2-1)] = 1/24$ for $gq \to gq$. The ratio gives a factor of $8/3$. Thus, up to a sign that I will discuss shortly, we have (dropping the prime notation)

$$\begin{aligned} \frac{d\sigma}{dt}^{gq\to gq}(s,t,u) &= \frac{8}{3}\frac{d\sigma}{dt}^{gg\to q\overline{q}}(t,s,u) \\ &= \frac{8}{3}\frac{\pi\alpha_s^2}{s^2}\left(\frac{1}{6}\frac{u^2+s^2}{su} - \frac{3}{8}\frac{u^2+s^2}{t^2}\right) \\ &= -\frac{\pi\alpha_s^2}{s^2}\left(\frac{u^2+s^2}{t^2} - \frac{4}{9}\frac{u^2+s^2}{us}\right)\,. \end{aligned} \tag{738}$$

This expression is obviously < 0, and so there was some subtlety that the above argument misses. To get the sign straight, we must return to the underlying procedures and recognize an important subtlety.

The subtlety is that in crossing the antiquark in $gg \to q\overline{q}$ to a quark in $gq \to gq$ we must change the sign of the quark momentum. Using the quark labeled with p for this and taking $p \to -p$, this means that

$$\sum_s v(p,s)\overline{v}(p,s) = \not{p} \to -\not{p} = -\sum_s u(p,s)\overline{u}(p,s) \tag{739}$$

whereas what should actually appear in $gq \to gq$ is

$$+\sum_s u(p,s)\overline{u}(p,s) = +\not{p}\,. \tag{740}$$

In terms of the notation we have adopted here, this means that the final outgoing antiquark momentum p_2 is crossed to an incoming quark with outgoing momentum $-p_2$, whereas (in our all outgoing notation) we want this momentum to be $+p_2$.

This means that we must also change the sign of $\sum_{\text{colors,spins}} |\mathcal{M}|^2$ at the same time that we perform the $s \leftrightarrow t$ interchange, yielding

$$\frac{d\sigma}{dt} = +\frac{\pi\alpha_s^2}{s^2}\left(\frac{u^2+s^2}{t^2} - \frac{4}{9}\frac{u^2+s^2}{us}\right)\,. \tag{741}$$