230ABC Problems

For 230B, Winter 2016 The next problems now assigned and due Wednesday of next week are problems 25 and 26. This problem file reflects the current due dates.

Grading: Unless indicated otherwise, each problem is due one week after it is assigned. Problems handed in up to the day of the class following the class day when the problems were first due (e.g. Wednesday if due on Monday, or Monday if due on Friday) will receive 60% credit. After that, no credit. Problems will constitute a sizable (about 50% to 60%) fraction of your final grade.

The problems 1-16 are those I normally assign during 230A. But, for Winter 2016 230B, we will be doing problems 1, 12, 14, followed by problems 19, ...

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1. Assigned 1/4/2016; Due 1/11/2016. (20 points)

   Mandl-Shaw Problem 1.2

   The Lagrangian of a particle of mass m and charge q, moving in an electromagnetic field, is given by
   $$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} m \dot{\vec{x}}^2 + \frac{q}{c} \vec{A} \cdot \dot{\vec{x}} - q\phi \tag{1}$$
   where $\vec{A} = \vec{A}(\vec{x}, t)$ and $\phi = \phi(\vec{x}, t)$ are the vector and scalar potentials of the electromagnetic field at the position $\vec{x}$ of the particle at time t.

   (i) Show that the momentum conjugate to $\vec{x}$ is given by
   $$\vec{p} = m\dot{\vec{x}} + \frac{q}{c}\vec{A} \tag{2}$$
   (i.e. the conjugate momentum $\vec{p}$ is not the kinetic momentum $m\dot{\vec{x}}$ in general) and that Lagrange's equations reduce to the equations of motion of the particle
   $$m\frac{d}{dt}\dot{\vec{x}} = q\left[\vec{E} + \frac{1}{c}\dot{\vec{x}} \wedge \vec{B}\right], \tag{3}$$
   where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at the instantaneous position of the charged particle.

   (ii) Derive the corresponding Hamiltonian:
   $$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi, \tag{4}$$
   and show that the resulting Hamilton equations again lead to Eqs. (2) and (3).

2. Assigned 4/10/12; Due 4/17/12.

   Mandl-Shaw Problems 2.2,2.3,2.4

   2.2: The real Klein-Gordon field is described by the Hamiltonian density
   $$\mathcal{H}(x) = \frac{1}{2}\left[c^2\pi^2(x) + (\vec{\nabla}\phi(x))^2 + \mu^2\phi^2(x)\right] \tag{5}$$
   Use the commutation relations
   $$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\hbar\delta^3(\vec{x} - \vec{x}'), \quad [\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \tag{6}$$

to show that

$$[H, \phi(x)] = -i\hbar c^2 \pi(x)\,, \quad [H, \pi(x)] = i\hbar(\mu^2 - \vec{\nabla}^2)\phi(x)\,, \tag{7}$$

where H is the Hamiltonian of the field. From this result and the Heisenberg equations of motion for the operators $\phi(x)$ and $\pi(x)$, show that

$$\dot{\phi}(x) = c^2\pi(x)\,, \quad (\Box + \mu^2)\phi(x) = 0\,. \tag{8}$$

2.3: Show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}[\partial_\alpha\phi_\beta(x)][\partial^\alpha\phi^\beta(x)] + \frac{1}{2}[\partial_\alpha\phi^\alpha(x)][\partial_\beta\phi^\beta(x)] + \frac{1}{2}\mu^2\phi_\alpha(x)\phi^\alpha(x) \tag{9}$$

for the real vector field $\phi^\alpha(x)$ leads to the field equations

$$[g_{\alpha\beta}(\Box + \mu^2) - \partial_\alpha\partial_\beta]\phi^\beta(x) = 0\,, \tag{10}$$

and that the field $\phi^\alpha(x)$ satisfies the Lorentz condition

$$\partial_\alpha\phi^\alpha(x) = 0,. \tag{11}$$

2.4: Use the commutation relations

$$[\phi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = i\hbar\delta^3(\vec{x} - \vec{x}')\delta_{rs}\,, \quad [\phi_r(\vec{x}, t), \phi_s(\vec{x}', t)] = [\pi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = 0 \tag{12}$$

to show that the momentum operator of the fields

$$P^j = \int d^3\vec{x}\pi_r(x)\frac{\partial\phi_r(x)}{\partial x_j} \tag{13}$$

satisfies the equations

$$[P^j, \phi_r(x)] = -i\hbar\frac{\partial\phi_r(x)}{\partial x_j}\,, \quad [P^j, \pi_r(x)] = -i\hbar\frac{\partial\pi_r(x)}{\partial x_j}\,. \tag{14}$$

Hence, show that any operator $F(x) = F(\phi_r(x), \pi_r(x))$, which can be expanded in a power series in the field operators $\phi_r(x)$ and $\pi_r(x)$, satisfies

$$[P^j, F(x)] = -i\hbar\frac{\partial F(x)}{\partial x_j}\,. \tag{15}$$

Note that we can combine these equations with the Heisenberg equation of motion for the operator $F(x)$

$$[H, F(x)] = -i\hbar\frac{\partial F(x)}{\partial x_0} \tag{16}$$

to obtain the covariant equations of motion

$$[P^\alpha, F(x)] = -i\hbar\frac{\partial F(x)}{\partial x_\alpha}\,, \tag{17}$$

where $P^0 = H/c$.

3. Assigned 4/12/2012; Due 4/19/2012.

   Mandl-Shaw Problem 3.1 plus proof that $H = \sum_{\vec{k}}\omega_{\vec{k}}\left[N(\vec{k}) + \frac{1}{2}\right]$ and $\vec{P} = \sum_{\vec{k}}\vec{k}N(\vec{k})$ for the real scalar field.

   Problem 3.1 reads:

From the expansion for the real Klein-Gordon field $\phi(x)$ in terms of the a and $a^\dagger$ operators, derive the following expression for the absorption operator $a(\vec{k})$:

$$a(\vec{k}) = \frac{1}{(2V\omega_{\vec{k}})^{1/2}} \int d^3\vec{x} e^{ik\cdot x}[i\dot{\phi}(x) + \omega_{\vec{k}}\phi(x)] \,, \tag{18}$$

where I am using $\hbar = c = 1$ notation. Use this expression to derive the commutation relations for the creation and annihilation operators, $a^\dagger$ and a, from the commutation relations for the fields, $\phi(x)$ and $\pi(x)$.

4. Assigned 4/17/12; Due 4/24/12.

   Mandl-Shaw Problem 3.2. Problem 3.2 reads as follows:

   With the complex Klein-Gordon fields $\phi(x)$ and $\phi^\dagger(x)$ expressed in terms of two *independent* real KG fields $\phi_1(x)$ and $\phi_2(x)$ by

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \,, \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \,, \tag{19}$$

   and with $\phi_r(x)$ $(r = 1, 2)$ expanded in the form

$$\phi_r(x) = \sum_{\vec{k}} \left(\frac{1}{2V\omega_{\vec{k}}}\right)^{1/2} \left[a_r(\vec{k})e^{-ik\cdot x} + a_r^\dagger(\vec{k})e^{ik\cdot x}\right] \,, \tag{20}$$

   show that

$$a(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) + ia_2(\vec{k})\right] \,, \quad b(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - ia_2(\vec{k})\right] \,. \tag{21}$$

   Next, derive the commutation relations for the ϕ and $\phi^\dagger$ fields and their conjugate momenta from those for the $\phi_{r=1,2}$ fields and their conjugate momenta (without reference to their operator decompositions).

   Finally, derive the commutation relations for the $a, a^\dagger, b, b^\dagger$ operators from those for $a_1, a_1^\dagger, a_2, a_2^\dagger$.

5. Assigned 4/17/12; Due 4/24/12.

   Derive the expression for the Q operator given in the notes,

$$Q = q\sum_{\vec{k}} \left[N_a(\vec{k}) - N_b(\vec{k})\right] \,, \tag{22}$$

   starting from its expression in terms of fields,

$$Q = -iq \int d^3\vec{x} : \left[\dot{\phi}^\dagger(x)\phi(x) - \dot{\phi}(x)\phi^\dagger(x)\right] : \,, \tag{23}$$

   where you should note the normal ordering instruction. However, please show what you get without normal ordering before you give the normal-ordered version.

   Now consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Using Noether's theorem, show that there are now four conserved charges, one given by the generalization of Q, and the other three given by

$$Q^i = \frac{i}{2}\int d^3x (\phi_a^\dagger(\sigma^i)_{ab}\pi_b^\dagger - \pi_a(\sigma^i)_{ab}\phi_b) \,, \tag{24}$$

   where the σ^i are the Pauli sigma matrices, and I have chosen a normalization for the Q^i that allows the connection to $SU(2)$ below to be obvious. Show that these three charges have the commutation relations of angular momentum, $SU(2)$. This extra part of the problem will give you experience with invariances of $\mathcal{L}$ in a space that has no physical interpretation (the $SU(2)$ is not actual spin, but some other spin — it could be, for example, isospin in one possible context).

6. Assigned 4/l9/2012; Due 4/26/2012.

   Compute a different Green's function or propagator for the scalar field theory defined by

$$\Delta_{new}(x) = \frac{1}{(2\pi)^4} \int_{C_{new}} d^4k \frac{e^{-ik\cdot x}}{k^2 - \mu^2} \tag{25}$$

   where C_{new} is a contour that passes *above* both the $-\omega_{\vec{k}}$ and the $+\omega_{\vec{k}}$ poles. Give an expression for Δ_{new} in terms of Δ^+ and Δ^-.

7. Assigned 4/26/12; Due 5/3/12.

   Demonstrate that the $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ matrices obey the commutation relationship required for them to be a representation of the Lorentz Lie group generator algebra:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\,[g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}]$$

8. Assigned 4/26/12; Due 5/3/12.

   Demonstrate that the structure $\overline{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)$ appearing in the Dirac $\mathcal{L}$ is: a) Hermitian; and b) Lorentz invariant. For the latter, you will need the discussion of the lecture notes.

9. Assigned 5/1/12; Due 5/14/12.

   Derive the expression for Q in terms of operators, namely $Q = q\sum_{r\vec{p}}[N_r(\vec{p}) - \overline{N}_r(\vec{p})]$, as given in the lecture notes.

10. Assigned 5/1/12; Due 5/14/12.

    Prove the result stated in the text. Namely, show that

$$[\psi^\pm_\alpha(x), \overline{\psi}^\mp_\beta(y)]_+ = i(i\partial\!\!\!/ + m)_{\alpha\beta}\Delta^\pm(x-y)\,. \tag{26}$$

    In principle, you could use this problem as an opportunity to learn a bit more about the $u_r(\vec{p})$ and $v_r(\vec{p})$ spinors, from Appendices A.4 and A.5. However, to do this problem you really only need to know two results, which I allow you to assume without proof. For my spinor normalization conventions (i.e. $2E_{\vec{p}}$ vs. $E_{\vec{p}}/m$) one has $\sum_r u_r(\vec{p})\overline{u}_r(\vec{p}) = p\!\!\!/ + m$ (vs. Mandl-Shaw $(p\!\!\!/ + m)/(2m)$) and $\sum_r v_r(\vec{p})\overline{v}_r(\vec{p}) = p\!\!\!/ - m$. We will prove this and other results at a later time.

11. Assigned 5/2/12; Due 5/14/12.

    Prove that $[\mathcal{O}^i(x), \mathcal{O}^j(y)] = 0$ if $(x-y)^2 < 0$, where the operators are of the form $\mathcal{O}^i(x) = \overline{\psi}(x)\Gamma^i\psi(x) = \overline{\psi}_\alpha(x)\Gamma^i_{\alpha\beta}\psi_\beta(x)$ with Γ^i being any 4×4 matrix. The final form exposes the Dirac indices on the fields and on the Γ^i matrix.

    Problem 4.3 of Mandl-Shaw is a special case of this more general result.

12. Assigned 1/4/2016; Due 1/11/2016. (20 points)

    Repeat the minimal substitution rule, $\partial_\mu \to \partial_\mu + iqA_\mu$, game for the charged scalar field Lagrangian and obtain the interaction Lagrangian describing the interactions between the charged scalar field and the electromagnetic field.

    Then define the appropriate *local* gauge transformation of the ϕ field that leaves the *full* Lagrangian invariant when $A_\mu \to A_\mu + \partial_\mu f(x)$, proving that what you define works.

13. Assigned 5/2/12; Due 5/14/12.

    Do problem 4.5 of Mandl-Shaw. Problem 4.5 reads as follows.

    For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x)\,, \quad \psi^\dagger(x) \to \psi^{\dagger\prime}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5}\,, \tag{27}$$

where α is an arbitrary real parameter, are called chiral phase transformations. Show that the Lagrangian density $\overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $j_A^\mu(x) = \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)$.

In the above, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. For this part of the problem, the only thing you need to know about γ^5 is that it anticommutes with all the regular gamma matrices: $[\gamma^\mu, \gamma^5]_+ = 0$ for any μ and that $(\gamma^5)^2 = 1$. (You should check that these statements are true given the definition of γ^5.)

Further, deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(1-\gamma^5)\psi(x)\,, \quad \psi_R(x) \equiv \frac{1}{2}(1+\gamma^5)\psi(x) \tag{28}$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Using this last result, show that the Lagrangian density

$$\mathcal{L}(x) = \overline{\psi}_L(x) i\partial\!\!\!/ \psi_L(x) \tag{29}$$

describes zero-mass fermions with negative helicity only, and zero-mass antifermions with positive helicity only. (This ψ_L field is called a Weyl field and it was used to describe the neutrinos in weak interactions when we thought they had zero mass. Because the neutrino masses are so small, this is still a useful starting point.) For this part of the problem, you need to know one more fact about the spinors u and v: namely, in the massless limit, $\gamma^5 u_r(\vec{p}) = \Sigma_{\vec{p}} u_r(\vec{p})$ and $\gamma^5 v_r(\vec{p}) = \Sigma_{\vec{p}} v_r(\vec{p})$, i.e. γ^5 and the helicity operator $\Sigma_{\vec{p}}$ are equivalent in the massless limit. This can be shown starting from the massless Dirac equation as follows. In the following, $w_r(\vec{p})$ stands for either $u_r(\vec{p})$ or $v_r(\vec{p})$. The massless Dirac equation says that

$$p\!\!\!/ w_r(\vec{p}) = 0\,. \tag{30}$$

This, with $E_{\vec{p}} = |\vec{p}|$, can be rewritten as

$$\gamma^0|\vec{p}|w_r(\vec{p}) = -p_k\gamma^k w_r(\vec{p}) = +p^k\gamma^k w_r(\vec{p})\,, \tag{31}$$

where $k = 1, 2, 3$ is summed over. Now, premultiply by $\gamma^5\gamma^0$ to obtain

$$|\vec{p}|\gamma^5 w_r(\vec{p}) = p^k\gamma^5\gamma^0\gamma^k w_r(\vec{p})\,. \tag{32}$$

Using the definition of γ^5 and the definition of $\Sigma^k = \sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j]$, where kij are in cyclic order, one can show that $\gamma^5\gamma^0\gamma^k = \Sigma^k$. (For instance, $\gamma^5\gamma^0\gamma^1 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1 = -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1 = -i(\gamma^1)^2\gamma^2\gamma^3 = +i\gamma^2\gamma^3 = +\frac{i}{2}[\gamma^2, \gamma^3] = \Sigma^1$.) Substituting this into the above equation gives

$$|\vec{p}|\gamma^5 w_r(\vec{p}) = \vec{\Sigma}\cdot\vec{p}\, w_r(\vec{p})\,, \tag{33}$$

which, after dividing by $|\vec{p}|$ becomes

$$\gamma^5 w_r(\vec{p}) = (\vec{\Sigma}\cdot\hat{p})\, w_r(\vec{p}) \equiv \Sigma_{\vec{p}}\, w_r(\vec{p})\,, \tag{34}$$

the required ingredient for the proof.

14. Assigned 1/6/2016; Due 1/13/2016.

Do problems 5.1 (20 points) and 5.3 (30 points) of Mandl-Shaw.

Problem 5.1 reads as follows.

Show that the Lagrangian density obtained from

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{35}$$

by adding the term $-\frac{1}{2}(\partial_\mu A^\mu(x))(\partial_\nu A^\nu(x))$, i.e.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}(\partial_\mu A^\mu(x))(\partial_\nu A^\nu(x)) \tag{36}$$

is equivalent to the Lagrangian density proposed by Fermi:

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x))\,. \tag{37}$$

Problem 5.3 reads as follows.

Let $|\Psi_T\rangle$ be a state which contains transverse photons only. Let

$$|\Psi'_T\rangle = \left\{1 + c\left[a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})\right]\right\}|\Psi_T\rangle\,, \tag{38}$$

where c is some constant. Show that replacing $|\Psi_T\rangle$ by $|\Psi'_T\rangle$ corresponds to a gauge transformation, *i.e.*

$$\langle\Psi'_T|A^\mu(x)|\Psi'_T\rangle = \langle\Psi_T|\left[A^\mu(x) + \partial^\mu\Lambda(x)\right]|\Psi_T\rangle\,, \tag{39}$$

where

$$\Lambda(x) = \left(\frac{2}{V\omega_{\vec{k}}^3}\right)^{1/2}\mathrm{R}e(ice^{-ik\cdot x})\,. \tag{40}$$

15. (15 pts) Not assigned Winter 2016.

    As an explicit check of Wicks theorem for four scalar fields, show explicitly that

    $$T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \sum \text{ all contractions :}\,,$$

    assuming $x_2^0 > x_1^0 > x_3^0 > x_4^0$.

16. (20 points) Not assigned Winter 2016

    Derive the expression given in the notes for the contribution of the left-hand "s-channel" diagram of Fig. 2 (in QFT-II.pdf) to $e^+\gamma \to e^+\gamma$:

    $$\begin{aligned} &+e^2(2\pi)^4\delta^4(p'+k'-p-k)\overline{v}_r(\vec{p})\not{\epsilon}_s(\vec{k})\left[i\frac{\not{q}+m}{q^2-m^2}\right]_{q=-k-p}\not{\epsilon}_{s'}(\vec{k}')v_{r'}(\vec{p}') \\ &\times\frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}}\,. \end{aligned} \tag{41}$$

    Make sure the derivation is as detailed as that done in the notes for the right-hand u-channel diagram.

Problems above are typically for 230A; problems below and down to about 29 are for 230B, but some are typically skipped.

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17. (40 pts) Not assigned Winter 2016

    This is an enhanced version of problem 7.1 of Mandl-Shaw.

    Problem 7.1 says:

    Derive the lowest-order non-vanishing S-matrix element and hence the corresponding Feynman amplitude for Bhabha scattering, i.e the process

    $$e^+(\vec{p}_1, r_1) + e^-(\vec{p}_2, r_2) \to e^+(\vec{p}_1{}', s_1) + e^-(\vec{p}_2{}', s_2) \tag{42}$$

The result you are supposed to prove for the lowest order S-matrix element is

$$S^{(2)}(e^+e^- \to e^+e^-) = S_a + S_b\,, \tag{43}$$

with

$$S_a = -e^2 \int d^4x_1\, d^4x_2 : (\overline{\psi}^- \gamma^\alpha \psi^+)_{x_1} (\overline{\psi}^+ \gamma^\beta \psi^-)_{x_2} : iD_{F\,\alpha\beta}(x_1 - x_2) \tag{44}$$

$$S_b = -e^2 \int d^4x_1\, d^4x_2 : (\overline{\psi}^- \gamma^\alpha \psi^-)_{x_1} (\overline{\psi}^+ \gamma^\beta \psi^+)_{x_2} : iD_{F\,\alpha\beta}(x_1 - x_2)\,. \tag{45}$$

The corresponding Feynman amplitudes you must derive are:

$$\mathcal{M}_a = -ie^2 \overline{u}_{s_2}(\vec{p}_2{}') \gamma_\alpha u_{r_2}(\vec{p}_2) \frac{1}{(p_1 - p_1{}')^2} \overline{v}_{r_1}(\vec{p}_1) \gamma^\alpha v_{s_1}(\vec{p}_1{}') \tag{46}$$

$$\mathcal{M}_b = +ie^2 \overline{u}_{s_2}(\vec{p}_2{}') \gamma_\alpha v_{s_1}(\vec{p}_1{}') \frac{1}{(p_1 + p_2)^2} \overline{v}_{r_1}(\vec{p}_1) \gamma^\alpha u_{r_2}(\vec{p}_2) \tag{47}$$

In other words, I want you to make clear (following the same kind of detailed derivation that I do in the notes for $e^-e^- \to e^-e^-$) that there is indeed a relative $-$ sign between $\mathcal{M}_a$ and $\mathcal{M}_b$ and that the expressions given are indeed the correct ones. This is the proof by example of the relative minus sign rule (in our list of Feynman rules) that is to be supplied if two diagrams differ only by interchanging an initial positron with a final electron or vice versa.

18. (40 pts) Not assigned for Winter 2016

    Compute the differential cross-section $\frac{d\sigma}{d\Omega}$ for $e^+(p')e^-(p) \to \gamma(k')\gamma(k)$ (please use indicated momenta labeling) in the laboratory frame $p = (m, 0, 0, 0)$ (using $k = (\omega, \vec{k})$, etc.) and in the center-of-mass frame at high energy, neglecting the electron mass. In the latter case, convert $\frac{d\sigma}{d\Omega}$ to $\frac{d\sigma}{dt}$ and express the result in terms of the Mandelstam invariants $s = (p+p')^2$, $t = (p-k)^2$ and $u = (p-k')^2$. Please be sure to do the computation using my fermion normalizations, as opposed to MS's.

    Compare this latter result to that obtained for $\gamma(k)e^-(p) \to \gamma(k')e^-(p')$ at high energy in the center-of-mass. For the latter, simply use the expression given in the notes, neglecting m. Again use the Mandelstam invariants for the $\gamma e^- \to \gamma e^-$ process ($s = (p + k)^2$, $t = (k - k')^2 = (p - p')^2$, and $u = (k - p')^2 = (p - k')^2$).

    Show that the high energy c.o.m expressions for the $e^+e^- \to \gamma\gamma$ and $e^-\gamma \to e^-\gamma$ cases are related by the appropriate "crossing relation" (state what the crossing relation should be).

19. (50 pts) Assigned 1/14/2016; due 1/20/2016.

    This is a problem regarding non-relativistic single particle quantum mechanics and path integrals.

    Imagine a Hamiltonian for a particle moving in one dimension given by $H_{op} = \frac{P_{op}^2}{2M} + \mu$ (μ can be thought of as a chemical potential). For this Hamiltonian, the Schroedinger equation for the one-particle wave function $\phi(x)$ takes the form $\mathcal{O}_{op}\phi(x) = 0$, where $\mathcal{O}_{op} = i\frac{\partial}{\partial t} + \frac{1}{2M}\frac{\partial^2}{\partial x^2} - \mu$.

    (a) Compute $\langle x_f, t_f | x_i, t_i\rangle$ without using path integrals and show that

    $$\langle x_f, t_f | x_i, t_i\rangle = \int \frac{dp}{2\pi} e^{ip(x_f - x_i) - i\left[\frac{p^2}{2M} + \mu\right](t_f - t_i)}\,. \tag{48}$$

    (b) Check that $P(x_f - x_i, t_f - t_i) = -i\langle x_f, t_f | x_i, t_i\rangle \theta(t_f - t_i)$ is a Greens function of the Schroedinger equation operator: i.e.

    $$\mathcal{O}_{op} P(x - x', t - t') = \delta(x - x')\delta(t - t')\,. \tag{49}$$

(c) Evaluate $\langle x_f, t_f | x_i, t_i \rangle$ explicitly by performing the $\int dp$ above. You should find

$$\langle x_f, t_f | x_i, t_i \rangle = e^{-i\mu(t_f - t_i) + \frac{i(x_f - x_i)^2 M}{2(t_f - t_i)}} \sqrt{\frac{M}{2\pi i (t_f - t_i)}} \,. \tag{50}$$

We will now obtain the same result following a path integral approach.

(d) Consider first the general case where we have a 1-dimensional $\mathcal{L}$ of form

$$\mathcal{L}(\dot{x}(t), x(t)) = c_1 \dot{x}^2(t) + c_2 x^2(t) \,, \tag{51}$$

with action

$$S[x(t)] = \int dt'' \mathcal{L}(\dot{x}(t''), x(t'')) \,, \tag{52}$$

where the time derivative is in the dummy integration variable t''. Expand $S[x(t)]$ about an arbitrary trajectory $x_0(t)$ using

$$\begin{aligned} S[x(t)] &= S[x_0(t) + y(t)] \\ &= S[x_0(t)] + \int dt \left.\frac{\delta S}{\delta x(t)}\right|_{x_0(t)} y(t) + \frac{1}{2} \int dt\, dt' y(t) \left.\frac{\delta^2 S}{\delta x(t) \delta x(t')}\right|_{x_0(t)} y(t') + \ldots \end{aligned} \tag{53}$$

Hint: Use rules like $\frac{\delta x(t)}{\delta x(t')} = \delta(t - t')$ and $\frac{\delta \dot{x}(t)}{\delta x(t')} = \frac{d}{dt}\delta(t - t')$ — sometimes it will be useful to partial integrate the latter form under some integral.

(e) Derive the Euler-Lagrange equation for the general $\mathcal{L}(\dot{x}, x)$ from the action principle $\frac{\delta S}{\delta x(t)} = 0$.

(f) Now compute $\langle x_f, t_f | x_i, t_i \rangle$ using path integral techniques. You must look back in our notes and you will find the expression

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{x(t_i) = x_i}^{x(t_f) = x_f} \left[\frac{dx(t)}{\sqrt{2\pi i \epsilon / M}} \right] e^{iS[x(t)]} \tag{54}$$

where the ϵ stuff is to remind you of what is actually present when you go to the discrete limit. Note that we have made M explicit here, whereas we had set $M = 1$ in the notes to simplify our expressions obtained after performing the dp_i integrals. In the discrete limit, the fully correct expression was (assuming the simple potential $V = \mu$ and using the notation adopted for this problem):

$$\langle x_f, t_f | x_i, t_i \rangle = \frac{1}{\sqrt{2\pi i \epsilon / M}} \lim_{n \to \infty} \int \prod_{k=1}^{n} \frac{dx_k}{\sqrt{2\pi i \epsilon / M}} \exp\Big\{ i\epsilon \sum_{j=1}^{n+1} \Big[\frac{M}{2} \left(\frac{x_j - x_{j-1}}{\epsilon} \right)^2 - \mu \Big] \Big\} \tag{55}$$

where the j indices label the intermediate points separated by ϵ in time and $x_{n+1} = x_f$ and $x_0 = x_i$ are fixed. In the above, I have assumed the same H and associated $\mathcal{L}$ as specified in the problem introduction. One can, of course, evaluate this directly. For pedagogical reasons, I want you to instead evaluate this by using the expansion techniques above, expanding around the classical path $x_0(t)$ which solves $\frac{\delta S}{\delta x(t)} = 0$. You wish to show that

$$\langle x_f, t_f | x_i, t_i \rangle = e^{iS[x_0(t)]} \langle 0, t_f | 0, t_i \rangle \,, \tag{56}$$

with

$$\langle 0, t_f | 0, t_i \rangle = \int_{y(t_i)=0}^{y(t_f)=0} \left[\frac{dy(t)}{\sqrt{2\pi i \epsilon / M}} \right] e^{i \int dt \; c_1 \dot{y}^2(t)}$$
$$\equiv \left(\frac{1}{\sqrt{2\pi i \epsilon / M}} \right)^{n+1} \int \prod_{k=1}^{n} dy_k \exp\left\{ i\epsilon \sum_{j=1}^{n+1} \left[\frac{M}{2} \left(\frac{y_j - y_{j-1}}{\epsilon} \right)^2 \right] \right\} , \quad (57)$$

where for this case we took $c_1 = \frac{M}{2}$ and $c_2 = 0$ and we have $y_{n+1} = y_0 = 0$.

(g) Now evaluate $\langle 0, t_f | 0, t_i \rangle$ explicitly. For this purpose, go back to the discrete version of Eq. (57) which we obtain by noting that

$$\int dt \; c_1 \dot{y}^2(t) = \lim_{\epsilon \to 0, n \to \infty} \left[\epsilon \sum_{j=1}^{n+1} c_1 \left(\frac{y_j - y_{j-1}}{\epsilon} \right)^2 \right] \quad (58)$$

and write the sum as a bilinear form

$$\kappa y^T A y \, , \quad (59)$$

where $y(t_f) = y_{n+1} = 0$ and $y(t_i) = y_0 = 0$ and A is a symmetric matrix containing only numbers. Then use the result derived in class that says

$$\int dx_1 \ldots dx_n \exp\left\{ -\frac{1}{2} x_i A_{ij} x_j + x_i J_i \right\} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left\{ \frac{1}{2} J_i A_{ij}^{-1} J_j \right\} , \quad (60)$$

setting $J_i = 0$. You will obviously need to compute $\det A$ — it should be fairly easy. Having done this, take the continuum limit of your result, $n \to \infty, \epsilon \to 0$.

(h) Finally, solve the Euler Lagrange equation for the boundary conditions

$$x(t_f) = x_f \, , \quad x(t_i) = x_i \quad (61)$$

and call the solution $x_0(t)$. Compute $S[x_0(t)]$. Then, plug this and your result for $\langle 0, t_f | 0, t_i \rangle$ into Eq. (56) and verify that this answer agrees with the answer you obtained in Eq. (50).

20. (15 pts) Assigned 1/14/2016; due 1/20/2016.

The simplest path integral there is, is the 0-dimensional path integral, which is just a Gaussian integral over one variable:

$$Z = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} \, . \quad (62)$$

In analogy with what we do for the full path integral to evaluate correlation functions, generalize this to be a function of an external current J:

$$Z[J] = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2 + J\phi} \, . \quad (63)$$

Evaluate each of the following:

$$\left. \frac{d^2 Z[J]}{dJ^2} \right|_{J=0} \quad \text{and} \quad \left. \frac{d^4 Z[J]}{dJ^4} \right|_{J=0} \quad (64)$$

by each of the following two methods:

(a) Carry out the derivatives with respect to J. Then, set $J = 0$. Then, do the ϕ integration.

(b) Solve for $Z[J]$ by actually doing the ϕ integration, first. Then, carry out the derivatives on the resulting expression, and set $J = 0$ at the end.

You should get the same answers. Verify by looking at the first method, that the answers are the two-point and four-point functions, that is, the integral with ϕ^2 and ϕ^4 inserted into the integrand.

21. (40 pts) Assigned 1/22/16; Due 2/3/16 (note extra time).

Do problem 9.2 of Peskin and Schroeder, parts (a), (b) and (c). This is a hard problem but a very interesting topic that we won't cover in class – namely the connection between path integrals and the partition function.

22. (20 pts) Assigned 1/27/16; Due 2/3/16.

Using the functional derivative technique, obtain the usual Wick's theorem result for

$$\langle 0|T\{\psi(x_1)\psi(x_2)\psi(x_3)\overline{\psi}(x_4)\overline{\psi}(x_5)\overline{\psi}(x_6)\}|0\rangle \tag{65}$$

for the free-field Dirac Lagrangian (i.e. no interactions). Of course, you want to first write down the "usual Wick's theorem result" obtained in the anticommutator 2nd quantization approach.

23. (20 pts) Assigned 2/3/16; Due 2/12/16.

Consider an abelian gauge theory with charge e in the standard covariant derivative.

Consider a closed contour C_0 through which passes a certain amount of flux, as shown in Fig. 1. Assume that $F_{\mu\nu} = 0$ on the contour C_0. Show that (at any fixed time)

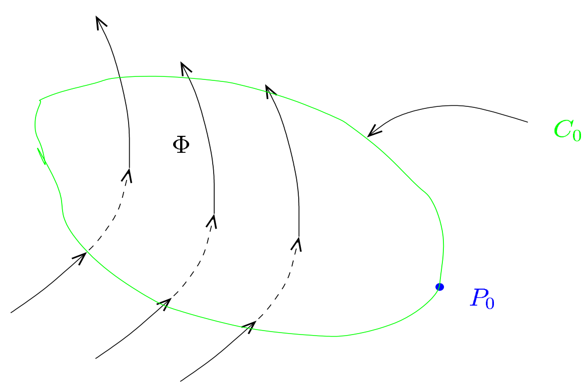


Figure 1: Picture of contour C_0 with flux passing through it.

$$\Phi \equiv \oint_{C_0} A_i dx^i = \frac{2\pi n}{e}\,, \tag{66}$$

where n is any integer, using the fact that A_μ is pure gauge on C_0 (i.e. $A_\mu \propto (\partial_\mu U)U^{-1}$) and the requirement that all charged fields be single-valued. P_0 is the point at which the integral above begins.

24. (20 pts) Assigned 2/3/16; Due 2/12/16.

Consider $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ in the case of a non-abelian gauge theory (recall $F_{\mu\nu} = \vec{L}\cdot\vec{F}_{\mu\nu}$). Show that

$$\mathrm{Tr}[\widetilde{F}_{\mu\nu}F^{\mu\nu}] = \partial^\mu \xi_\mu\,, \tag{67}$$

where

$$\xi_\mu = 2\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left[A^\nu\partial^\alpha A^\beta + \frac{2g}{3i}A^\nu A^\alpha A^\beta\right]\,, \tag{68}$$

where g appears in our standard equation of motion, for example,

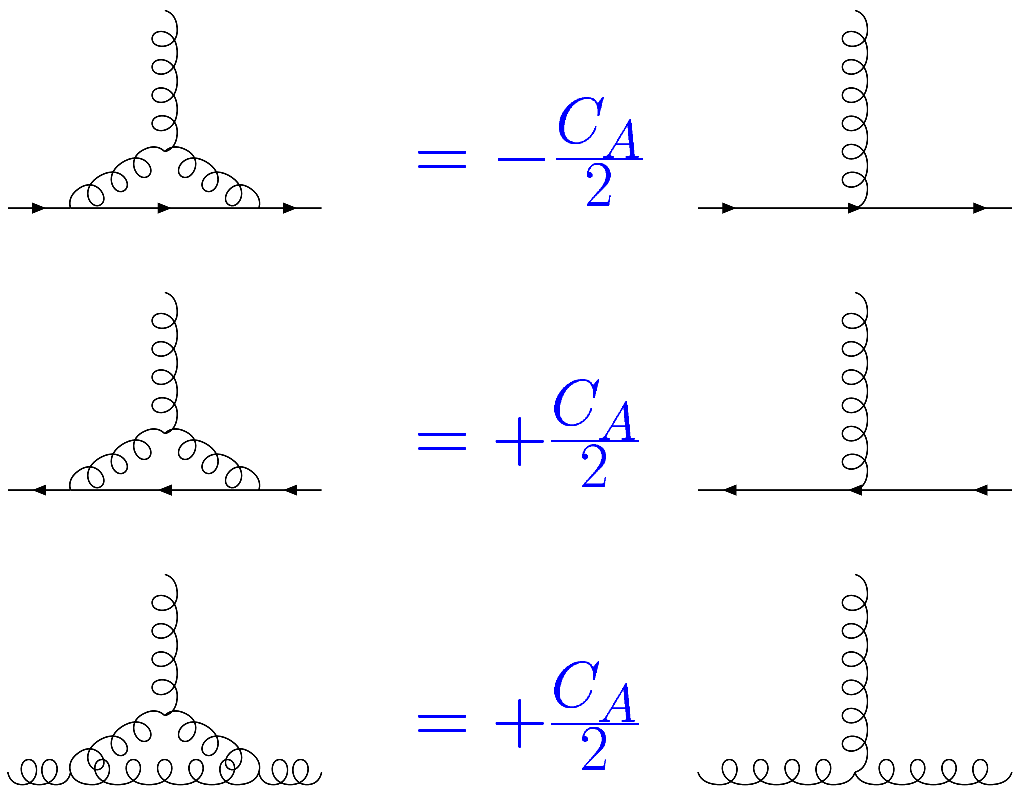
$$[D^\mu, F_{\mu\nu}] = \partial^\mu F_{\mu\nu} - ig[A^\mu, F_{\mu\nu}] = 0\,. \tag{69}$$

25. (20 pts) Assigned 2/17/16; Due 2/24/16.

    Obtain the 3-gluon vertex (only the first term analogous to path integral approach 1st term done in class) using the 2nd quantization commutator "killing" approach.

26. (20 pts) Assigned 2/17/16; Due 2/24/16.

    Obtain the 4-gluon vertex (only two inequivalent terms need to be obtained, let us say the $a \to f, b \to g, c \to f', d \to g'$ and $a \to g, b \to f, c \to f', d \to g'$ terms) using the path integral/functional derivative approach, assuming that the reduction projection works as it did in the 3-gluon vertex case.



27. (20 pts) Assigned 2/29/16; Due 3/7/16.

    Prove the first and third of the color relations above.

28. (30 pts) Assigned 3/2/16; Due 3/9/16. This problem is likely to be important for the final.

    Compute the cross section $d\sigma/dt$ for $u\overline{u} \to d\overline{d}$, where the u and $\overline{u}$ represent up-quark and anti-up-quark (and similarly for d and $\overline{d}$) in the fundamental triplet representation. Express the cross section in terms of the s, t, u variables in the massless limit. The cross section you should give is that obtained by using the Feynman rules of QCD and by summing over final spins *and* final colors for the quark and antiquark and averaging over initial spins *and* initial colors. You will probably want to employ the color graph techniques to get the appropriate color factors for the various terms, although other techniques can also be used. I also hope you will use the spinor techniques to do the Feynman/momentum-space part of the computations, but this is not required since we will only barely have got to them.

Problems above this point are those typically assigned for 230B. They are not relevant for 230C. Problems below this point (but above the next division) have been assigned for 230C.

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29. (30 pts) Assigned 1/16/13; Due 1/28/13

    This problem concerns renormalization of ϕ^4 theory at two loops.

    As we have discussed in class, at the one loop level $Z_\phi = 1$ since the one loop diagram that has a bubble attached to the propagator is such that the external momentum does not enter into the bubble. This is no longer true at two loops.

    a) (10 pts) Find the one diagram at two loop level (order λ^2) which gives a momentum dependent contribution to $\Sigma(p^2)$. Set up the expression for the associated contribution to $\Sigma(p^2)$ in terms of two loop integrals, verifying that indeed the expression will be p^2-dependent.

    b) (20 pts) Carry out the loop integrations using Feynman parameter techniques and dimensional regularization. There are some useful hints for this part.

    Your diagram should involve three propagators that you will combine all at once using the identity given as Eq. 131 in the web notes. You should then shift momenta so that you end up with an expression that looks like

$$\int\int\int dx\, dy\, dz\, \delta(x+y+z-1) \int \frac{d^4 l_1}{(2\pi)^4} \int \frac{d^4 l_2}{(2\pi)^4} \frac{1}{[\alpha l_1^2 + \beta l_2^2 + \gamma p^2 - m^2 + i\epsilon]^3} \qquad (70)$$

    where the α, β, γ depend on x, y, z. Now wick rotate l_1 and l_2 and dimensionally regularize and evaluate the momentum integrals. Do not yet try to integrate over x, y, z. Once you have gotten an expression, compute $d\Sigma/dp^2$ (which is what you need to get Z_ϕ). Then, go to the limit of $\epsilon \to 0$ isolating the terms of order $1/\epsilon$ and terms that are finite as $\epsilon \to 0$. Take $p^2 \to m^2$ in the resulting expression. You should find that the counter term required for $Z_\phi - 1$ takes the form

$$\int\int\int dxdydz\delta(x+y+z-1)F(x,y,z) \times \left(\frac{1}{\epsilon} + const. + \log G(x,y,z)\right) \qquad (71)$$

    for some rational functions $F(x,y,z)$ and $G(x,y,z)$. Now go to Mathematica and evaluate these integrals using substitutions like $x = \xi w$, $y = (1-\xi)w$, $z = 1-w$ and integrating over w first and then over ξ. (Be careful to get the Jacobian for the change of variables into the game.)

30. (20 pts) Assigned 1/16/13; Due 1/28/13. More ϕ^4 renormalization.

    In order to carry out the full 2-loop renormalization of ϕ^4 theory, many additional diagrams aside from the one above would need to be considered.

    a) Draw all order λ^2 diagrams for $-i\Sigma(p^2)$. Be sure to include diagrams generated by including $\Delta\mathcal{L}^{(1)}$.

    b) Draw all order λ^3 diagrams for $\Gamma(p^2)$ (i.e. all diagrams that correct the basic vertex of order λ that have up to an additional two powers of λ. Again, be sure to include diagrams generated by including $\Delta\mathcal{L}^{(1)}$.

    This exercise should make it clear why the full two-loop renormalization of ϕ^4 theory (in principle, the simplest theory we can consider) is fairly non-trivial.

31. (15 pts) Assigned 1/16/13, Due 1/28/13. This, and the next problem are relevant for renormalization issues in QED. They show that certain potentially infinite diagrams are either zero or at least not divergent. You must use dimensionalregularization for them.

    Show that the "tadpole" fermion-loop correction to the electron propagator is zero. This diagram is that in which a single photon attaches to a propagating electron and the other end of the photon line attaches to a closed fermion loop.

32. (15 pts) Assigned 1/16/13, Due 1/28/13.

Show that $D < 0$ for the fermion-loop contributions to a 4-photon vertex (after summing all of them).

33. (50 pts) Assigned 2/6/13; Due 2/13/13

Exotic contributions to $g - 2$.

As discussed in class, new physics can give rise to extra contributions to the $g - 2$ of the electron and muon. One such type of new physics is Higgs bosons and axions.

(a) A scalar Higgs boson interacts with the electron or muon according to

$$H_{int} = \int d^3x \frac{\lambda}{\sqrt{2}} h\overline{\psi}\psi \,. \tag{72}$$

Compute the contribution of a virtual Higgs boson exchange to the electron $g-2$ in terms of λ and the mass m_h of the Higgs boson.

(b) QED accounts extremely well for the electron's anomalous magnetic moment, a calculation we did in class at one loop. One finds, defining $a = \frac{g-2}{2}$ that

$$|a^e_{expt} - a^e_{QED}| < 1 \times 10^{-10} \,. \tag{73}$$

What limit does this place on λ and m_h? In the simplest version of the electroweak theory, $\lambda_e = 3 \times 10^{-6}$ (where λ_e goes with ψ_e for the electron) and $m_h > 114$ GeV. Show that these values are not excluded.

The coupling of the Higgs boson to the muon is larger by a factor of m_μ/m_e: $\lambda_\mu = 6 \times 10^{-4}$ and

$$|a^\mu_{expt} - a^\mu_{QED}| < 3 \times 10^{-8} \tag{74}$$

is the experimental limit. Does this place a limit on m_h?

(c) Some more complex versions of this theory contain a pseudoscalar particle called the axion, which couples to the electron or muon according to

$$H_{int} = \int d^3x \frac{i\lambda}{\sqrt{2}} a\overline{\psi}\gamma^5\psi. \tag{75}$$

The axion may be as light as the electron, or lighter, and may couple more strongly than the Higgs boson. Compute the contribution of a virtual axion to the $g - 2$ of the electron, and work out the excluded values of λ and m_a.

34. (20 pts) Assigned 2/6/13; Due 2/13/13

Show explicitly (i.e. not using the charge conjugation operator but rather using explicit expressions for the one-loop diagrams) that the sum of the two diagrams for the possible fermion-loop contribution to a 3-photon vertex combine to give 0. (Furry's Theorem)

35. (40 pts) Assigned 2/6/13; Due 2/18/13

Do Peskin Problem 10.4. Asymptotic behavior of diagrams in ϕ^4 theory.

Compute the leading terms in the S-matrix element for boson-boson scattering in ϕ^4 theory in the limit of $s \to \infty$, t fixed. Ignore all masses on internal lines, and keep external masses non-zero only as infrared regulators when needed. Show that the sum of the zero-loop, one-loop and two-loop diagrams (all of them) give

$$\mathcal{M}(s,t) \sim -i\lambda - i\frac{\lambda^2}{(4\pi)^2}\log s - i\frac{3\lambda^3}{2(4\pi)^4}\log^2 s + \ldots \tag{76}$$

Notice that ignoring the internal masses allows some pleasing simplifications of the Feynman parameter integrals. Also note that my answer above differs from Peskin's answer. See if you can verify my answer or not. I believe Peskin left out a contribution coming from the expansion of his Eq. (10.63). Actually, he now confirms that my answer is correct.

36. (30 pts) Assigned 3/4/13, due 3/11/13.

a) Verify the c_{10}, c_{11} and c_{22} coefficients in

$$\alpha_s^{(n_f+1)}(\mu^2) = \alpha_s^{(n_f)}(\mu^2)\left(1 + \sum_{n=1}^{\infty}\sum_{l=0}^{n} c_{nl}\left[\alpha_s^{(n_f)}(\mu^2)\right]^n \ln^l \frac{\mu^2}{m_h^2}\right), \tag{77}$$

where m_h is the mass of the $(n_f+1)^{\text{th}}$ flavor, and the first few c_{nl} coefficients are $c_{11} = \frac{1}{6\pi}$, $c_{10} = 0$, $c_{22} = c_{11}^2$, $c_{21} = \frac{19}{24\pi^2}$, and $c_{20} = -\frac{11}{72\pi^2}$ when m_h is the $\overline{MS}$ mass at scale m_h ($c_{20} = \frac{7}{24\pi^2}$ when m_h is the pole mass.)

b) Verify the $\frac{b_1}{b_0^2}$ term of

$$\alpha_s(\mu^2) \simeq \frac{1}{b_0 t}\left(1 - \frac{b_1}{b_0^2}\frac{\ln t}{t} + \frac{b_1^2(\ln^2 t - \ln t - 1) + b_0 b_2}{b_0^4 t^2}\right.$$

$$\left. - \frac{b_1^3(\ln^3 t - \frac{5}{2}\ln^2 t - 2\ln t + \frac{1}{2}) + 3 b_0 b_1 b_2 \ln t - \frac{1}{2} b_0^2 b_3}{b_0^6 t^3}\right), \tag{78}$$

where $t \equiv \ln \frac{\mu^2}{\Lambda^2}$.

37. (30 pts) Assigned 3/8/13, due 3/15/11.

In equation (556) of the notes, you will find that b_1 and b_2 contain the terms

$$-b_1 \ni \frac{3}{5} N_{T_2} \quad \text{and} \quad -b_2 \ni \frac{2}{3} N_{T_2}, \tag{79}$$

where N_{T_2} is the number of weak isospin *triplets* with hypercharge $Y = \pm 2$. And, there is no N_{T_2} term in b_3. Using the discussions of the notes, but thinking carefully about how going from a doublet to a triplet representation of the weak isospin group changes things, derive these results. For the $SU(2)$ you might find it useful to look at the QFT-III notes where I first discussed $SU(2)$ and the doublet and triplet representations thereof.

38. (30 pts) Assigned 3/8/13, due 3/15/13.

a) One of our experimentalists asked me last year whether there is a top-loop induced contribution to $gg \to \gamma + Higgs$, where g of course is a gluon. In fact, this potentially very important process (since there are so many gluons in a proton effectively, especially in LHC collisions) is zero at the one-loop level. The vision might have been that gg attaches to a top quark loop from which the γ and $Higgs$ are emitted. Prove that the sum of all top quark loop diagrams in which two gluons, a photon and a scalar Higgs boson attach to the quark loop is zero. Hint: you will want to use the fact that there is a matrix C, where $C \equiv i\gamma^2\gamma^0$ is such that $-C = C^{-1} = C^T = C^\dagger$.

b) There are non-zero diagrams that give rise to $q\overline{q} \to \gamma + Higgs$. For simplicity, just consider the case where the q from the colliding proton is $q = u$ and the $\overline{q}$ from the other colliding proton at the LHC (or from the colliding anti-proton if at the Tevatron) is $\overline{q} = \overline{u}$.

At tree-level there are two possible diagrams. Draw them and explain why they will be quite small. For this, and the following part, you will have to make use of your knowledge of the SM, including the Higgs.

Now consider the possible one-loop diagrams. Try to think of all the contributing diagrams using Feynman gauge for the vector bosons involved (which includes things like the W and Z) and associated ghosts. There are a lot of diagrams. The idea is to give you a taste of the complexity of a real calculation that is of interest for a potential Higgs signal that because of the photon being present in the final state might have small background.

If you have not fully derived Feynman rules for the weak interactions in the Feynman gauge, you will find it helpful to look at an appropriate reference, e.g. the Appendix on Feynman rules for the electroweak interactions in Cheng and Li.

The evaluation of the one-loop diagrams is messy, so don't attempt it. Just give the diagrams and for your own good it would be valuable to think about writing down some of the starting expressions. In fact, by doing so, it is possible to argue that some of the one-loop diagrams are again very small. But others are not necessarily small. See if you can give at least one example of each type.