Class Notes for Quantum Field Theory: Section II

S-Matrix, Wick's Theorem, Feynman Rules, Specific Sample Calculations

Primary Reference: Mandl & Shaw, "Quantum Field Theory"

See also: Peskin and Schroeder, "Quantum Field Theory"

Jack Gunion
U.C. Davis

230A – Part II

The S-matrix

- Ideally, we would like to solve the coupled differential equations that result from introducing interactions among the free fields, as for instance E&M interactions found by the minimal substitution rules.

- An exact solution has not been found, but we have made much progress using a perturbative approach. So far, we have been fortunate that nature has always allowed such a perturbative approach to be fruitfully compared to experiment.

  QED, in particular, has passed extremely precise tests because we can compute very precisely due to the small size of $\alpha \sim 1/137$.

- The perturbative approach is most conveniently derived in the interaction picture in which most of the time evolution of the states (namely, that associated with the free particle part of the Hamiltonian) is removed leaving behind only the (small) time evolution from the perturbatively treated interaction(s).

So far, we have been using the Heisenberg picture in which state vectors (in the Foch space sense) are constant in time while the field operators contain all the time dependence.

- The solution to the full problem can be formulated using the Dyson expansion which is ideally suited for obtaining perturbation results systematically

The Interaction Picture

- Consider QED as the example theory.

- We divide up the full $\mathcal{L}$ into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{1}$$

with

$$\mathcal{L}_0 =: \overline{\psi}(x)(i\partial^\mu\gamma_\mu - m)\psi(x) - \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) : \tag{2}$$

(using the Lorentz condition formulation for the A_μ fields) and

$$\mathcal{L}_I =: e\overline{\psi}(x)A^\mu\gamma_\mu\psi(x) : \ . \tag{3}$$

- **A full (non-perturbative) solution to the theory would arise if we could compute**

$$U \equiv U(t, t_0) = e^{-iH(t-t_0)} \tag{4}$$

where U defines the time evolution of the Schroedinger picture states,

$$|A, t\rangle_S = U|A, t_0\rangle_S \tag{5}$$

from which one has

$$i\frac{d}{dt}|A, t\rangle_S = H|A, t\rangle_S \,. \tag{6}$$

U also defines the relation between the Schroedinger and Heisenberg pictures:

$$|A, t\rangle_H = U^\dagger|A, t\rangle_S = |A, t_0\rangle_S \tag{7}$$

and

$$\mathcal{O}^H(t) = U^\dagger \mathcal{O}^S U \,, \tag{8}$$

where the Schroedinger picture is defined by $\mathcal{O}^S$ being independent of time. Of course, $\mathcal{O} = H$ is time-independent and is the same in both pictures.

At $t = t_0$, the states and operators in the two pictures are the same. However, the Schroedinger state changes with time whereas the Heisenberg picture state is constant in time: one could write $|A, t\rangle_H = |A\rangle_H$, but MS doesn't.

- **Of course, expectation values are the same in the two pictures since U is unitary; i.e.**

$$ {}_S\langle B,t|\mathcal{O}^S|A,t\rangle_S = {}_H\langle B,t|\mathcal{O}^H(t)|A,t\rangle_H \,. \tag{9} $$

- **From Eq. (8),**

$$ i\frac{d}{dt}\mathcal{O}^H(t) = [\mathcal{O}^H(t), H] \,. \tag{10} $$

- **Now write $H = H_0 + H_I$, where the I subscript is for interaction and not for interaction picture!**

- **The I.P. is defined relative to the S.P. by using**

$$ U_0 \equiv U_0(t,t_0) = e^{-iH_0(t-t_0)} \tag{11} $$

to define the I.P. state

$$ |A,t\rangle_I = U_0^\dagger |A,t\rangle_S \tag{12} $$

and I.P. operators

$$\mathcal{O}^I(t) = U_0^\dagger \mathcal{O}^S U_0 \,. \tag{13}$$

Here the subscript I on $|A,t\rangle_I$ and the superscript I on $\mathcal{O}^I(t)$ refer to the interaction picture (I.P.) and not to the interaction part of the Hamiltonian.

Note that $H_0^I = H_0^S \equiv H_0$.

- Differentiating Eq. (13) gives

$$i\frac{d}{dt}\mathcal{O}^I(t) = [\mathcal{O}^I(t), H_0] \tag{14}$$

and we also find

$$\begin{aligned}
i\frac{d}{dt}|A,t\rangle_I &= i\frac{d}{dt}\left[e^{+iH_0(t-t_0)}|A,t\rangle_S\right] \\
&= -H_0 e^{+iH_0(t-t_0)}|A,t\rangle_S + e^{+iH_0(t-t_0)} i\frac{d}{dt}|A,t\rangle_S \\
&= -e^{+iH_0(t-t_0)} H_0^S|A,t\rangle_S + e^{+iH_0(t-t_0)} H^S|A,t\rangle_S \\
&= e^{+iH_0(t-t_0)} H_I^S|A,t\rangle_S
\end{aligned}$$

$$
\begin{aligned}
&= e^{+iH_0(t-t_0)} H_I^S e^{-iH_0(t-t_0)} |A,t\rangle_I \\
&= H_I^I |A,t\rangle_I \,, \qquad (15)
\end{aligned}
$$

where: 1) we used the fact that $H^S \equiv H$ in this notation; 2) we used the fact that $H_0^S = H_0$ (see below Eq. (13)); 3) we used Eq. (6); and 4) we were very careful to preserve the placing of the H operator next to $|A,t\rangle_S$.

- Note that we can rewrite the last few lines above as

$$
\begin{aligned}
H_I^I &= e^{+iH_0(t-t_0)} H_I^S e^{-iH_0(t-t_0)} \\
&= e^{+iH_0(t-t_0)} e^{-iH(t-t_0)} H_I^H(t) e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \\
&= U'^{\dagger} H_I^H U' \,, \qquad (16)
\end{aligned}
$$

where U' is a unitary operator that takes us from the H.P. to the I.P. for any operator (not just the Hamiltonian).

The S-matrix expansion

- This expansion is based upon the I.P. reviewed above, in which:
  - the operators satisfy the Heisenberg-like equations of motion but involving the free Hamiltonian H_0 only, not the complete H, as algebraically stated in Eq. (14):

$$i\frac{d}{dt}\mathcal{O}^I = [\mathcal{O}^I(t), H_0]\,; \tag{17}$$

  - if the interaction $\mathcal{L}_I$ does not involve field derivatives (as for all cases of interest to us), the fields canonically conjugate to the interacting fields and those conjugate to the free fields are identical, e.g. in QED $\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = \frac{\partial\mathcal{L}_0}{\partial\dot{\psi}_\alpha}$, where we are in the H.P. when writing such equations and thinking about conjugate fields.
    This implies that since the I.P. and the H.P. are related by a unitary transformation, then in the I.P. the interacting fields satisfy the same commutation or anticommutation relations as the free fields. Using scalar field notation, this is shown algebraically as follows:

$$[\phi^I(\vec{x},t), \pi^I(\vec{y},t)] \;\; = \;\; [U'^\dagger\phi^H(\vec{x},t)U', U'^\dagger\pi^H(\vec{y},t)U']$$

$$
\begin{aligned}
&= U'^{\dagger}[\phi^H(\vec{x},t),\pi^H(\vec{y},t)]U' \\
&= U'^{\dagger} i\delta^3(\vec{x}-\vec{y})U' \\
&= i\delta^3(\vec{x}-\vec{y})\,. \qquad (18)
\end{aligned}
$$

This in turn implies that we will 2nd quantize the I.P. fields in a manner that is essentially identical to what we did in the free field case.

– **Note:** there is a subtlety since the asymptotic states in the I.P. can have a different mass and different normalization than the asymptotic states in the free-particle case.
  This difference arises because the asymptotic states must contain not just the free particle behavior, but the full collection of virtual processes associated with the presence of interactions (think bare line + line with all possible insertions, loops, ...).
  This is a subtlety that we will eventually address, but not in detail until we come to renormalization theory.

- At this point, MS simplifies the notation a bit and writes in the I.P.

$$
i\frac{d}{dt}|\Phi(t)\rangle = H_I(t)|\Phi(t)\rangle \qquad (19)
$$

where

$$H_I(t) \equiv e^{iH_0(t-t_0)} H_I^S e^{-iH_0(t-t_0)} = H_I^I(t)\,; \tag{20}$$

that is, we will be dropping the ^{I} superscript reminding us that everything is in the I.P.

- It will also be important to note that if H_I^S is a product of fields in the S.P. (i.e product of operators in the S.P.), then H_I will be the same product of fields/operators where all the fields/operators are in the I.P. (This is shown by inserting a whole bunch of 1's in between the operators in the form $1 = e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)}$.)

- If we start with a state at some initial time, $|\Phi(t_i)\rangle = |i\rangle$, the solution of Eq. (19) gives the state $|\Phi(t)\rangle$ at any other time t.

  Since $H_I(t)$ is hermitian, this time development is given by a unitary transformation. Accordingly, it preserves the normalization of the state:

$$\langle\Phi(t)|\Phi(t)\rangle = const. \tag{21}$$

- This formalism we are going to develop will not be appropriate for bound states. It will apply to the situation where we consider an initial state in which there are a certain number of widely separated particles which then converge upon a small interaction region, interact via the interactions, and then at large times a (usually different) set of widely separated particles emerge from the interaction region.

Eq. (19) determines the state $|\Phi(t=\infty)\rangle$ into which $|\Phi(t=-\infty)\rangle$ evolves long after the scattering is over and all particles are far apart again.

- The S-matrix is defined by

$$|\Phi(t=\infty)\rangle = S|\Phi(t=-\infty)\rangle = S|i\rangle\,. \tag{22}$$

Note that Eq. (21), conservation of normalization, implies

$$\langle\Phi(\infty)|\Phi(\infty)\rangle = \langle\Phi(-\infty)|S^\dagger S|\Phi(-\infty)\rangle = \langle\Phi(-\infty)|\Phi(-\infty)\rangle \tag{23}$$

which requires that $S^\dagger S = 1$, *i.e.* S is a unitary operator.

- $S|i\rangle$ can consist of a large selection of possible final states $|f\rangle$.

The probability that after the collision (i.e. at $t=\infty$) the system is in state $|f\rangle$ is given by

$$|\langle f|\Phi(\infty)\rangle|^2\,, \tag{24}$$

assuming unity normalizations for $|\Phi(-\infty)\rangle = |i\rangle$ (and hence also for $|\Phi(\infty)\rangle$ given conservation of normalization as noted above). The corresponding probability amplitude is

$$\langle f|\Phi(\infty)\rangle = \langle f|S|i\rangle \equiv S_{fi}\,. \tag{25}$$

in terms of which

$$|\Phi(\infty)\rangle = \sum_f |f\rangle S_{fi}\,. \tag{26}$$

The unitarity of the S-matrix can be written in this basis as

$$\sum_f |S_{fi}|^2 = 1\,, \tag{27}$$

as follows from

$$1 = \langle i|i\rangle = \langle i|S^\dagger S|i\rangle = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = S^*_{fi}S_{fi}\,. \tag{28}$$

This equation expresses the conservation of probability. It is more general than the corresponding conservation of particles in NRQM since now particles can be created and/or destroyed.

Calculating the S-matrix

- **We must solve**

$$i\frac{d}{dt}|\Phi(t)\rangle = H_I(t)|\Phi(t)\rangle \tag{29}$$

with initial condition $|\Phi(-\infty)\rangle = |i\rangle$. Equivalently we must compute

$$|\Phi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\Phi(t_1)\rangle\,. \tag{30}$$

This form obviously satisfies the above differential equation and has the correct initial boundary condition at $t = -\infty$.

The perturbative series is based on the iterative solution of this equation where we plug in for $|\Phi(t_1)\rangle$ the same form as given above for $|\Phi(t)\rangle$ and so forth.

$$|\Phi(t)\rangle \;=\; |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|i\rangle$$

$$+(-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) |i\rangle + \ldots \quad (31)$$

yielding (as $t \to \infty$)

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \ldots H_I(t_n) \,. \quad (32)$$

- Now comes a very crucial trick that partly motivates the use of time ordering. The above expression can be rewritten as

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n T\{H_I(t_1) H_I(t_2) \ldots H_I(t_n)\} \,. \quad (33)$$

  Here, the time ordering operation has been generalized to include an arbitrary list of operators. It is necessary to insert the T instruction, since in the original form, Eq. (32), the H_I operators (which do not commute with one another when evaluated at different times) were very definitively

ordered so that H_I's with earlier times were always written to the right of H_I's with later times.

Let me show how one arrives at this using the 2nd order term as an example. We write (dropping the subscript I for the moment):

$$
\begin{aligned}
& \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H(t_1) H(t_2) \\
= & \int_{-\infty}^{t} dt_2 \int_{t_2}^{t} dt_1 H(t_1) H(t_2) \quad \text{interchange of integration order} \\
= & \int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2 H(t_2) H(t_1) \quad \text{relabel } t_1 \leftrightarrow t_2 \\
= & \int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2 T\{H(t_1) H(t_2)\} \quad \text{no change because of } T \text{ instruction} \, .
\end{aligned}
\tag{34}
$$

Of course, it is also trivially true that

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H(t_1) H(t_2) = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 T\{H(t_1) H(t_2)\} \tag{35}$$

since the H's are already in the correct time ordering.

Thus, we can write

$$\begin{aligned} & \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H(t_1) H(t_2) \\ = \; & \frac{1}{2} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 T\{H(t_1) H(t_2)\} \\ & + \frac{1}{2} \int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2 T\{H(t_1) H(t_2)\} \\ = \; & \frac{1}{2} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 T\{H(t_1) H(t_2)\} \, . \end{aligned} \tag{36}$$

Given that we want to include $-$ signs in the definition of T when fermionic fields are involved, the absence of any extra signs in the 2nd version of the above requires that H_I contains an even number of fermion factors (as in QED) and that H_I be written in terms of a local interaction (all fields at same space-time x point). With this latter point in mind we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n T\{H_I(t_1) H_I(t_2) \ldots H_I(t_n)\} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n T\{\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \ldots \mathcal{H}_I(x_n)\} \,. \end{aligned} \tag{37}$$

where the $\int d^4x$ integrals are over all space and all times.

- MS now embarks on a discussion of why it is justified to use non-interacting states to specify $|i\rangle$ and $|f\rangle$ instead of those including the photon and loop clouds. This discussion is simply incorrect. In fact, in the interacting theory, $|i\rangle$ and $|f\rangle$ are not described using a free-particle state Foch state picture and basis. They will be defined using creation and annihilation operators that create single particle states in the sense that the states can exist on their own even in the presence of interactions. Such states will have a mass that is different from the bare mass appearing in $\mathcal{H}_0$. This difference arises from the fact that as an isolated particle moves along in the interacting theory, it is continually emitting and reabsorbing virtual quanta in a manner dictated by Feynman diagrams. This cloud of virtual processes alters the physical particle mass.

It is only these states which include these virtual particle clouds that can be used to construct a proper basis for the theory after including interactions. So let us imagine that we have creation and annihilation operators for these isolated single particle states including their separated virtual clouds. The states $|i\rangle$ and $|f\rangle$ will be defined by using these operators. Effectively, we will mimic this by using I.P. basis operators to form $|i\rangle$ and $|f\rangle$, but with the mass appearing in the $\omega_{\vec{p}}$ contained in the $\phi^I(x)$ exponential factors being the full mass obtained after including the virtual particle clouds.

In addition, the creation operator, used to quantize $\phi^I(x)$ in the usual way, will lead to a strange normalization of the states $|i\rangle$ and $|f\rangle$ when they are defined using this same creation operator. The state created will have a single particle component present with some reduced normalization $Z < 1$. Very roughly this means that $|\vec{p}\rangle = a^\dagger(\vec{p})|\Omega\rangle$, where $|\Omega\rangle$ is now the vacuum *in the presence of interactions*, **has normalization $\langle\vec{p}|\vec{p}\rangle = Z < 1$. This factor of Z arises because the creation operator $a^\dagger$ appearing in the field $\phi^I(x)$ (which we are assuming has conventional normalization, as required for the standard canonical commutation 2nd quantization) also 'knows' that the particle it is creating has connections to multi-particle states as well as to a single particle state, and it is only after summing over all such possibilities that full unity normalization holds.**

The $\sqrt{Z}$ factor arising for each single particle state used to construct $|i\rangle$ or $|f\rangle$ must be carefully treated. It used to be that Peskin and Schroeder tried to give a naive discussion of this in their earlier chapters. I see that in the latest edition of their book they have given up trying to do this, just as I will and as the Mandl-Shaw book implicitly does.

Thus, we will simply drop the $\sqrt{Z}$ factors that should really be present in the scattering formulas. Perturbatively, this can actually be justified if one is only interested in computing so-called tree-level amplitudes in which no closed loops of virtual particles appear. The reason is that in perturbation theory $Z \sim 1 + \mathcal{O}(\alpha)$ (e.g. in QED) while the tree-level amplitudes define the lowest order in perturbation theory at which a process can take place. Thus, including the correction to $Z \sim 1$ is a higher order correction.

In contrast, we *must* **use the physical mass (which includes loops ....) of**

a free-particle-like state in defining our I.P. states and fields. It is not the same as the mass of the free-particle theory.

Wicks Theorem

- We must now develop a computational technique for writing down an expression for S given a certain form for $\mathcal{H}_I(x)$.

- To get a non-zero contribution to S_{fi} we must have an appropriate number of creation and/or annihilation operators to match those appearing in the definitions of the states $|i\rangle$ and $|f\rangle$.

  Creation operators in the product of $\mathcal{H}_I$'s for particles that do not appear in $|i\rangle$ or $|f\rangle$ must be compensated by annihilation operators; the result is emission and later absorption of a *virtual* particle that only appears in an intermediate state (i.e. one that mediates between $|i\rangle$ and $|f\rangle$, but is not part of either).

- The only way to make this language clear is to give a specific example.

  MS chooses $e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')$ at lowest possible order in perturbation theory.

You thus start with an initial state

$$|i\rangle = c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \tag{38}$$

and end with a final state

$$|f\rangle = c_{r'}^\dagger(\vec{p}\,')a_{s'}^\dagger(\vec{k}')|0\rangle\,. \tag{39}$$

- So, now the trick is to go to the form of S given in Eq. (33) and to use an n such that there are enough H_I's in between $\langle f|$ and $|i\rangle$ as to get a non-zero result. If you insert the minimal number, i.e. use the smallest n, then this defines the "tree-level" amplitude.

  Since $\langle f|$ has a different photon in it than does $|i\rangle$, you will need two A fields, one to annihilate the initial photon and one to create the final photon. This means you will need two H_I's.

- Since the minimal substitution interaction is $\mathcal{H}_I = -\mathcal{L}_I = -e :\overline{\psi}\not{A}\,\psi:$, this also implies that that you will have an extra ψ and $\overline{\psi}$ in addition to the ψ and $\overline{\psi}$ needed to annihilate the electron present in the initial state and create the one present in the final state.

These extra ψ and $\overline{\psi}$ will have inside them creation and annihilation operators that will basically annihilate one another.

- For now, let us not worry about these two extra guys and focus only on the fields necessary to get from the initial state to the final state.

  The only non-zero contribution to

$$\langle f| \left(-e : \overline{\psi} \not{A}\, \psi :_{x_1}\right) \left(-e : \overline{\psi} \not{A}\, \psi :_{x_2}\right) |i\rangle \tag{40}$$

  will come from the structure (notice I am not yet specifying whether the fields below are at x_1 or x_2 – we will come to the appropriate mixtures):

$$\langle f|\overline{\psi}^- A^- \psi^+ A^+|i\rangle = \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\overline{\psi}^- A^- \psi^+ A^+ c^\dagger_r(\vec{p})a^\dagger_s(\vec{k})|0\rangle \tag{41}$$

  where the A^+ has an a in it to "kill" the $a^\dagger_s(\vec{k})$ used to define $|i\rangle$ and ψ^+ has the c in it to "kill" the $c^\dagger_r(\vec{p})$ that is also part of defining $|i\rangle$. Similarly, the $c^\dagger$ in $\overline{\psi}^-$ will "kill" the $c_{r'}(\vec{p}')$ and the $a^\dagger$ in A^- will "kill" the $a_{s'}(\vec{k}')$.

  By "kill" I mean the operation (to give one of the 4 kills)

$$a_i(\vec{l})a^\dagger_s(\vec{k})|0\rangle = \left([a_i(\vec{l}), a^\dagger_s(\vec{k})] + a^\dagger_s(\vec{k})a_i(\vec{l})\right)|0\rangle = \delta_{is}\delta_{\vec{l},\vec{k}}|0\rangle + 0\,. \tag{42}$$

Here $i, \vec{l}$ are the dummy summation variables used in defining

$$A^{\mu +}(x_2) = \sum_{i,\vec{l}} \frac{1}{\sqrt{2V|\vec{l}|}} \epsilon_i^{\mu}(\vec{l}) a_i(\vec{l}) e^{-il\cdot x_2} \,. \tag{43}$$

- **Note:** The ordering which makes the killing simple, as given above, is the normal ordering of the required operators.

  Thus, in general, we will want to expand the S-matrix as a sum of normal products. The method for doing so is due to Dyson and Wick.

- First, we repeat the general definition of a normal product. Let $Q, R, \ldots, W$ be operators of the type $\psi^{\pm}$, $A^{\pm}$, etc. Each contains a creation or annihilation operator. Then,

$$: QR\ldots W := (-1)^P \times (reordering) \tag{44}$$

  where the reordering is such that all absorption operators (i.e. $^+$ components) are to the right of all creation operators (i.e. $^-$ components).

The exponent P is the number of interchanges of fermion operators required for this reordering.

We also require that normal ordering be a distributive operation: $: RS\ldots + VW\ldots :=: RS\ldots : + : VW\ldots :$.

- Now, we note that in all cases $\mathcal{H}_I(x) =: A(x)B(x)\ldots :$ is already normal ordered. (Remember if $A = A^+ + A^-$ then use distributive rule to do the normal ordering.)

  What appears in S is $T\{\mathcal{H}_I(x_1)\ldots\mathcal{H}_I(x_n)\}$. We must learn how to deal with this structure.

- First, note that (with $A = A(x_1)$ and $B = B(x_2)$)

$$AB - : AB := \begin{cases} [A^+, B^-]_+ , & \text{for two fermion fields} \\ [A^+, B^-] , & \text{otherwise} \end{cases} \tag{45}$$

Proof, e.g. for two fermions fields:

$$\begin{aligned} AB - : AB : &= \left[A^+B^+ + A^+B^- + A^-B^+ + A^-B^-\right] - \left[A^+B^+ - B^-A^+ + A^-B^+ + A^-B^-\right] \\ &= A^+B^- + B^-A^+ = \left[A^+, B^-\right]_+ . \end{aligned} \tag{46}$$

- **Now, the above commutators or anticommutators are c-numbers and so their vacuum expectation values are the same as the numbers themselves.**

  So consider, for example, (remembering $A^+|0\rangle = 0$ since A^+ contains the annihilation operator)

$$\begin{aligned} \langle 0|[A^+, B^-]_+|0\rangle &= \langle 0|[A^+B^- + B^-A^+]|0\rangle \\ &= \langle 0|A^+B^-|0\rangle \\ &= \langle 0|(A^+ + A^-)(B^+ + B^-)|0\rangle \\ &= \langle 0|AB|0\rangle \end{aligned} \tag{47}$$

  where the next to last line follows since $\langle 0|A^- = 0$ and $B^+|0\rangle = 0$. The commutator case follows a similar sequence of steps:

$$\begin{aligned} \langle 0|[A^+, B^-]|0\rangle &= \langle 0|[A^+B^- - B^-A^+]|0\rangle \\ &= \langle 0|A^+B^-|0\rangle \\ &= \langle 0|(A^+ + A^-)(B^+ + B^-)|0\rangle \\ &= \langle 0|AB|0\rangle \end{aligned} \tag{48}$$

As a result, in both cases we have

$$AB =: AB : + \langle 0|AB|0\rangle. \tag{49}$$

- Further, we note that $: AB := \pm : BA :$, with the $-$ sign applying in the case of two fermion fields, and the plus sign otherwise.

  Just to make sure you understand this statement, lets do a fermion case. We must remember that two creation operators *anticommute* as do two annihilation operators.

$$\begin{aligned} : AB : &= \; : (A^+ + A^-)(B^+ + B^-) : \\ &= A^+B^+ + A^-B^+ + A^-B^- - B^-A^+ \\ &= -B^+A^+ + A^-B^+ - B^-A^- - B^-A^+ \\ &= - : (B^+ + B^-)(A^+ + A^-) : \\ &= - : BA :, \end{aligned} \tag{50}$$

  where we used $- : B^+A^- := -(-)A^-B^+ = A^-B^+$.

- **Using the above, it follows that if $x_1^0 > x_2^0$ we have**

$$\begin{aligned} T\{A(x_1)B(x_2)\} &= A(x_1)B(x_2) \\ &= :A(x_1)B(x_2): + \langle 0|A(x_1)B(x_2)|0\rangle \\ &= :A(x_1)B(x_2): + \langle 0|T\{A(x_1)B(x_2)\}|0\rangle \end{aligned} \quad (51)$$

whereas if $x_1^0 < x_2^0$, then

$$\begin{aligned} T\{A(x_1)B(x_2)\} &= \pm B(x_2)A(x_1) \\ &= \pm [:B(x_2)A(x_1): + \langle 0|B(x_2)A(x_1)|0\rangle] \\ &= \pm [\pm :A(x_1)B(x_2): \pm \langle 0|T\{A(x_1)B(x_2)\}|0\rangle] \\ &= :A(x_1)B(x_2): + \langle 0|T\{A(x_1)B(x_2)\}|0\rangle\,, \end{aligned} \quad (52)$$

i.e. exactly the same result since the $\pm$ sign changes cancel one another. In short, we have

$$T\{A(x_1)B(x_2)\} =: A(x_1)B(x_2) : + \overbracket{A(x_1)B}(x_2) \quad (53)$$

where we have introduced the shorthand notation:

$$\overbracket{A(x_1)B}(x_2) \equiv \langle 0|T\{A(x_1)B(x_2)\}|0\rangle \tag{54}$$

which is simply the Feynman propagator that we have discussed.

This Feynman propagator is, of course, only non-zero when A and B have compensating creation and annihilation operators. Sometimes the above structure is also called the "contraction" of the two fields.

- The Feynman propagators we have so far considered are:

$$\begin{aligned}
\overbracket{\phi(x_1)\phi}(x_2) &= i\Delta_F(x_1-x_2) \\
\overbracket{\phi(x_1)\phi^\dagger}(x_2) &= \overbracket{\phi^\dagger(x_2)\phi}(x_1) = i\Delta_F(x_1-x_2) \\
\overbracket{\psi_\alpha(x_1)\overline{\psi}}_\beta(x_2) &= -\overbracket{\overline{\psi}_\beta(x_2)\psi}_\alpha(x_1) = iS_{F\alpha\beta}(x_1-x_2) \\
\overbracket{A^\mu(x_1)A}^\nu(x_2) &= iD_F^{\mu\nu}(x_1-x_2)\,.
\end{aligned} \tag{55}$$

- We now need to generalize this beyond just 2 operators in the following way. We define a generalized normal product containing many operators, some of which have been contracted, as

$$\begin{aligned} &: ABCDEF\ldots JKLM\ldots : \\ = \ & (-1)^P AK\,BC\,EL \ldots : DF \ldots JM \ldots : . \end{aligned} \tag{56}$$

  Here, P is the number of interchanges of neighboring fermion operators required to change the order $ABC\ldots$ to $AKB\ldots$; for example,

$$\begin{aligned} &: \psi_\alpha(x_1)\psi_\beta(x_2)A^\mu(x_3)\overline{\psi}_\gamma(x_4)\overline{\psi}_\delta(x_5) : \\ = \ & (-1)\psi_\beta(x_2)\overline{\psi}_\delta(x_5) : \psi_\alpha(x_1)A^\mu(x_3)\overline{\psi}_\gamma(x_4) : \end{aligned} \tag{57}$$

- With this definition, Wick has proven the following generalization of $T\{AB\} =: AB : + AB$ (assuming that none of the times are the same in the operators):

$$T\{ABCD\ldots WXYZ\}$$

$$
\begin{aligned}
= \;& :ABCD\ldots WXYZ: \\
& + :\overbrace{AB}C\ldots YZ: + \text{all other single contraction cases} \\
& + :\overbrace{AB}\,\overbrace{CD}EF\ldots YZ: + \text{all other double contraction cases} \\
& + \ldots \\
& + \text{all maximal contraction cases}\,.
\end{aligned} \tag{58}
$$

In the maximal contraction cases, if there are matching fields then no field will be left over.

The proof of this theorem is by induction and can be found in the old book by Bjorken and Drell on Relativistic Field Theory (2nd volume). Another form of the inductive proof appears at the end of Peskin and Schroeder, section 4.3. We will not go through the proof here, but will give some illustration below of how it works in a specific case of interest.

- Finally, we must figure out what the correct procedure is for the case of interest where we are dealing with

$$T\{\mathcal{H}_I(x_1)\ldots\mathcal{H}_I(x_n)\} = T\{:AB\ldots:_{x_1}\ldots:AB\ldots:_{x_n}\}\,. \tag{59}$$

Let us consider a simplified version of this sort of structure. For $x_1^0 > x_2^0$ we have, assuming bosonic fields for simplicity,

$$
\begin{aligned}
& T\{A(x_1) : B(x_2)C(x_2) :\} \\
= \; & (A^+ + A^-)(B^+C^+ + B^-C^- + B^-C^+ + C^-B^+) \\
= \; & A^-(B^+C^+ + B^-C^- + B^-C^+ + C^-B^+) \\
& +(B^+C^+ + B^-C^- + B^-C^+ + C^-B^+)A^+ \\
& +[A^+, B^-]C^- + B^-[A^+, C^-] + [A^+, B^-]C^+ + [A^+, C^-]B^+ \\
= \; & : ABC : +[A^+, B^-]C + [A^+, C^-]B \\
= \; & : ABC : +\langle 0|T\{AB\}|0\rangle C + \langle 0|T\{AC\}|0\rangle B \\
= \; & : ABC : +\overbracket{AB}C + \overbracket{AC}B
\end{aligned} \tag{60}
$$

where we of course wrote $:BC:$ in normal ordered form to begin with, and then moved the A^+ part of the A field from left to right until we also got complete normal ordering. In so doing, we had to cross the B^- and C^- fields with which A^+ has non-zero contractions (assuming the same kind of field). But, it was never necessary to commute a B component past a C component, since they were already in normal ordering. In the above,

we also used the fact that $[A^+, B^-] = \langle 0|T\{AB\}|0\rangle$ by virtue of the fact that (by our initial assumption) A had later time than B. A very similar proof would have given the same general result had the time ordering been reversed.

The net result in all cases is that Eq. (58) applies with the proviso that no equal time contractions are to be included, which in the present context means that fields inside the same $\mathcal{H}_I$ are not to be contracted.

- You should note that this result is critically dependent upon using $:\mathcal{H}_I:$ (i.e. normal ordering the interaction Hamiltonian from the beginning) rather than $\mathcal{H}_I$. In the latter case, you would include equal time contractions in applying Wicks theorem. Using normal ordering for $\mathcal{H}_I$, in a certain sense, obscures what is really happening.

  Peskin and Schroeder discuss the situation without normal ordering $\mathcal{H}_I$. In this PS approach (also quite common in other texts), all the contractions of fields in the same $\mathcal{H}_I$ with one another are kept. Such contractions, as we already know, are associated with infinities. We have discussed how such infinities can be thrown away in the linear free particle context (i.e. when considering just one $\mathcal{H}_0$ in the free particle case). It is quite another matter to show that they don't matter when considering $\mathcal{H}_I$ multiply repeated

inside the S matrix.

In fact, such a proof can be carried out and is contained in Sec. 4.4 of Peskin-Schroeder. In order to understand this proof, you must follow the development of the concept of the true vacuum state. The true vacuum state contains all sorts of infinities associated with not normal ordering $\mathcal{H}_I$. But, when computing an S matrix element, the infinities arising in the S matrix calculation associated with not normal ordering $\mathcal{H}_I$ are canceled by the infinities associated with simply defining the true interacting vacuum $|\Omega\rangle$ relative to the non-interacting vacuum $|0\rangle$.

Pictorially, the vacuum should be visualized as a bare vacuum plus all kinds of Feynman graphs containing various sorts of closed bubbles, the latter arising only when we include contractions of fields within the same $\mathcal{H}_I$. Meanwhile any given S matrix calculation will have a "connected" part associated with the process at hand, multiplied by this same collection of closed bubble graphs. But the closed bubble factor is simply absorbed into the definition of the "true" vacuum and thus does not affect the final result for the physical calculation.

Well, I am sure that all this discussion is somewhat mysterious; it is only in 230C that it will really be clarified. For now, we must be satisfied with the fact that at tree-level all the naive manipulations are ok.

Feynman Diagrams and QED rules

Calculation of the Matrix element

- **Let us return to the the process**

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k') \tag{61}$$

process. We wish to evaluate

$$\langle f|S|i\rangle = \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')Sc_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \tag{62}$$

with the part of S that we keep being the minimal version capable of yielding the process. As described earlier this is what MS calls $S^{(2)}$, i.e. the 2nd order component:

$$S^{(2)} = -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 T\{: \overline{\psi}_1 \not{A}_1 \psi_1 :: \overline{\psi}_2 \not{A}_2 \psi_2 :\} \tag{63}$$

and, in fact, only certain components of this $S^{(2)}$ that emerge in the Wick expansion survive: namely, those with two contracted fermion fields with the others left over to "kill" the creation and annihilation operators defining the initial state and the hermitian conjugate of the final state:

$$
\begin{aligned}
S^{(2)} \ni & -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 : \overline{\psi}_1 \not{A}_1 \psi_1 \overline{\psi}_2 \not{A}_2 \psi_2 : \\
& -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 : \overline{\psi}_2 \not{A}_2 \psi_2 \overline{\psi}_1 \not{A}_1 \psi_1 :
\end{aligned} \tag{64}
$$

From the symmetry under $1 \leftrightarrow 2$ above, it should be obvious that these two terms are equal and that we need only evaluate one of these two terms with a factor of $-e^2$ — let us take the 1st term to evaluate.

- In this 1st term we must employ the particular fields components $\overline{\psi}_1^- \ni c^\dagger$ and $\psi_2^+ \ni c$. We also need one $A^+ \ni a$ and one $A^- \ni a^\dagger$, leaving:

$$
\begin{aligned}
S^{(2)} \ni & -e^2 \int d^4x_1 \int d^4x_2 : \overline{\psi}_1^- \not{A}_1^- \psi_1 \overline{\psi}_2 \not{A}_2^+ \psi_2^+ : \\
& -e^2 \int d^4x_1 \int d^4x_2 : \overline{\psi}_1^- \not{A}_1^+ \psi_1 \overline{\psi}_2 \not{A}_2^- \psi_2^+ :
\end{aligned} \tag{65}
$$

- Now we need a few intermediate results related to the "killing" process. Our first example is:

$$\begin{aligned}
\psi^+(x)c_r^\dagger(\vec{p})|0\rangle &= \sum_{\vec{q},t} \frac{1}{\sqrt{2VE_{\vec{q}}}} c_t(\vec{q}) u_t(\vec{q}) e^{-iq\cdot x} c_r^\dagger(\vec{p})|0\rangle \\
&= \frac{1}{\sqrt{2VE_{\vec{p}}}} u_r(\vec{p}) e^{-ip\cdot x}|0\rangle \qquad (66)
\end{aligned}$$

after using $[c_t(\vec{q}), c_r^\dagger(\vec{p})]_+ = \delta_{rt}\delta_{\vec{q},\vec{p}}$.

Similarly, we find

$$\begin{aligned}
\langle 0|c_{r'}(\vec{p}\,')\overline{\psi}^-(x) &= \langle 0|c_{r'}(\vec{p}\,') \sum_{\vec{q},t} \frac{1}{\sqrt{2VE_{\vec{q}}}} c_t^\dagger(\vec{q}) \overline{u}_t(\vec{q}) e^{+iq\cdot x} \\
&= \frac{1}{\sqrt{2VE_{\vec{p}\,'}}} \overline{u}_{r'}(\vec{p}\,') e^{+ip'\cdot x}\langle 0|\,. \qquad (67)
\end{aligned}$$

Finally, we have

$$
\begin{aligned}
A^{\mu\,+}(x)a_s^\dagger(\vec{k})|0\rangle &= \sum_{t,\vec{q}} \frac{1}{\sqrt{2V|\vec{q}|}}\epsilon_t^\mu(\vec{q})a_t(\vec{q})e^{-iq\cdot x}a_s^\dagger(\vec{k})|0\rangle \\
&= \frac{1}{\sqrt{2V|\vec{k}|}}\epsilon_s^\mu(\vec{k})e^{-ik\cdot x}|0\rangle\,, \qquad (68)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle 0|a_{s'}(\vec{k}')A^{\mu\,-}(x) &= \langle 0|a_{s'}(\vec{k}')\sum_{t,\vec{q}} \frac{1}{\sqrt{2V|\vec{q}|}}\epsilon_t^\mu(\vec{q})a_t^\dagger(\vec{q})e^{+iq\cdot x} \\
&= \frac{1}{\sqrt{2V|\vec{k}'|}}\epsilon_{s'}^\mu(\vec{k}')e^{ik'\cdot x}\langle 0|\,. \qquad (69)
\end{aligned}
$$

So let us use these results to first evaluate the contribution of the first term in the part of $S^{(2)}$ that appears in Eq. (65). Using $\langle 0|0\rangle = 1$, we obtain, dropping temporarily the $\frac{1}{\sqrt{2V\ldots}}$ factors:

$$
\begin{aligned}
& \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 : \overline{\psi}_1^{\,-}\, \not{A}_1^{\,-}\,\psi_1\overline{\psi}_2 \not{A}_2^{\,+}\psi_2^+ :\Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 \overline{\psi}_1^{\,-}\, \not{A}_1^{\,-}\,\psi_1\overline{\psi}_2 \not{A}_2^{\,+}\psi_2^+ \Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 \overline{\psi}_1^{\,-}\gamma_\mu A_1^{\mu\,-}\,\psi_1\overline{\psi}_2\gamma_\nu A_2^{\nu\,+}\psi_2^+ \Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \overline{u}_{r'}(\vec{p}')\gamma_\mu\epsilon^{\mu}_{s'}(\vec{k}')\Big[-e^2\int d^4x_1\int d^4x_2 \psi_1\overline{\psi}_2 e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)}\Big]\gamma_\nu\epsilon^{\nu}_{s}(\vec{k})u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}')\not{\epsilon}_{\,s'}(\vec{k}')\Big[-e^2\int d^4x_1\int d^4x_2 \psi_1\overline{\psi}_2 e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)}\Big]\not{\epsilon}_{\,s}(\vec{k})u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}')\not{\epsilon}_{\,s'}(\vec{k}')\Big[-e^2\int d^4x_1 d^4x_2 \left\{\frac{i}{(2\pi)^4}\int d^4q\frac{\not{q}+m}{q^2-m^2}e^{-iq\cdot(x_1-x_2)}\right\} \\
& \qquad e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)}\Big]\not{\epsilon}_{\,s}(\vec{k})u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}')\not{\epsilon}_{\,s'}(\vec{k}')\Big[-e^2\frac{i}{(2\pi)^4}\int d^4q\frac{\not{q}+m}{q^2-m^2}(2\pi)^4\delta^4(k'+p'-q)(2\pi)^4\delta^4(q-p-k)\Big] \\
& \not{\epsilon}_{\,s}(\vec{k})u_r(\vec{p}) \\
= \quad & -e^2(2\pi)^4\delta^4(p'+k'-p-k)\overline{u}_{r'}(\vec{p}')\not{\epsilon}_{\,s'}(\vec{k}')\left[i\frac{\not{q}+m}{q^2-m^2}\right]_{q=p+k}\not{\epsilon}_{\,s}(\vec{k})u_r(\vec{p}) \\
& \quad \times\frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}} \qquad (70)
\end{aligned}
$$

where we have extended the d^4x_1 and d^4x_2 integrals over all space and time, but have temporarily kept the volume normalization for the external factors (writing them explicitly in the last line of the equation after having dropped them earlier). This latter will be convenient when it comes time to compute a cross section.

Of course, we could have kept finite volume in all the non-external stuff as well and arrived at an exactly analogous answer. The integral over continuous $\vec{q}$ values traces all the way back to QFT-I where we wrote for the scalar field case

$$\begin{aligned}[\phi^+(x), \phi^-(y)] &= \frac{1}{2V}\sum_{\vec{k}\vec{k}'}\frac{1}{\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}}[a(\vec{k}), a^\dagger(\vec{k}')]e^{-ik\cdot x+ik'\cdot y} \\ &\overset{V\to\infty}{\to} \frac{1}{2(2\pi)^3}\int\frac{d^3\vec{k}}{\omega_{\vec{k}}}e^{-ik\cdot(x-y)} \\ &\equiv i\Delta^+(x-y)\,. \end{aligned} \tag{71}$$

We could have simply written $i\Delta^+(x-y) = \frac{1}{2V}\sum_{\vec{k}}\frac{1}{\omega_{\vec{k}}}e^{-ik\cdot(x-y)}$. When we reexpressed as a contour integral, the dk^0 would still have been a continuous integral, but the $\frac{d^3\vec{k}}{(2\pi)^3}$ would have been replaced by $\frac{1}{V}\sum_{\vec{k}}$. And in the algebra on the preceding page, the $\int d^3x_1 \int d^3x_2$ would yield

$$\int d^3x_1 \int d^3x_2 e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)-iq\cdot(x_1-x_2)} = V^2 \delta_{\vec{k}\,'+\vec{p}\,',\vec{q}}\ \delta_{\vec{q},\vec{p}+\vec{k}}\,, \tag{72}$$

(the $dx_1^0 dx_2^0$ would still be over all time and you would have Dirac delta functions from these integrals still). The final stages of the calculation would have then been:

$$\begin{aligned}
& \overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}')\left[-e^2\int d^4x_1\int d^4x_2 \overbrace{\psi_1\overline{\psi}_2} e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)}\right]\not\epsilon_s(\vec{k})u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}')\left[-e^2\int d^4x_1 d^4x_2\left\{\frac{i}{(2\pi)V}\int dq^0\sum_{\vec{q}}\frac{\not q+m}{q^2-m^2}e^{-iq\cdot(x_1-x_2)}\right\}\right. \\
& \qquad \left. e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)}\right]\not\epsilon_s(\vec{k})u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}')\left[-e^2\frac{i}{(2\pi)V}\int dq^0\sum_{\vec{q}}\frac{\not q+m}{q^2-m^2}(2\pi)\delta([k'+p'-q]^0)(2\pi)\delta([q-p-k]^0)\right] \\
& \times\left\{V^2\delta_{\vec{k}\,'+\vec{p}\,',\vec{q}}\ \delta_{\vec{q},\vec{p}+\vec{k}}\right\}\not\epsilon_s(\vec{k})u_r(\vec{p}) \\
= \quad & -e^2(2\pi)V\delta([p'+k'-p-k]^0)\delta_{\vec{p}\,'+\vec{k}\,',\vec{p}+\vec{k}}\ \overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}')\left[i\frac{\not q+m}{q^2-m^2}\right]_{q=p+k}\not\epsilon_s(\vec{k})u_r(\vec{p}) \\
& \qquad \times\frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}\,'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}}
\end{aligned} \tag{73}$$

This shows that the infinite volume limit corresponds to

$$V\delta_{\vec{a},\vec{b}} \to (2\pi)^3\delta(\vec{a}-\vec{b})\,. \tag{74}$$

In any case, putting aside the normalization factors and exact version of three momentum conservation delta function, the structure we have obtained makes a lot of sense. We observe an over all coupling strength of e^2, overall 4-momentum conservation δ functions, and a very specific algebraic structure that can be associated with a momentum-space Feynman diagram as shown in the first diagram of the figure.

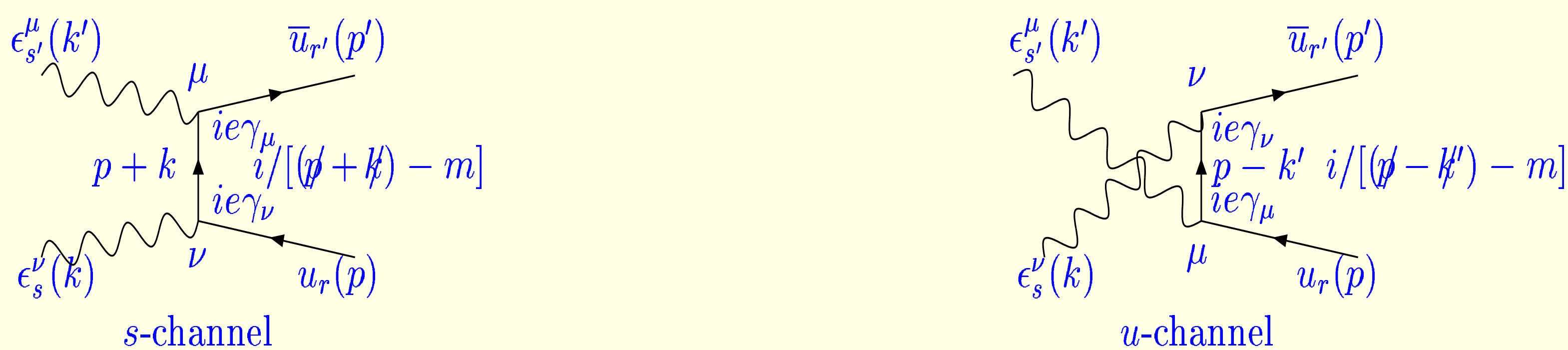

Figure 1: The two Feynman diagrams contributing to $e^-\gamma \to e^-\gamma$.

The 2nd term of Eq. (65) gives a similar result. We just have to be a little more careful with the Dirac structure. We have

$$
\begin{aligned}
& \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 :\overline{\psi}_1^{\,-} \not{A}_1^{\,+}\overbracket{\psi_1\overline{\psi}_2}\not{A}_2^{\,-}\psi_2^+ :\Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 :\overline{\psi}_1^{\,-}\gamma_\nu A_1^{\nu\,+}\overbracket{\psi_1\overline{\psi}_2}\gamma_\mu A_2^{\mu\,-}\psi_2^+ :\Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}')c_{r'}(\vec{p}')\Big[-e^2\int d^4x_1\int d^4x_2 A_2^{\mu\,-}\overline{\psi}_1^{\,-}\gamma_\nu\overbracket{\psi_1\overline{\psi}_2}\gamma_\mu\psi_2^+A_1^{\nu\,+}\Big]c_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \epsilon_{s'}{}^{\mu}(\vec{k}')\overline{u}_{r'}(\vec{p}')\gamma_\nu\Big[-e^2\int d^4x_1\int d^4x_2\overbracket{\psi_1\overline{\psi}_2}e^{ix_2\cdot k'+ix_1\cdot p'-ix_1\cdot k-ix_2\cdot p}\Big]\gamma_\mu u_r(\vec{p})\epsilon_s{}^{\nu}(\vec{k}) \\
= \quad & \overline{u}_{r'}(\vec{p}')\not{\epsilon}_s(\vec{k})\Big[-e^2\int d^4x_1 d^4x_2\left\{\frac{i}{(2\pi)^4}\int d^4q\frac{\not{q}+m}{q^2-m^2}e^{-iq\cdot(x_1-x_2)}\right\} \\
& \qquad e^{ix_2\cdot k'+ix_1\cdot p'-ix_1\cdot k-ix_2\cdot p}\Big]\not{\epsilon}_{s'}(\vec{k}')u_r(\vec{p}) \\
= \quad & \overline{u}_{r'}(\vec{p}')\not{\epsilon}_s(\vec{k})\Big[-e^2\frac{i}{(2\pi)^4}\int d^4q\frac{\not{q}+m}{q^2-m^2}(2\pi)^4\delta^4(p'-k-q)(2\pi)^4\delta^4(q+k'-p)\Big] \\
& \not{\epsilon}_{s'}(\vec{k}')u_r(\vec{p}) \\
= \quad & -e^2(2\pi)^4\delta^4(p'+k'-p-k)\overline{u}_{r'}(\vec{p}')\not{\epsilon}_s(\vec{k})\left[i\frac{\not{q}+m}{q^2-m^2}\right]_{q=p-k'}\not{\epsilon}_{s'}(\vec{k}')u_r(\vec{p}) \\
& \qquad \times\frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}} \qquad (75)
\end{aligned}
$$

- **So, the rules that you would use to get the two contributions to $\langle f|S|i\rangle$ are the following.**

1. **Draw the diagrams involving two external photons, a final e^- and an initial e^- (the latter connected by a continuously uni-directional e^--fermion flow line) and only $e^- \to e^-\gamma$ "vertices" as specified by the form of $\mathcal{H}_I$ and having no loops.**

   You should quickly understand that you must have exactly two vertices and that there are only the two *topologically distinct* **(including the arrow on the electron line) diagrams that we have drawn.**

2. **Now write down an algebraic expression for each of the two diagrams using the following rules. For diagram 1 we do the following.**

   (a) **For the outgoing $e^-(p', r')$ write on the far left of the Dirac structure a $\overline{u}_{r'}(\vec{p}\,')$.**

   (b) **With the outgoing external photon associate a polarization vector $\epsilon^{\mu}_{s'}(\vec{k}')$ (in a complex polarization basis this would actually be ϵ^*).**

   (c) **Attach the outgoing photon $\epsilon^{\mu}_{s'}(\vec{k}')$ to a $ie\gamma_\mu$ written just to the right of the $\overline{u}_{r'}(\vec{p}\,')$.**

   (d) **Then write down the momentum space virtual e^- propagator**

$$i\frac{\not{q} + m}{q^2 - m^2 + i\epsilon} = \frac{i}{\not{q} - m + i\epsilon} \tag{76}$$

with the momentum q carried by the virtual e^- given by momentum conservation, i.e. $q = p + k$ for the first diagram.

(e) Then comes the $ie\gamma_\nu$ to which we attach the incoming photon $\epsilon_s^\nu(\vec{k})$.

(f) Finally write down to the far right of the Dirac structure the spinor $u_r(\vec{p})$ associated with the incoming e^-.

3. For the 2nd diagram, we follow a very similar procedure except for the following.

(a) We first attach the incoming $\epsilon_s^\nu(\vec{k})$ to a $ie\gamma_\nu$ just to the right of the $\overline{u}_{r'}(\vec{p}\,')$.

(b) We employ the e^- propagator using the appropriate momentum conservation value of $q = p - k'$.

(c) We then attach $\epsilon_{s'}^\mu(\vec{k}')$ for the outgoing photon to a $ie\gamma_\mu$ written to the right of the Feynman propagator.

(d) We finish the Dirac structure with the $u_r(\vec{p})$.

4. These rules define the amplitudes (denoted $\mathcal{M}_a$ and $\mathcal{M}_b$ in MS).

$$\mathcal{M}_a = -e^2 \overline{u}_{r'}(\vec{p}\,') \not\epsilon_{s'}(\vec{k}') \left[i \frac{\not q + m}{q^2 - m^2} \right]_{q=p+k} \not\epsilon_s(\vec{k}) u_r(\vec{p}) \tag{77}$$

$$\mathcal{M}_b = -e^2 \overline{u}_{r'}(\vec{p}\,') \not\epsilon_s(\vec{k}) \left[i \frac{\not q + m}{q^2 - m^2} \right]_{q=p-k'} \not\epsilon_{s'}(\vec{k}') u_r(\vec{p}) \tag{78}$$

The full amplitude is the sum $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$, and is to be multiplied by $(2\pi)^4\delta^4(p' + k' - p - k)$, the overall momentum conservation delta function, and the product of the $1/\sqrt{2VE}$ normalization factors.

- One of the key ingredients above is the $ie\gamma_\mu$ "vertex". Algebraically, you can think of this factor as coming from:

  1. $-i$ for each power of $\mathcal{H}_I$ appearing in the required level of S-matrix expansion;
  2. $-e\gamma_\mu$ as appearing in each $\mathcal{H}_I$ in association with constructing the $\not{A}$ appearing therein.

  This vertex is associated with each interaction of a photon and a fermion whether these are real or virtual.

  In the above case, the initial or final e^- is real, the photons are real, but the intermediate e^- is virtual.

  Regardless, one writes down (assuming incoming or outgoing electrons, and not positrons)

$$\overline{u}_r(p) ie\gamma_\mu u_s(q)\epsilon_t^\mu(k) \tag{79}$$

  for each vertex, even when one of the particles is virtual.

In addition, each internal virtual propagator is then associated with the $i/(q^2 - m^2)$ factor. We will shortly understand why I have left out the $\not{q} + m$ part.

In the above process, this procedure works as follows, using the example of diagram 1.

1. We have a $\epsilon^{\mu}_{s'}(k')\overline{u}_{r'}(\vec{p}\,')ie\gamma_\mu u_t(q)$ for the upper vertex. (I am free to write the ϵ, which is just a number, anywhere I want.)
2. We have a $\overline{u}_t(q)ie\gamma_\nu u_r(\vec{p})\epsilon^{\nu}_{s}(\vec{k})$ for the lower vertex.
3. We have the $i/(q^2 - m^2)$ factor with $q = p + k$ by momentum conservation.
4. We sum over possible intermediate spin states t for the virtual electron.
5. We use $\sum_t u_t(q)\overline{u}_t(q) = \not{q} + m$ which is true even if we allow the q to have an off-shell energy.
6. The result,

$$\sum_t \epsilon^{\mu}_{s'}(\vec{k}')\overline{u}_{r'}(\vec{p}\,')ie\gamma_\mu u_t(q)\frac{i}{q^2 - m^2}\overline{u}_t(q)ie\gamma_\nu u_r(\vec{p})\epsilon^{\nu}_{s}(\vec{k})$$

$$= \epsilon^{\mu}_{s'}(\vec{k}')\overline{u}_{r'}(\vec{p}\,')ie\gamma_\mu i\left[\frac{\not{q} + m}{q^2 - m^2}\right]_{q=p+k=p'+k'} ie\gamma_\nu u_r(\vec{p})\epsilon^{\nu}_{s}(\vec{k})$$

$$= \quad \overline{u}_{r'}(\vec{p}\,')ie\not{\epsilon}_{s'}(\vec{k}')i\left[\frac{\not{q}+m}{q^2-m^2}\right]_{q=p+k=p'+k'} ie\not{\epsilon}_s(\vec{k})u_r(\vec{p})\,, \quad (80)$$

is precisely what we had before.

- There are a few more rules that we can only get by considering a few other processes. For the rules for external positrons, we need the appropriate "killing" operations. For an incoming positron contained in $|i\rangle$ we have:

$$\begin{aligned}\overline{\psi}^{\,+}(x)d_r^\dagger(\vec{p})|0\rangle &= \sum_{\vec{q},t}\frac{1}{\sqrt{2VE_{\vec{q}}}}d_t(\vec{q})\overline{v}_t(\vec{q})e^{-iq\cdot x}d_r^\dagger(\vec{p})|0\rangle \\ &= \frac{1}{\sqrt{2VE_{\vec{p}}}}\overline{v}_r(\vec{p})e^{-ip\cdot x}|0\rangle \qquad (81)\end{aligned}$$

after using $[d_t(\vec{q}), d_r^\dagger(\vec{p})]_+ = \delta_{rt}\delta_{\vec{q},\vec{p}}$.

Similarly, for an outgoing positron in $|f\rangle$, we find

$$\langle 0|d_{r'}(\vec{p}\,')\psi^{-}(x) = \langle 0|d_{r'}(\vec{p}\,')\sum_{\vec{q},t}\frac{1}{\sqrt{2VE_{\vec{q}}}}d_t^\dagger(\vec{q})v_t(\vec{q})e^{+iq\cdot x}$$

$$= \quad \frac{1}{\sqrt{2VE_{\vec{p}\,'}}} v_{r'}(\vec{p}\,')e^{+ip'\cdot x}\langle 0| \,. \tag{82}$$

Note how these differ from the corresponding electron results. Where we have a $u(\vec{p})$ for an incoming electron we find $\overline{v}(\vec{p})$ for an incoming positron, and where we had $\overline{u}(\vec{p})$ for an outgoing electron, we find $v(\vec{p})$ for an outgoing positron. As a result, the $\overline{v}$ and $\overline{u}$ spinors both appear at the end of the line directed in the direction of *electron* (i.e. fermion vs. antifermion) flow. For instance, an incoming positron has the arrow of fermion flow directed in the outgoing direction, and one writes $\overline{v}$.

- As an example of how to use the above results, let us consider $e^+(p)\gamma(k) \to e^+(p')\gamma(k')$ scattering. There would be the same two types of contributions, but we would have to focus on different contributions to the S matrix. One contribution would be:

$$\begin{aligned}
& \langle 0|a_{s'}(\vec{k}\,')d_{r'}(\vec{p}\,')\Big[-e^2\int d^4x_1\int d^4x_2 : \overline{\psi}{}_1^+{}_\alpha \left[\not{A}_1^- \psi_1\overline{\psi}_2 \not{A}_2^+\right]_{\alpha\beta}\psi_2^-{}_\beta :\Big] d_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}\,')d_{r'}(\vec{p}\,')\Big[+e^2\int d^4x_1\int d^4x_2 \psi_2^-{}_\beta \left[\not{A}_1^- \psi_1\overline{\psi}_2 \not{A}_2^+\right]_{\alpha\beta}\overline{\psi}{}_1^+{}_\alpha\Big] d_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle \\
= \quad & \langle 0|a_{s'}(\vec{k}\,')d_{r'}(\vec{p}\,')\Big[+e^2\int d^4x_1\int d^4x_2 \psi_2^-{}_\beta \left[\gamma_\mu A_1^{\mu\,-} \psi_1\overline{\psi}_2 \gamma_\nu A_2^{\nu\,+}\right]_{\alpha\beta}\overline{\psi}{}_1^+{}_\alpha\Big] d_r^\dagger(\vec{p})a_s^\dagger(\vec{k})|0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= v_{r'}(\vec{p}')_\beta \Big[\gamma_\mu \epsilon^\mu_{s'}(\vec{k}')\Big[+e^2 \int d^4x_1 \int d^4x_2 \overline{\psi}_1\overline{\psi}_2 e^{ix_2\cdot p'+ix_1\cdot k'-ix_1\cdot p-ix_2\cdot k}\Big]\gamma_\nu \epsilon^\nu_s(\vec{k})\Big]_{\alpha\beta}\overline{v}_r(\vec{p})_\alpha \\
&= \overline{v}_r(\vec{p})\not\epsilon_{s'}(\vec{k}')\Big[+e^2 \int d^4x_1 \int d^4x_2 \overline{\psi}_1\overline{\psi}_2 e^{ix_2\cdot p'+ix_1\cdot k'-ix_1\cdot p-ix_2\cdot k}\Big]\not\epsilon_s(\vec{k})v_{r'}(\vec{p}') \\
&= \overline{v}_r(\vec{p})\not\epsilon_{s'}(\vec{k}')\Big[+e^2 \int d^4x_1 d^4x_2 \left\{\frac{i}{(2\pi)^4}\int d^4q \frac{\not q + m}{q^2-m^2} e^{-iq\cdot(x_1-x_2)}\right\} \\
&\qquad e^{ix_2\cdot p'+ix_1\cdot k'-ix_1\cdot p-ix_2\cdot k}\Big]\not\epsilon_s(\vec{k})v_{r'}(\vec{p}') \\
&= \overline{v}_r(\vec{p})\not\epsilon_{s'}(\vec{k}')\Big[+e^2 \frac{i}{(2\pi)^4}\int d^4q \frac{\not q + m}{q^2-m^2}(2\pi)^4\delta^4(k'-p-q)(2\pi)^4\delta^4(q+p'-k)\Big] \\
&\quad \not\epsilon_s(\vec{k})v_{r'}(\vec{p}') \\
&= +e^2(2\pi)^4\delta^4(p'+k'-p-k)\overline{v}_r(\vec{p})\not\epsilon_{s'}(\vec{k}')\left[i\frac{\not q + m}{q^2-m^2}\right]_{q=k'-p}\not\epsilon_s(\vec{k})v_{r'}(\vec{p}') \\
&\qquad \times \frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}}
\end{aligned}
\tag{83}
$$

Note how, in going from the 1st line to the 2nd line above, I moved the $\overline{\psi}^+_{1\ \alpha}$ all the way to the right, past the $[\ldots]_{\alpha\beta}$ matrix and past $\psi^-_{2\ \beta}$. This is allowed since I retain proper Dirac matrix contractions so long as I keep track of the α index sums later, as I do. A similar statement applies to moving the $\psi^-_{2\ \beta}$ over to the left.

The final expression above can be mapped to the *second* **of the Feynman**

diagrams shown in the figure below.

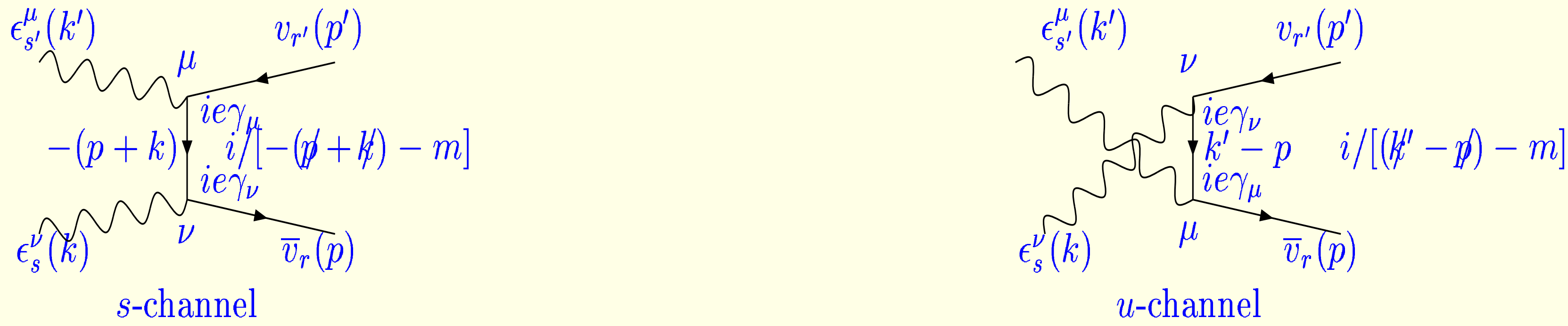

Figure 2: The two Feynman diagrams contributing to $e^+\gamma \to e^+\gamma$.

This figure will be confusing at first. The arrows show the direction of *fermion* **flow. But we must remember that the incoming (antifermion) e^+ has momentum p** *entering* **the diagram from the bottom right, and the outgoing e^+ has momentum p'** *exiting* **the diagram. I.e. anti-fermion momenta and fermion flow are oppositely directed. Momentum conservation for the virtual propagator is correctly given (for the direction of** *fermion* **flow) by $k'-p$ for the 2nd diagram. That is, we always write the Feynman propagator in the canonical form where q is given by the momentum carried in the fermionic direction, rather than the antifermionic direction.**

I will not work out the details of the 2nd contribution, which gives you the first (left-hand) diagram of the above figure. Using our Feynman rules we

will get

$$+e^2(2\pi)^4\delta^4(p'+k'-p-k)\overline{v}_r(\vec{p})\not\epsilon_s(\vec{k})\left[i\frac{\not q+m}{q^2-m^2}\right]_{q=-k-p}\not\epsilon_{s'}(\vec{k}')v_{r'}(\vec{p}')$$
$$\times\frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{p}'}}\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}'|}}. \tag{84}$$

A homework problem will be to verify this result in the same kind of detail as given for the right-hand diagram.

- **One important note on the above. We were careful to keep the sign resulting from strict application of Wick's theorem, which ended up giving us a $+e^2$ overall sign for this positron case as opposed to the $-e^2$ overall sign for the electron case.**

  Such an overall sign is not actually observable, since we will eventually square these amplitudes.

- **However, one should be very careful with signs in cases where different diagrams correspond to the exchange of identical fermions in the final state**

  An example of this, which is also useful for getting the correct result for a virtual photon propagator, is provided by $e^-e^-\to e^-e^-$ scattering.

For this process, we need two ψ fields and two $\overline{\psi}$ fields in order "kill" the creation operators defining the incoming and outgoing states. More precisely, we will need 2 ψ^+'s and 2 $\overline{\psi}^-$'s. This means we are again dealing with $S^{(2)}$ the 2nd order term in the S-matrix expansion, which I repeat below. Note that since each of the two $\mathcal{H}_I$'s required contains an A field, we will have to contract these two A fields against one another in order to generate a non-zero contribution to the scattering process of interest. The relevant stuff is thus:

$$
\begin{aligned}
S^{(2)} &= -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 T\{:\overline{\psi}_1 \not{A}_1 \psi_1 :: \overline{\psi}_2 \not{A}_2 \psi_2 :\} \\
&\ni -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 : \overline{\psi}_1^{\,-}{}_\alpha (\gamma_\mu)_{\alpha\beta} \psi_1^+{}_\beta \overbracket{A_1^\mu A_2^\nu} \overline{\psi}_2^{\,-}{}_\delta (\gamma_\nu)_{\delta\rho} \psi_2^+{}_\rho : \\
&= -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 \overline{\psi}_2^{\,-}{}_\delta \overline{\psi}_1^{\,-}{}_\alpha (\gamma_\mu)_{\alpha\beta} \overbracket{A_1^\mu A_2^\nu} (\gamma_\nu)_{\delta\rho} \psi_1^+{}_\beta \psi_2^+{}_\rho \,, \qquad (85)
\end{aligned}
$$

where two changes of sign occurred when we passed two fermionic operators past one another in order to get to the explicitly normal ordered form given.

We must now adopt a convention for our $|i\rangle$ and $|f\rangle$ states. We use p, r and k, s for the incoming guys and p', r' and k', s' for the outgoing guys so that we define

$$|i\rangle = c_r^\dagger(\vec{p})c_s^\dagger(\vec{k})|0\rangle\,, \quad |f\rangle = c_{r'}^\dagger(\vec{p}\,')c_{s'}^\dagger(\vec{k}')|0\rangle\,,$$
$$\Rightarrow \quad \langle f| = \langle 0|c_{s'}(\vec{k}')c_{r'}(\vec{p}\,')\,, \tag{86}$$

where the precise order of the $c^\dagger$ operators must be carefully adhered to because of the minus signs from various anticommutators.

Now, let us consider the structure

$$\begin{aligned} &\psi_{2\ \rho}^{+} c_r^\dagger(\vec{p})c_s^\dagger(\vec{k})|0\rangle \\ &= \sum_{t,\vec{l}} \frac{1}{\sqrt{2VE_{\vec{l}}}} c_t(\vec{l})u_t(\vec{l})_\rho e^{-il\cdot x_2} c_r^\dagger(\vec{p})c_s^\dagger(\vec{k})|0\rangle \end{aligned} \tag{87}$$

Contained within this sequence is the structure

$$\begin{aligned} c_t(\vec{l})c_r^\dagger(\vec{p})c_s^\dagger(\vec{k})|0\rangle &= \left[\delta_{tr}\delta_{\vec{l}\vec{p}}c_s^\dagger(\vec{k}) - c_r^\dagger(\vec{p})c_t(\vec{l})c_s^\dagger(\vec{k})\right]|0\rangle \\ &= \left[\delta_{tr}\delta_{\vec{l}\vec{p}}c_s^\dagger(\vec{k}) - c_r^\dagger(\vec{p})\delta_{ts}\delta_{\vec{l}\vec{k}}\right]|0\rangle\,. \end{aligned} \tag{88}$$

If we now bring into play the

$$\psi^{+}_{1\ \beta} = \sum_{u,\vec{q}} \frac{1}{\sqrt{2VE_{\vec{q}}}} c_u(\vec{q}) u_u(\vec{q})_\beta e^{-iq\cdot x_1} \tag{89}$$

we will need the structure

$$\begin{aligned} c_u(\vec{q})c_t(\vec{l})c_r^\dagger(\vec{p})c_s^\dagger(\vec{k})|0\rangle &= c_u(\vec{q})\left[\delta_{tr}\delta_{\vec{l}\vec{p}}c_s^\dagger(\vec{k}) - c_r^\dagger(\vec{p})\delta_{ts}\delta_{\vec{l}\vec{k}}\right]|0\rangle \\ &= \left[\delta_{tr}\delta_{\vec{l}\vec{p}}\delta_{us}\delta_{\vec{q}\vec{k}} - \delta_{ur}\delta_{\vec{q}\vec{p}}\delta_{ts}\delta_{\vec{l}\vec{k}}\right]|0\rangle\,. \end{aligned} \tag{90}$$

Obviously, the two terms differ by the interchange of $t,\vec{l} \leftrightarrow u,\vec{q}$ or, equivalently, $r,\vec{p} \leftrightarrow s,\vec{k}$ and the minus sign which is keeping track of the fermionic statistics.

Using these δ functions to eliminate the $\sum_{u,\vec{q}}$ and $\sum_{t,\vec{l}}$ yields

$$\begin{aligned} \frac{1}{\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{k}}}}\Big[& e^{-ik\cdot x_1 - ip\cdot x_2} u_s(\vec{k})_\beta u_r(\vec{p})_\rho \\ & - e^{-ip\cdot x_1 - ik\cdot x_2} u_r(\vec{p})_\beta u_s(\vec{k})_\rho \Big] \end{aligned} \tag{91}$$

These clearly differ by the interchange $r, \vec{p} \leftrightarrow s, \vec{k}$ and the fermionic statistics sign.

We would now have to carry out an exactly equivalent game for the left hand part of the expression in Eq. (85). Without giving the details, I hope it is relatively straightforward to understand the result. One simply has to switch everything to primes and do the barring (which includes a complex conjugation of the exponential factors and $u \to \overline{u}$) and change the Dirac indices appropriately ($\beta \to \alpha$ and $\rho \to \delta$):

$$
\begin{aligned}
& \langle 0 | c_{s'}(\vec{k}') c_{r'}(\vec{p}') \overline{\psi}_2^{\,-}{}_\delta \overline{\psi}_1^{\,-}{}_\alpha \\
= \quad & \frac{1}{\sqrt{2VE_{\vec{p}'}}\sqrt{2VE_{\vec{k}'}}} \Big[e^{+ik'\cdot x_1 + ip'\cdot x_2} \overline{u}_{s'}(\vec{k}')_\alpha \overline{u}_{r'}(\vec{p}')_\delta \\
& \qquad - e^{+ip'\cdot x_1 + ik'\cdot x_2} \overline{u}_{r'}(\vec{p}')_\alpha \overline{u}_{s'}(\vec{k}')_\delta \Big] . \qquad (92)
\end{aligned}
$$

Note the antisymmetry under $r', \vec{p}\,' \leftrightarrow s', \vec{k}\,'$ corresponding to Fermi statistics. This and the initial-state antisymmetry under $r, \vec{p} \leftrightarrow s, \vec{k}$ guarantee that we get zero if the initial or final electrons are in exactly the same physical state.

We now combine the results of Eqs. (91) and (92) with the remainder of the stuff in Eq. (85) to obtain the result

$$\begin{aligned}\langle f|S^{(2)}|i\rangle &= \frac{-e^2}{2}\frac{1}{\sqrt{2VE_{\vec{p}'}}\sqrt{2VE_{\vec{k}'}}\sqrt{2VE_{\vec{p}}}\sqrt{2VE_{\vec{k}}}}\int d^4x_1 \int d^4x_2 \times \\ &\left[e^{+ik'\cdot x_1+ip'\cdot x_2}\overline{u}_{s'}(\vec{k}')_\alpha \overline{u}_{r'}(\vec{p}')_\delta - e^{+ip'\cdot x_1+ik'\cdot x_2}\overline{u}_{r'}(\vec{p}')_\alpha \overline{u}_{s'}(\vec{k}')_\delta\right] \\ &(\gamma_\mu)_{\alpha\beta}\left[\frac{i}{(2\pi)^4}\int d^4q e^{-iq\cdot(x_1-x_2)}\frac{-g^{\mu\nu}}{q^2+i\epsilon}\right](\gamma_\nu)_{\delta\rho} \\ &\left[e^{-ik\cdot x_1-ip\cdot x_2}u_s(\vec{k})_\beta u_r(\vec{p})_\rho - e^{-ip\cdot x_1-ik\cdot x_2}u_r(\vec{p})_\beta u_s(\vec{k})_\rho\right] \end{aligned} \tag{93}$$

where we inserted the Fourier representation of $\overbrace{A_1^\mu A_2^\nu} = iD_F^{\mu\nu}(x_1 - x_2)$.

Of the four terms in the above expression, two are equal to corresponding ones of the remaining two. This will cancel the $\frac{1}{2}$ appearing in $e^2/2$. This always happens in QED. We saw another example of this in our earlier calculations. There is equivalence under interchange of the x_1 and x_2 vertices (i.e under $\mathcal{H}_1 \leftrightarrow \mathcal{H}_2$). Let us focus on two inequivalent terms. We begin with (dropping some external factors for the moment) taking the 1st term in each of the large brackets of the above expression:

$$\begin{aligned}
&\int d^4x_1 \int d^4x_2 \left[e^{+ik'\cdot x_1 + ip'\cdot x_2} \overline{u}_{s'}(\vec{k}')_\alpha \overline{u}_{r'}(\vec{p}')_\delta \right] (\gamma_\mu)_{\alpha\beta} \\
&\qquad \left[\frac{i}{(2\pi)^4} \int d^4q e^{-iq\cdot(x_1 - x_2)} \frac{-g^{\mu\nu}}{q^2 + i\epsilon} \right] (\gamma_\nu)_{\delta\rho} \left[e^{-ik\cdot x_1 - ip\cdot x_2} u_s(\vec{k})_\beta u_r(\vec{p})_\rho \right] \\
&= (2\pi)^4 \delta^4(p + k - p' - k') \overline{u}_{s'}(\vec{k}') \gamma_\mu u_s(\vec{k}) \left[\frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \right]_{q = k' - k = p - p'} \overline{u}_{r'}(\vec{p}') \gamma_\nu u_r(\vec{p}) \qquad (94)
\end{aligned}$$

where we did the d^4x_1 and d^4x_2 integrals to get $(2\pi)^4\delta^4(k' - q - k)$ and $(2\pi)^4\delta^4(p' + q - p)$ respectively and used one of these to do the d^4q integral. Now, we take the 2nd term of the 1st large bracket and the 1st term of the 2nd large bracket. We have

$$\begin{aligned}
&\int d^4x_1 \int d^4x_2 \left[-e^{+ip'\cdot x_1 + ik'\cdot x_2} \overline{u}_{r'}(\vec{p}')_\alpha \overline{u}_{s'}(\vec{k}')_\delta \right] (\gamma_\mu)_{\alpha\beta} \\
&\qquad \left[\frac{i}{(2\pi)^4} \int d^4q e^{-iq\cdot(x_1 - x_2)} \frac{-g^{\mu\nu}}{q^2 + i\epsilon} \right] (\gamma_\nu)_{\delta\rho} \left[e^{-ik\cdot x_1 - ip\cdot x_2} u_s(\vec{k})_\beta u_r(\vec{p})_\rho \right] \\
&= -(2\pi)^4 \delta^4(p + k - p' - k') \overline{u}_{r'}(\vec{p}') \gamma_\mu u_s(\vec{k}) \left[\frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \right]_{q = p' - k = p - k'} \overline{u}_{s'}(\vec{k}') \gamma_\nu u_r(\vec{p}) \qquad (95)
\end{aligned}$$

where we did the d^4x_1 and d^4x_2 integrals to get $(2\pi)^4\delta^4(p'-q-k)$ and $(2\pi)^4\delta^4(k'+q-p)$ respectively and used one of these to do the d^4q integral. A quick inspection of Eq. (94) compared to Eq. (95) shows that the relation corresponds to interchanging the $r', \vec{p}\,'$ final state e^- with the $s', \vec{k}'$ final state e^- and introducing a relative minus sign between the two contributions. The remaining two contributions obtained by using the 2nd term of the 2nd large bracket in Eq. (93) simply duplicate the two contributions that we have already obtained. Thus, after extracting the standard $(2\pi)^4\delta^4(p'+k'-p-k)$ overall momentum conservation factor with its $(2\pi)^4$ factor and the four $1/\sqrt{2VE}$ factors, we end up with the two invariant amplitudes:

$$
\begin{aligned}
\mathcal{M}_a &= -e^2\overline{u}_{s'}(\vec{k}')\gamma_\mu u_s(\vec{k})\left[\frac{-ig^{\mu\nu}}{q^2+i\epsilon}\right]_{q=k'-k=p-p'}\overline{u}_{r'}(\vec{p}\,')\gamma_\nu u_r(\vec{p}) \\
\mathcal{M}_b &= +e^2\overline{u}_{r'}(\vec{p}\,')\gamma_\mu u_s(\vec{k})\left[\frac{-ig^{\mu\nu}}{q^2+i\epsilon}\right]_{q=p'-k=p-k'}\overline{u}_{s'}(\vec{k}')\gamma_\nu u_r(\vec{p}) \qquad (96)
\end{aligned}
$$

the factor of $\frac{1}{2}$ having been canceled by the above-noted factor of 2 duplication. The relative $-$ sign becomes part of our Feynman rules:

– Whenever there are two contributing diagrams that are related to one another by interchanging fermions of the same type, one writes the same type of expression for each but introduces a relative minus sign.

We note that while the relative sign between $\mathcal{M}_a$ and $\mathcal{M}_b$ is determined, the signs of both could be switched without any physical impact in computing probabilities (related to $|\mathcal{M}_a + \mathcal{M}_b|^2$). The overall sign is pure convention, related to how we defined $|i\rangle$ and $|f\rangle$. Had we reordered the $c^\dagger$ operators in the definition of one of these two states, both $\mathcal{M}_a$ and $\mathcal{M}_b$ would have changed sign.

The Feynman diagrams associated with $\mathcal{M}_a$ and $\mathcal{M}_b$ are shown in the figure below.

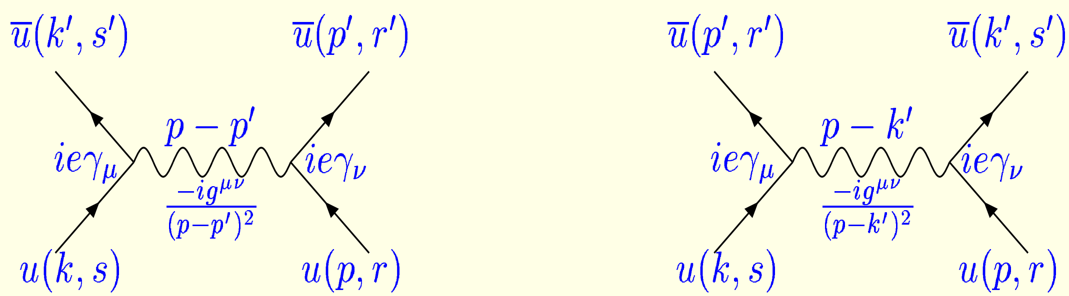

Figure 3: The two Feynman diagrams contributing to $e^- e^- \to e^- e^-$.

- So, by combining the results for the various processes we have computed we obtain an almost complete set of Feynman rules for QED.

1. Draw all possible tree-level (lowest possible order) diagrams that can give rise to the process of interest making use of just the fermion $\rightarrow$ fermion + photon "vertex".

   In so doing, keep track of the fermion line direction. An incoming fermion line must be traced continuously to an outgoing fermion line (or, equivalently, to an incoming antifermion line).
2. For each vertex, write $ie\gamma_\mu$, where a different Lorentz index μ must be used for each vertex and this μ must be connected either to the μ on a virtual photon propagator or to the μ of an external photon polarization vector.
3. For an outgoing photon γ, p, s, write $\epsilon_s^*(\vec{p})$, where I here allow for a complex polarization basis.
4. For an incoming photon γ, p, s write $\epsilon_s(\vec{p})$.
5. For an incoming e^-, p, r write $u_r(\vec{p})$.
6. For an outgoing e^-, p, r write $\overline{u}_r(\vec{p})$.
7. For an incoming e^+, p, r write $\overline{v}_r(\vec{p})$.
8. For an outgoing e^+, p, r write $v_r(\vec{p})$.
9. For the positron cases, note that the momentum direction will be opposite the $fermion$ arrow direction.

10. For an internal (virtual) electron propagator write

$$i\frac{\not{q} + m}{q^2 - m^2 + i\epsilon} \tag{97}$$

where q is the momentum *in the direction of the fermion arrow* and the value of q is obtained by momentum conservation at the vertices.

11. For an internal photon propagator, write

$$i\frac{-g^{\mu\nu}}{q^2 + i\epsilon} \tag{98}$$

where q can be chosen in any direction and is given by momentum conservation at the vertices.

12. A fully contracted Dirac structure should be constructed for each fermion line, beginning with a $\overline{u}$ or $\overline{v}$ to the far left of the expression and working back *against* the fermion arrow flow direction until terminating on a u or v spinor.

    Along the way, one will encounter first a vertex and then, possibly, an internal fermion line, and then, possibly, another vertex, and so forth until ending with a vertex and then the final u or v.

13. If there are diagrams that differ by simply interchanging:
   - two initial e^-'s;
   - or two initial e^+'s;
   - or two final state e^-'s;
   - or two final state e^+'s;
   - or an initial e^+ with a final e^-;
   - or a final e^+ with an initial e^-;

   then a relative minus sign should be introduced between the corresponding Feynman amplitudes.
14. For each closed loop, there will be an integration over an unconstrained momentum, call it q_i. One should perform the integral $\frac{1}{(2\pi)^4}\int d^4q_i$ for each such unconstrained loop momentum.

   If the closed loop is a continuous fermion line, an explicit minus sign should be introduced.

- A note on quickly checking the sign of the vertex. We began with $\mathcal{L} = \overline{\psi} i \not{D} \psi$ with $D_\mu = \partial_\mu + iqA_\mu$ to obtain $\mathcal{L}_I = -q\overline{\psi}\not{A}\psi$. The convention for QED is that $q = -e$ so we end up with $\mathcal{L}_I = e\overline{\psi}\not{A}\psi = -\mathcal{H}_I$. The evolution operator appearing in the S matrix of the interaction picture is basically e^{-iH_It} which implies that S contains $(-iH_I)^n$ powers. The lowest order is just $-iH_I = +ie\overline{\psi}\not{A}\,\psi$. If you remove the external $\psi, \overline{\psi}, A^\mu$ fields the corresponding vertex in Dirac space is just $+ie\gamma_\mu$.

- It is the last of the Feynman rules above that we have yet to derive using an explicit example.

  We choose to use the virtual e^- loop correction to the photon propagator as our basic example. The basic Feynman diagram that corresponds to our calculation is below. There, I use the label p for the free momentum associated with the loop.

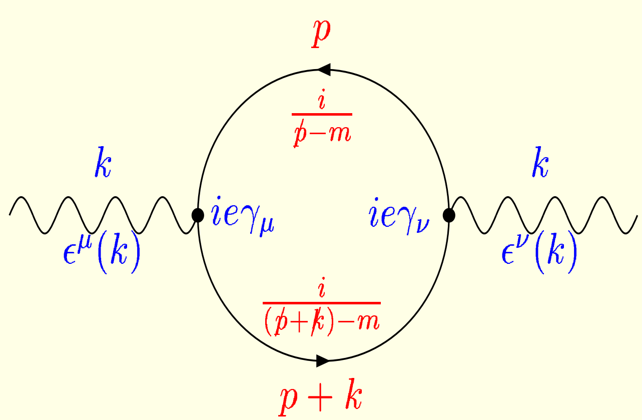

Figure 4: The e^- loop correction to the photon propagator.

The computation begins with $|i\rangle = a_r^\dagger(k)|0\rangle$ and ends with $|f\rangle = a_r^\dagger(k)|0\rangle$ and contains two vertices, so that means we are looking at $S^{(2)}$. We then have

$$\langle 0|a_r(\vec{k})\left[-\frac{e^2}{2}\int d^4x_1\int d^4x_2 T\left\{:\overline{\psi}_1\gamma_\mu A_1^\mu\psi_1::\overline{\psi}_2\gamma_\nu A_2^\nu\psi_2:\right\}\right]a_r^\dagger(\vec{k})|0\rangle \tag{99}$$

To reduce this expression, we must employ Wick's theorem. In this case, ψ_1 must contract with $\overline{\psi}_2$ and vice versa. (Recall: no contractions between fields in same $\mathcal{H}_I$.) In getting to the contraction configuration given below, I must do an odd number of fermion field interchanges — hence the very crucial extra $-$ sign! Keeping in mind that we must have one A^+ and one A^- in order to kill the a and $a^\dagger$ operators, the above reduces to (using the 1st line to simply display all the Dirac indices of the previous form above, and the 2nd line to display the particular contraction term in Wicks theorem that will survive as well as the two different choices as to which A is A^+ and which is A^- – we need one of each and there are two possibilities)

$$\begin{aligned}
&\langle 0|a_r(\vec{k})\left[-\frac{e^2}{2}\int d^4x_1\int d^4x_2 T\left\{:\overline{\psi}_{1\alpha}(\gamma_\mu)_{\alpha\beta}A_1^\mu\psi_{1\beta}::\overline{\psi}_{2\rho}(\gamma_\nu)_{\rho\sigma}A_2^\nu\psi_{2\sigma}:\right\}\right]a_r^\dagger(\vec{k})|0\rangle \\
= \quad &\langle 0|a_r(\vec{k})\left[+\frac{e^2}{2}\int d^4x_1\int d^4x_2 :\psi_{2\,\sigma}\overline{\psi}_{1\,\alpha}(\gamma_\mu)_{\alpha\beta}A_1^{\mu\,+}\psi_{1\,\beta}\overline{\psi}_{2\,\rho}(\gamma_\nu)_{\rho\sigma}A_2^{\nu\,-}:\right]a_r^\dagger(\vec{k})|0\rangle \\
&+\langle 0|a_r(\vec{k})\left[+\frac{e^2}{2}\int d^4x_1\int d^4x_2 :\psi_{2\,\sigma}\overline{\psi}_{1\,\alpha}(\gamma_\mu)_{\alpha\beta}A_1^{\mu\,-}\psi_{1\,\beta}\overline{\psi}_{2\,\rho}(\gamma_\nu)_{\rho\sigma}A_2^{\nu\,+}:\right]a_r^\dagger(\vec{k})|0\rangle
\end{aligned}$$

$$
\begin{aligned}
= & \int d^4x_1 \int d^4x_2 \langle 0|a_r(\vec{k}) A_2^{\nu\,-} \left[+ \frac{e^2}{2} \psi_{2\,\sigma} \bar{\psi}_{1\,\alpha} (\gamma_\mu)_{\alpha\beta} \psi_{1\,\beta} \bar{\psi}_{2\,\rho} (\gamma_\nu)_{\rho\sigma} \right] A_1^{\mu\,+} a_r^\dagger(\vec{k})|0\rangle \\
& + \int d^4x_1 \int d^4x_2 \langle 0|a_r(\vec{k}) A_1^{\mu\,-} \left[+ \frac{e^2}{2} \psi_{2\,\sigma} \bar{\psi}_{1\,\alpha} (\gamma_\mu)_{\alpha\beta} \psi_{1\,\beta} \bar{\psi}_{2\,\rho} (\gamma_\nu)_{\rho\sigma} \right] A_2^{\nu\,+} a_r^\dagger(\vec{k})|0\rangle \\
= & \int d^4x_1 \int d^4x_2 \epsilon_r^\nu(\vec{k}) e^{ik\cdot x_2} \left[+ \frac{e^2}{2} \psi_{2\,\sigma} \bar{\psi}_{1\,\alpha} (\gamma_\mu)_{\alpha\beta} \psi_{1\,\beta} \bar{\psi}_{2\,\rho} (\gamma_\nu)_{\rho\sigma} \right] \epsilon_r^\mu(\vec{k}) e^{-ik\cdot x_1} \\
& + \int d^4x_1 \int d^4x_2 \epsilon_r^\mu(\vec{k}) e^{ik\cdot x_1} \left[+ \frac{e^2}{2} \psi_{2\,\sigma} \bar{\psi}_{1\,\alpha} (\gamma_\mu)_{\alpha\beta} \psi_{1\,\beta} \bar{\psi}_{2\,\rho} (\gamma_\nu)_{\rho\sigma} \right] \epsilon_r^\nu(\vec{k}) e^{-ik\cdot x_2}
\end{aligned}
\tag{100}
$$

where I used the standard formulae for $A^+ a^\dagger|0\rangle$ and $\langle 0|aA^-$:

$$
\begin{aligned}
A^{\mu\,+}(x) a_s^\dagger(\vec{k})|0\rangle &= \frac{1}{\sqrt{2V|\vec{k}|}} \epsilon_s^\mu(\vec{k}) e^{-ik\cdot x}|0\rangle \,, \\
\langle 0|a_s(\vec{k}) A^{\mu\,-}(x) &= \frac{1}{\sqrt{2V|\vec{k}|}} \epsilon_s^\mu(\vec{k}) e^{ik\cdot x}\langle 0| \,.
\end{aligned}
\tag{101}
$$

derived earlier, dropping the two $1/\sqrt{2V|\vec{k}|}$ factors. If I interchange the x_1 and x_2 variables and relabel dummy indices ($\mu \leftrightarrow \nu$, $\alpha \leftrightarrow \rho$ and $\beta \leftrightarrow \sigma$), then it is easy to see that the 2nd term is identical to the first term.

We thus need only evaluate

$$
\begin{aligned}
&\int d^4x_1 \int d^4x_2 \epsilon_r^{\nu}(\vec{k}) e^{ik\cdot x_2} \left[+e^2 \psi_{2\,\sigma} \overline{\psi}_{1\,\alpha} (\gamma_\mu)_{\alpha\beta} \psi_{1\,\beta} \overline{\psi}_{2\,\rho} (\gamma_\nu)_{\rho\sigma} \right] \epsilon_r^{\mu}(\vec{k}) e^{-ik\cdot x_1} \\
= \quad &\int d^4x_1 \int d^4x_2 \epsilon_r^{\nu}(\vec{k}) e^{ik\cdot x_2} \Bigg[+e^2 \left[\frac{i}{(2\pi)^4} \int d^4q e^{-iq\cdot(x_2-x_1)} \left(\frac{\not{q} + m}{q^2 - m^2} \right)_{\sigma\alpha} \right] (\gamma_\mu)_{\alpha\beta} \\
&\qquad \left[\frac{i}{(2\pi)^4} \int d^4p e^{-ip\cdot(x_1-x_2)} \left(\frac{\not{p} + m}{p^2 - m^2} \right)_{\beta\rho} \right] (\gamma_\nu)_{\rho\sigma} \Bigg] \epsilon_r^{\mu}(\vec{k}) e^{-ik\cdot x_1} \\
= \quad &\epsilon_r^{\nu}(\vec{k}) \Bigg[+e^2 \left[\frac{i}{(2\pi)^4} \int d^4q \left(\frac{\not{q} + m}{q^2 - m^2} \right)_{\sigma\alpha} \right] (\gamma_\mu)_{\alpha\beta} \\
&\qquad \left[\frac{i}{(2\pi)^4} \int d^4p \left(\frac{\not{p} + m}{p^2 - m^2} \right)_{\beta\rho} \right] (2\pi)^4 \delta^4(k - q + p) (2\pi)^4 \delta^4(q - p - k) (\gamma_\nu)_{\rho\sigma} \Bigg] \epsilon_r^{\mu}(\vec{k}) \\
= \quad &(2\pi)^4 \delta^4(k - k)(+e^2) \int \frac{d^4p}{(2\pi)^4} \left[i \left(\frac{\not{q} + m}{q^2 - m^2} \right)_{q=p+k} \not{\epsilon}_r(\vec{k}) i \left(\frac{\not{p} + m}{p^2 - m^2} \right) \not{\epsilon}_r(\vec{k}) \right]_{\sigma\sigma}
\end{aligned}
\tag{102}
$$

After including the $1/\sqrt{2V|\vec{k}|}$ factors we can summarize our final result

as

$$\langle\gamma(r,\vec{k})|S^{(2)}|\gamma(r,\vec{k})\rangle = (2\pi)^4\delta^4(k-k)\frac{1}{\sqrt{2V|\vec{k}|}\sqrt{2V|\vec{k}|}}\mathcal{M}\,, \tag{103}$$

with

$$\mathcal{M} \;=\; +e^2\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[i\left(\frac{\not{q}\;+m}{q^2-m^2}\right)_{q=p+k}\not{\epsilon}_r(\vec{k})\,i\left(\frac{\not{p}\;+m}{p^2-m^2}\right)\not{\epsilon}_r(\vec{k})\right] \tag{104}$$

Note that this result has the opposite sign compared to what you would write down just naively using $ie\gamma_\mu$ and $ie\gamma_\nu$ for the two vertices and $i\frac{\not{q}\,+m}{q^2-m^2}$ and $i\frac{\not{p}\,+m}{p^2-m^2}$ for the internal fermion propagators.

Note also how the Dirac structure is written by starting to the left with the $p+k$ propagator and then working "backward" to the p propagator as shown in the earlier figure.

In this final form, we have also used the Trace notation. The Dirac index structure was such that one ended up with a continuous connection of

Dirac indices to one another as we moved along the fermion line, with the result corresponding to the trace of the complicated Dirac matrix product indicated.

Tree-level QED Processes

Computing a cross section

- We will consider colliding two particles in the initial state $|i\rangle$ with momenta $p_i = (E_i, \vec{p}_i)$, $i = 1, 2$. (Sorry about the confusion of notation; I am simply following MS here.) The final state $|f\rangle$ is assumed to contain N particles with $p'_f = (E'_f, \vec{p}'_f)$, $f = 1, \ldots N$.

- The S-matrix, which always has an overall δ^4 momentum conservation factor, can be written as

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(\sum_f p'_f - \sum_i p_i) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE'_f}\right)^{1/2} \mathcal{M} \tag{105}$$

The δ^4 function in the above is obtained in the limit of an infinite time interval $T \to \infty$ and an infinite volume, $V \to \infty$. For finite T and V, we would have obtained the same expression with

$$(2\pi)^4\delta^4(\sum_f p'_f - \sum_i p_i) = \lim_{\substack{T\to\infty \\ V\to\infty}} \delta_{TV}(\sum_f p'_f - \sum_i p_i)$$

$$\equiv \quad \lim_{\substack{T\to\infty \\ V\to\infty}} \int_{-T/2}^{+T/2} dt \int_V d^3\vec{x} \exp\left[ix\cdot(\sum_f p'_f - \sum_i p_i)\right] \qquad (106)$$

replaced by $\delta_{TV}(\sum_f p'_f - \sum_i p_i)$. In deriving the cross section, it is helpful to take T and V finite temporarily.

- For finite T and V, we can define the transition probability per unit of time as

$$w = |S_{fi}|^2/T \qquad (107)$$

which will contain the factor $\left[\delta_{TV}(\sum_f p'_f - \sum_i p_i)\right]^2$. For one of these factors, we revert back to the continuum result $\delta_{TV}(\sum_f p'_f - \sum_i p_i) \to (2\pi)^4\delta^4(\sum_f p'_f - \sum_i p_i)$. For the other factor, we can take $(\sum_f p'_f - \sum_i p_i) = 0$ and use $\delta_{TV}(0) = TV$, keeping the finite volume and time interval.

- **We then get**

$$w = V(2\pi)^4 \delta^4 (\sum_f p'_f - \sum_i p_i) \prod_i \left(\frac{1}{2VE_i} \right) \prod_f \left(\frac{1}{2VE'_f} \right) |\mathcal{M}|^2 . \quad (108)$$

This is the result for transition from state i to a single final state f. It is of course vanishingly small in the continuum limit of $V \to \infty$ as the final state possibilities become a continuum or possibilities.

In the $V \to \infty$ limit, it only makes sense to compute the transition probability to a group of states centered on some central state; one takes states in the interval $(\vec{p}'_f, \vec{p}'_f + d\vec{p}'_f)$, $f = 1, \ldots, N$. The number states in such an interval is given by

$$\prod_f \frac{V d^3 \vec{p}'_f}{(2\pi)^3} . \quad (109)$$

The factors of V, above, cancel those in the denominator in the final state part of the expression for w, leaving one V in the numerator from the δ_{TV} and two V's in the denominator from the initial state killing operations.

- **The differential cross section, $d\sigma$, is defined to be the transition rate into this group of final states for one scattering center divided by the flux of incident particles incident on the volume containing this one scattering center.**

  With our choice of normalization for the states, the volume V which we are considering contains just one scattering center. To show this (again), just compute

$$\langle \vec{p} | \vec{p} \rangle = \langle 0 | a(\vec{p}) a^\dagger(\vec{p}) | 0 \rangle = \delta_{\vec{p}\vec{p}} \langle 0 | 0 \rangle = 1 \, , \tag{110}$$

  where we have used the finite volume result for the commutator,

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = \delta_{\vec{p}\vec{p}\,'} \, . \tag{111}$$

  And, the incident flux is v_{rel}/V where v_{rel} is the relative velocity of the colliding particles. To see this, picture a box of volume V traveling with velocity v_{rel} towards a fixed (for convenience) box of volume V. Orient the traveling box so that one face is parallel to the front face of the fixed box. Start at $t = 0$ with the front face of the traveling box touching the front face of the fixed box. If the facing faces of the two boxes have area

A, then the fraction of the traveling volume V that passes the front face of the fixed box per unit time is $v_{rel}A/V$. Since there is only one particle in the volume V, this is also the number of particles that pass through the front face of the fixed box (the number is fractional). The flux of particles is the number of particles per unit time passing through per unit area, i.e. it is this number divided by A, giving $flux = v_{rel}/V$.

Thus, we obtain

$$\begin{aligned} d\sigma &= w\frac{1}{[\frac{v_{rel}}{V}]}\prod_f \frac{Vd^3\vec{p}_f'}{(2\pi)^3} \\ &= (2\pi)^4\delta^4(\sum_f p_f' - \sum_i p_i)\frac{1}{4E_1E_2v_{rel}}\prod_f \frac{d^3\vec{p}_f'}{(2\pi)^3 2E_f'}|\mathcal{M}|^2 . \end{aligned} \tag{112}$$

Note that this differs from the result in MS (with the extra $2m$ factors for fermions) by virtue of the different normalization for fermion fields and operators. Using my normalizations (which are the same as essentially all modern treatments), there is no distinction in the basic normalizations for bosons and fermions.

Note that the δ^4 function above implies that not all of the $\vec{p}_f'$ are independent. In any given situation, we will have to integrate out a certain number of the $\vec{p}_f'$ to eat up the δ^4 function.

- The above result holds in any Lorentz fame in which the colliding particles move collinearly.

  In such a frame, and assuming the particles are moving in opposite direction to one another (it is also ok if one is a rest), the relative velocity is given by

$$
\begin{aligned}
v_{rel} &= \left| \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_2|}{E_2} \right| \\
&= \frac{1}{E_1 E_2} \left| E_2 |\vec{p}_1| + E_1 |\vec{p}_2| \right| . \qquad (113)
\end{aligned}
$$

  Let us compare this expression with (remember opposite directions for the colliding particles are assumed)

$$
\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (E_1 E_2 + |\vec{p}_1||\vec{p}_2|)^2 - m_1^2 m_2^2 \\
&= E_1^2 E_2^2 + |\vec{p}_1|^2 |\vec{p}_2|^2 + 2E_1 E_2 |\vec{p}_1||\vec{p}_2| - m_1^2 m_2^2 \\
&= (|\vec{p}_1|^2 + m_1^2)(|\vec{p}_2|^2 + m_2^2) + |\vec{p}_1|^2 |\vec{p}_2|^2 + 2E_1 E_2 |\vec{p}_1||\vec{p}_2| - m_1^2 m_2^2
\end{aligned}
$$

$$
\begin{aligned}
&= 2|\vec{p}_1|^2|\vec{p}_2|^2 + m_1^2|\vec{p}_2|^2 + m_2^2|\vec{p}_1|^2 + 2E_1E_2|\vec{p}_1||\vec{p}_2| \\
&= E_1^2|\vec{p}_2|^2 + E_2^2|\vec{p}_1|^2 + 2E_1E_2|\vec{p}_1||\vec{p}_2| \\
&= (E_2|\vec{p}_1| + E_1|\vec{p}_2|)^2
\end{aligned}
\tag{114}
$$

In short, we have

$$
E_1E_2v_{rel} = \left[(p_1 \cdot p_2)^2 - m_1^2m_2^2\right]^{1/2} . \tag{115}
$$

This last result, substituted in the general form of $d\sigma$, implies that $d\sigma$ is a Lorentz invariant. (Recall that $\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}$ is a Lorentz invariant.)

- Some useful special cases are the com system, with $|\vec{p}_1| = |\vec{p}_2|$ resulting in

$$
E_1E_2v_{rel} = |E_2|\vec{p}_1| + E_1|\vec{p}_2|| = |\vec{p}_{\rm com}|E_{\rm tot} , \tag{116}
$$

(where $E_{\rm tot} \equiv E_1 + E_2$) and the laboratory frame in which $\vec{p}_2 = 0$ and

$$
E_1E_2v_{rel} = |E_2|\vec{p}_1| + E_1|\vec{p}_2|| = |\vec{p}_1|m_2 , \tag{117}
$$

- Since the center of mass system is so common and since we often consider $2 \to 2$ processes, it is useful to give some results specific to this case.

We use Eq. (116) and, after eliminating $\vec{p}_2'$, the result that $|\vec{p}_1'| = |\vec{p}_2'| \equiv |\vec{p}_{\rm com}'|$ to obtain the form

$$
\begin{aligned}
d\sigma &= \frac{1}{16|\vec{p}_{\rm com}|E_{\rm tot}(2\pi)^2E_1'E_2'}\delta^4(p_1'+p_2'-p_1-p_2)d^3\vec{p}_1'd^3\vec{p}_2'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2|\vec{p}_{\rm com}|E_{\rm tot}E_1'E_2'}\delta(E_1'+E_2'-E_{\rm tot})|\vec{p}_1'|^2d|\vec{p}_1'|d\Omega_1'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2|\vec{p}_{\rm com}|E_{\rm tot}E_1'E_2'}\left[\frac{\partial(E_1'+E_2')}{\partial|\vec{p}_{\rm com}'|}\right]^{-1}|\vec{p}_{\rm com}'|^2d\Omega_1'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2|\vec{p}_{\rm com}|E_{\rm tot}E_1'E_2'}\left[\frac{\partial\left(\sqrt{(m_1')^2+|\vec{p}_{\rm com}'|^2}+\sqrt{(m_2')^2+|\vec{p}_{\rm com}'|^2}\right)}{\partial|\vec{p}_{\rm com}'|}\right]^{-1}|\vec{p}_{\rm com}'|^2d\Omega_1'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2|\vec{p}_{\rm com}|E_{\rm tot}E_1'E_2'}\left[\frac{|\vec{p}_{\rm com}'|}{E_1'}+\frac{|\vec{p}_{\rm com}'|}{E_2'}\right]^{-1}|\vec{p}_{\rm com}'|^2d\Omega_1'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2E_{\rm tot}(E_1'+E_2')}\frac{|\vec{p}_{\rm com}'|}{|\vec{p}_{\rm com}|}d\Omega_1'|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2E_{\rm tot}^2}\frac{|\vec{p}_{\rm com}'|}{|\vec{p}_{\rm com}|}d\Omega_1'|\mathcal{M}|^2\,,
\end{aligned}
\tag{118}
$$

where in the last step we used $E_1' + E_2' = E_{\rm tot}$. Thus, our final result is

$$\left(\frac{d\sigma}{d\Omega_1'}\right)_{\rm com} = \frac{1}{64\pi^2 E_{\rm tot}^2}\frac{|\vec p\,'_{\rm com}|}{|\vec p_{\rm com}|}|\mathcal{M}|^2 \,. \tag{119}$$

- The above and other results for differential cross sections apply irregardless of whether or not we are dealing with identical particles.

  However, if there are two or more identical particles in the final state, we must not duplicate integration regions in obtaining the total cross section. For example, in the $2 \to 2$ process just computed, if the final particles were 2 e^-'s, then the case where $(\theta_1', \phi_1') = (\alpha, \beta)$ is not physically distinguishable from the case where $(\theta_1', \phi_1') = (\pi - \alpha, \pi + \beta)$. In this case, we should only integrate over $0 \leq \theta_1' \leq \frac{1}{2}\pi$, or, equivalently, we could integrate over all θ_1' and then divide the final result by $2!$:

$$\sigma^{\rm tot}_{\rm com} = \frac{1}{2}\int_{4\pi} d\Omega_1' \left(\frac{d\sigma}{d\Omega_1'}\right)_{\rm com} \,. \tag{120}$$

  In the more general case of n identical particles in the final state, we can obtain the correct result for $\sigma^{\rm tot}$ by integrating over all of phase space and then dividing by $n!$.

In a very rough sense, the $1/2!$ is partly compensating for the fact that for two identical particles there will be two Feynman diagram amplitudes that differ only by interchanging the two final particles (as in $e^- e^- \to e^- e^-$), which, again very roughly, means that the amplitude-squared would be 4 times as large as compared to the case where the final particles are not identical (e.g. $e^- \mu^- \to e^- \mu^-$).

(As we shall discuss, the e^- and μ^- are not identical particles. They carry a lepton identifier quantum number that distinguishes them. See MS for more details.)

- As an example of a particularly useful cross section, let us consider the process $e^+(p) e^-(k) \to \mu^+(p') \mu^-(k')$.

  We consider how to deal with two different types of leptons. As noted above muons and electrons are experimentally distinguishable — e.g. they have different mass and have a different lepton number.

  What this means in our theoretical framework is that there are creation and annihilation operators for electrons that are distinct from those for muons.

Each lepton has its own free-particle Lagrangian so that

$$\mathcal{L}_0 = \sum_{l=e,\mu,\tau} \overline{\psi}_l(x)(i\partial\!\!\!/ - m_l)\psi_l(x)\,. \tag{121}$$

Making the minimal substitution (with $q = -e$ for each) gives

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) = -e\sum_l :\overline{\psi}_l(x)A\!\!\!/(x)\psi_l(x): \tag{122}$$

Note that in the above there is no term like

$$-e:\overline{\psi}_\mu(x)A\!\!\!/(x)\psi_e(x): \tag{123}$$

Such a term would constitute what is called a flavor-changing neutral current (FCNC) interaction. Electromagnetism does not have such FCNC interactions when constructed using the minimal substitution rule from free-particle Lagrangians for the individual leptons. FCNC interactions for the leptons are predicted in the context of the weak interactions now that we know that neutrinos have mass and that the mass eigenstates are clearly not the same as the Lagrangian eigenstates.

- **So, now let's get back to the cross section computation making the absolute lepton number conservation assumption.**

  The initial and final states are:

$$|i\rangle = c_e^\dagger(\vec{k}) d_e^\dagger(\vec{p})|0\rangle \,, \quad |f\rangle = c_\mu^\dagger(\vec{k}\,') d_\mu^\dagger(\vec{p}\,')|0\rangle \,. \tag{124}$$

  The relevant Feynman diagram is

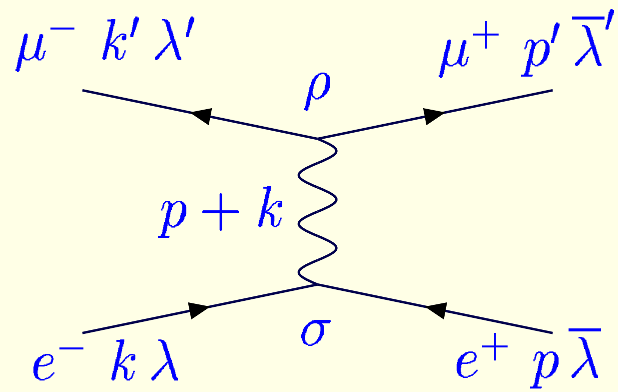

Figure 5: The one Feynman diagram contributing to $e^+e^- \to \mu^+\mu^-$. The arrows show the directions for the *momenta*. For the μ^+ and e^+, the fermion-flow direction is opposite the indicated momentum direction.

Can you figure out from the Feynman 'rules' (generalized to include μ vertices as well as e vertices) what to write down without going through the derivation below?

Now, since we need a e^+ and e^- in the initial state we will need a ψ_e^+ and a $\overline{\psi}_e^{\,+}$ for the killing operations. For the final state μ^+ and μ^- we will need a $\overline{\psi}_\mu^{\,-}$ and a ψ_μ^- for the killing operations. This means we must go to $S^{(2)}$ which now includes

$$
\begin{aligned}
S^{(2)} \quad \ni \quad & -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 \Big[T\{:\overline{\psi}_\mu(x_1) \not{A}(x_1)\psi_\mu(x_1) :: \overline{\psi}_e(x_2)\not{A}(x_2)\psi_e(x_2):\} \\
& + T\{\mu \leftrightarrow e, x_1 \leftrightarrow x_2\}\Big]
\end{aligned} \tag{125}
$$

To see this, we must remember that $S^{(2)}$ contains

$$
T\{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\} = T\{[\mathcal{H}_I^e(x_1) + \mathcal{H}_I^\mu(x_1)][\mathcal{H}_I^e(x_2) + \mathcal{H}_I^\mu(x_2)]\} \tag{126}
$$

and that the terms of interest then arise as the two cross terms containing the product of $\mathcal{H}_I^e$ with $\mathcal{H}_I^\mu$. Of course, the $x_1 \leftrightarrow x_2$ 2nd term above gives exactly the same contribution as the 1st term and so we will work with the equivalent form

$$
S^{(2)} \ni -e^2 \int d^4x_1 \int d^4x_2 T\{:\overline{\psi}_\mu(x_1)\not{A}(x_1)\psi_\mu(x_1) :: \overline{\psi}_e(x_2)\not{A}(x_2)\psi_e(x_2):\} \tag{127}
$$

For the process being considered we keep the already delineated parts of $S^{(2)}$, namely:

$$
\begin{aligned}
S^{(2)} &\ni -e^2 \int d^4x_1 \int d^4x_2 T\{:\overline{\psi}_\mu^-(x_1) \not{A}(x_1)\psi_\mu^-(x_1) :: \overline{\psi}_e^+(x_2)\not{A}(x_2)\psi_e^+(x_2):\} \\
&\to -e^2 \int d^4x_1 \int d^4x_2 \overline{\psi}_\mu^-(x_1)\gamma^\rho\psi_\mu^-(x_1)\overline{A_\rho(x_1)A}_\sigma(x_2)\overline{\psi}_e^+(x_2)\gamma^\sigma\psi_e^+(x_2) \qquad (128)
\end{aligned}
$$

We now insert this between our initial and final states and use the standard killing operations. Thus, we compute (you will have noted that I am not attempting to keep spin indices on the creation and annihilation operators defining $|i\rangle$ and $|f\rangle$)

$$
\begin{aligned}
\langle f|S^{(2)}|i\rangle &= \langle 0|d_\mu(\vec{p}\,')c_\mu(\vec{k}\,')\Big[-e^2 \int d^4x_1 \int d^4x_2 \overline{\psi}_\mu^-(x_1)\gamma^\rho\psi_\mu^-(x_1)\overline{A_\rho(x_1)A}_\sigma(x_2) \\
&\qquad \overline{\psi}_e^+(x_2)\gamma^\sigma\psi_e^+(x_2)\Big] c_e^\dagger(\vec{k})d_e^\dagger(\vec{p})|0\rangle \\
&= -e^2\overline{u}_\mu(\vec{k}\,')\gamma^\rho v_\mu(\vec{p}\,')\overline{v}_e(\vec{p})\gamma^\sigma u_e(\vec{k}) \times \\
&\qquad \int d^4x_1 \int d^4x_2 e^{ix_1\cdot(k'+p')-ix_2\cdot(k+p)} \left[\frac{i}{(2\pi)^4}\int d^4q \frac{-g_{\rho\sigma}}{q^2+i\epsilon}e^{-iq\cdot(x_1-x_2)}\right] \\
&= -e^2\overline{u}_\mu(\vec{k}\,')\gamma^\rho v_\mu(\vec{p}\,')\overline{v}_e(\vec{p})\gamma^\sigma u_e(\vec{k})\frac{1}{(2\pi)^4}\int d^4q \times \\
&\qquad (2\pi)^4\delta^4(k'+p'-q)(2\pi)^4\delta^4(q-p-k)\left[i\,\frac{-g_{\rho\sigma}}{q^2+i\epsilon}\right]_{q=p+k}
\end{aligned}
$$

$$= \quad (2\pi)^4 \delta^4(p' + k' - p - k)\mathcal{M}\,, \qquad \text{with}$$

$$\mathcal{M} \quad = \quad \overline{u}_\mu(\lambda', \vec{k}\,')(ie\gamma^\rho)v_\mu(\overline{\lambda}', \vec{p}\,')\overline{v}_e(\overline{\lambda}, \vec{p})(ie\gamma^\sigma)u_e(\lambda, \vec{k}) \left[i\,\frac{-g_{\rho\sigma}}{q^2 + i\epsilon}\right]_{q=p+k} \tag{129}$$

which is what you would get by "naive" application of the Feynman rules, generalized to include muons as well as electrons. In the very last expression, we have included the spin/helicities of the various particles that had been dropped until now.

- So, now let's insert this into our $2 \to 2$ cross section expression:

$$\left(\frac{d\sigma}{d\Omega_1'}\right)_{\rm com} = \frac{1}{64\pi^2 E_{\rm tot}^2}\frac{|\vec{p}\,'_{\rm com}|}{|\vec{p}_{\rm com}|}|\mathcal{M}|^2 \tag{130}$$

using the approximation of neglecting m_e and m_μ, so that $|\vec{p}\,'|_{\rm com} = |\vec{p}|_{\rm com} = E_{\rm tot}/2$, to obtain (dropping the $'_1$ notation on Ω, as is conventional)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm com} = \frac{1}{64\pi^2 E_{\rm tot}^2}\Big|\overline{u}_\mu(\lambda', \vec{k}\,')(ie\gamma^\rho)v_\mu(\overline{\lambda}', \vec{p}\,')$$

$$\overline{v}_e(\overline{\lambda}, \vec{p})(ie\gamma_\rho)u_e(\lambda, \vec{k}) \left[\frac{-i}{q^2 + i\epsilon}\right]_{q=p+k}\Big|^2 \tag{131}$$

The above result is for fixed spins for all the initial and final particles. More typically, we might wish to average over initial spins (corresponding to unpolarized e^+ and e^- incoming beams) and sum over final spins (corresponding to not measuring the final spins). In this case, we can make some simplifications. To show how this works, we consider for the moment the electron subcomponent of this process:

$$\sum_{\overline{\lambda},\lambda}[\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_\rho)u_e(\lambda,\vec{k})][\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_{\rho'})u_e(\lambda,\vec{k})]^* \qquad (132)$$

where, since ρ is a dummy index eventually to be summed over, we must use ρ' for the other half of the absolute square. Now,

$$\begin{aligned}
[\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_{\rho'})u_e(\lambda,\vec{k})]^* &= u_e(\lambda,\vec{k})^\dagger(ie\gamma_{\rho'})^\dagger(v_e(\overline{\lambda},\vec{p})^\dagger\gamma^0)^\dagger \\
&= u_e(\lambda,\vec{k})^\dagger\gamma^0\gamma^0(-ie\gamma^\dagger_{\rho'})(\gamma^0)^\dagger v_e(\overline{\lambda},\vec{p}) \\
&= \overline{u}_e(\lambda,\vec{k})(-ie\gamma_{\rho'})v_e(\overline{\lambda},\vec{p}) \qquad (133)
\end{aligned}$$

so that we have (exposing Dirac indices in the 3rd line before proceeding)

$$
\begin{aligned}
& \sum_{\overline{\lambda},\lambda} [\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_\rho)u_e(\lambda,\vec{k})][\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_{\rho'})u_e(\lambda,\vec{k})]^* \\
= & \sum_{\overline{\lambda},\lambda} [\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_\rho)u_e(\lambda,\vec{k})][\overline{u}_e(\lambda,\vec{k})(-ie\gamma_{\rho'})v_e(\overline{\lambda},\vec{p})] \\
= & \sum_{\overline{\lambda},\lambda} e^2\overline{v}_e(\overline{\lambda},\vec{p})_\gamma(\gamma_\rho)_{\gamma\delta}u_e(\lambda,\vec{k})_\delta\overline{u}_e(\lambda,\vec{k})_\epsilon(\gamma_{\rho'})_{\epsilon\beta}v_e(\overline{\lambda},\vec{p})_\beta \\
= & \sum_{\overline{\lambda},\lambda} e^2 v_e(\overline{\lambda},\vec{p})_\beta\overline{v}_e(\overline{\lambda},\vec{p})_\gamma(\gamma_\rho)_{\gamma\delta}u_e(\lambda,\vec{k})_\delta\overline{u}_e(\lambda,\vec{k})_\epsilon(\gamma_{\rho'})_{\epsilon\beta} \\
= & \; e^2(\not{p} - m_e)_{\beta\gamma}(\gamma_\rho)_{\gamma\delta}(\not{k} + m_e)_{\delta\epsilon}(\gamma_{\rho'})_{\epsilon\beta} \\
= & \; e^2\mathrm{Tr}\left[(\not{p} - m_e)\gamma_\rho(\not{k} + m_e)\gamma_{\rho'}\right]. \qquad (134)
\end{aligned}
$$

Similarly, we find (note the matching of the ρ and ρ' Lorentz indices is maintained within $\mathcal{M}$ and $\mathcal{M}^*$)

$$
\begin{aligned}
& \sum_{\lambda',\overline{\lambda}'} [\overline{u}_\mu(\lambda',\vec{k}\,')(ie\gamma^\rho)v_\mu(\overline{\lambda}',\vec{p}\,')][\overline{u}_\mu(\lambda',\vec{k}\,')(ie\gamma^{\rho'})v_\mu(\overline{\lambda}',\vec{p}\,')]^* \\
= & \; e^2\mathrm{Tr}\left[(\not{k}\,' + m_\mu)\gamma^\rho(\not{p}\,' - m_\mu)\gamma^{\rho'}\right]. \qquad (135)
\end{aligned}
$$

- **Thus, we now need to figure out how to evaluate this kind of trace of Dirac matrices. There are a series of rules that we will slowly add to that one needs to derive.**

1. **First, just by the definition of the trace, we have**

$$\mathrm{Tr}[\boldsymbol{\Gamma}_1\boldsymbol{\Gamma}_2] = \mathrm{Tr}[\boldsymbol{\Gamma}_2\boldsymbol{\Gamma}_1] \tag{136}$$

   for any two of our 16 matrices.

2. **Next, we can show that the trace of an odd number of γ matrices is zero:**

$$\mathrm{Tr}[\gamma^\alpha\gamma^\beta\ldots\gamma^\nu] = 0 \tag{137}$$

   if there are an odd number of matrices in $[\ldots]$. This is shown as follows:

$$\begin{aligned} & \mathrm{Tr}[\gamma^\alpha\gamma^\beta\ldots\gamma^\nu] \\ = & \ \mathrm{Tr}[\gamma_5^2\gamma^\alpha\gamma^\beta\ldots\gamma^\nu] \quad \textbf{(since } \gamma_5^2 = 1\textbf{)} \\ = & \ -\mathrm{Tr}[\gamma_5\gamma^\alpha\gamma^\beta\ldots\gamma^\nu\gamma_5] \quad \textbf{(since } \gamma_5 \textbf{ anticommutes with all the } \gamma^\alpha,\ldots\textbf{)} \\ = & \ -\mathrm{Tr}[\gamma_5\gamma_5\gamma^\alpha\gamma^\beta\ldots\gamma^\nu] \quad \textbf{(by the cyclic property of the trace)} \\ = & \ -\mathrm{Tr}[\gamma^\alpha\gamma^\beta\ldots\gamma^\nu] \quad \textbf{(since } \gamma_5^2 = 1\textbf{)} \end{aligned} \tag{138}$$

so that the only possibility is that the original Tr was equal to $\mathbf{0}$.

3. Next, we consider

$$\begin{aligned}
\mathrm{Tr}[\gamma^\alpha\gamma^\beta] &= \frac{1}{2}\mathrm{Tr}[\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha] = \frac{1}{2}\mathrm{Tr}[[\gamma^\alpha,\gamma^\beta]_+] \\
&= \frac{1}{2}\mathrm{Tr}[2g^{\alpha\beta}\mathbf{1}_{4\times4}] = 4g^{\alpha\beta} \qquad (139)
\end{aligned}$$

4. A particularly important one is:

$$\begin{aligned}
& \mathrm{Tr}[\gamma^\alpha\gamma^\beta\gamma^\epsilon\gamma^\delta] \\
= \; & \mathrm{Tr}[(2g^{\alpha\beta}-\gamma^\beta\gamma^\alpha)\gamma^\epsilon\gamma^\delta] \\
= \; & \mathrm{Tr}[2g^{\alpha\beta}\gamma^\epsilon\gamma^\delta] - \mathrm{Tr}[\gamma^\beta(2g^{\alpha\epsilon}-\gamma^\epsilon\gamma^\alpha)\gamma^\delta] \\
= \; & \mathrm{Tr}[2g^{\alpha\beta}\gamma^\epsilon\gamma^\delta] - \mathrm{Tr}[\gamma^\beta 2g^{\alpha\epsilon}\gamma^\delta] + \mathrm{Tr}[\gamma^\beta\gamma^\epsilon(2g^{\alpha\delta}-\gamma^\delta\gamma^\alpha)] \\
= \; & \mathrm{Tr}[2g^{\alpha\beta}\gamma^\epsilon\gamma^\delta] - \mathrm{Tr}[\gamma^\beta 2g^{\alpha\epsilon}\gamma^\delta] + \mathrm{Tr}[\gamma^\beta\gamma^\epsilon 2g^{\alpha\delta}] - \mathrm{Tr}[\gamma^\beta\gamma^\epsilon\gamma^\delta\gamma^\alpha] \\
= \; & 8[g^{\alpha\beta}g^{\epsilon\delta} - g^{\beta\delta}g^{\alpha\epsilon} + g^{\beta\epsilon}g^{\alpha\delta}] - \mathrm{Tr}[\gamma^\alpha\gamma^\beta\gamma^\epsilon\gamma^\delta]\,. \qquad (140)
\end{aligned}$$

Since the last term above duplicates the starting Tr, we can move it to the lhs of the equation and divide both sides by a factor of 2 to obtain

$$\mathrm{Tr}[\gamma^\alpha\gamma^\beta\gamma^\epsilon\gamma^\delta] = 4[g^{\alpha\beta}g^{\epsilon\delta} - g^{\beta\delta}g^{\alpha\epsilon} + g^{\beta\epsilon}g^{\alpha\delta}] \tag{141}$$

Note that if we contract some momenta with some of the free γ indices we will get things like

$$\begin{aligned} \mathrm{Tr}[\not p\, \gamma^\beta \not k\, \gamma^\delta] &= p_\alpha k_\epsilon \mathrm{Tr}[\gamma^\alpha\gamma^\beta\gamma^\epsilon\gamma^\delta] \\ &= 4p_\alpha k_\epsilon[g^{\alpha\beta}g^{\epsilon\delta} - g^{\beta\delta}g^{\alpha\epsilon} + g^{\beta\epsilon}g^{\alpha\delta}] \\ &= 4[p^\beta k^\delta - g^{\beta\delta}p\cdot k + p^\delta k^\beta] \end{aligned} \tag{142}$$

Well, that's all we need for the moment. The Appendix of MS has more.

- We now return to the initial spin average, final spin sum version of our cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{com}} = \frac{1}{64\pi^2 E_{\mathrm{tot}}^2}\frac{1}{4}\sum_{\lambda,\overline{\lambda},\lambda',\overline{\lambda}'}\left|\overline{u}_\mu(\lambda',\vec{k}\,')(ie\gamma^\rho)v_\mu(\overline{\lambda}',\vec{p}\,')\overline{v}_e(\overline{\lambda},\vec{p})(ie\gamma_\rho)u_e(\lambda,\vec{k})\left[\frac{-i}{q^2+i\epsilon}\right]_{q=p+k}\right|^2 \tag{143}$$

We use our conversion of the spin sums to traces in the form

$$\begin{aligned}
& e^2 \mathrm{Tr}\,[(\not p - m_e)\gamma_\rho(\not k + m_e)\gamma_{\rho'}] e^2 \mathrm{Tr}\,[(\not k\,' + m_\mu)\gamma^\rho(\not p\,' - m_\mu)\gamma^{\rho'}] \\
= \quad & 16 e^4 [p_\rho k_{\rho'} - g_{\rho\rho'}(p \cdot k + m_e^2) + p_{\rho'} k_\rho][k'^\rho p'^{\rho'} - g^{\rho\rho'}(k' \cdot p' + m_\mu^2) + k'^{\rho'} p'^\rho] \\
= \quad & 32 e^4 [p \cdot p' k \cdot k' + p \cdot k' p' \cdot k]
\end{aligned} \tag{144}$$

where we used the above theorems, including $\mathrm{Tr}[\textbf{odd number}] = 0$, then neglected m_e^2 and m_μ^2 and performed the remaining Lorentz index contractions.

We insert this to obtain:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{com}} = \frac{1}{8\pi^2 E_{\mathrm{tot}}^2} e^4 [p \cdot p' k \cdot k' + p \cdot k' p' \cdot k] \left| \frac{1}{(p+k)^2 + i\epsilon} \right|^2 \tag{145}$$

- We now do a little bit of kinematics, writing (neglecting masses so that $E = E' = E_{\mathrm{tot}}/2$ for all particles — for inclusion of masses in all this, see

MS)

$$p = (E, 0, 0, E)\,, \quad k = (E, 0, 0, -E)$$
$$p' = (E, E\sin\theta, 0, E\cos\theta)\,, \quad k' = (E, -E\sin\theta, 0, -E\cos\theta) \tag{146}$$

yielding

$$\begin{aligned} p\cdot p' &= k\cdot k' = E^2(1-\cos\theta) \\ p\cdot k' &= k\cdot p' = E^2(1+\cos\theta) \\ (p+k)^2 &= 2p\cdot k = 4E^2 \end{aligned} \tag{147}$$

so that we obtain

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\rm com} &= \frac{1}{8\pi^2 E_{\rm tot}^2} e^4 E^4 [(1-\cos\theta)^2 + (1+\cos\theta)^2] \left[\frac{1}{4E^2}\right]^2 \\ &= \frac{1}{8\pi^2 E_{\rm tot}^2} (4\pi\alpha)^2 \frac{2}{16} [1+\cos^2\theta] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^2}{4E_{\rm tot}^2}[1+\cos^2\theta]\,, \quad \textbf{and} \\
\sigma_{\rm tot} &= \frac{4\pi}{3}\frac{\alpha^2}{E_{\rm tot}^2} = \frac{\pi\alpha^2}{3E^2}\,.
\end{aligned}
\tag{148}
$$

It is common convention to write

$$
s \equiv (p+k)^2 = (p'+k')^2\,, \tag{149}
$$

where $s = E_{\rm tot}^2 = 4E^2$, the latter holding when masses are neglected. Also, we frequently write

$$
t \equiv (p'-p)^2 = (k'-k)^2\,, \quad u \equiv (k'-p)^2 = (k-p')^2\,. \tag{150}
$$

In the massless limit, $t = -2p'\cdot p = -2k'\cdot k$ and $u = -2k'\cdot p = -2k\cdot p'$. Further, since $t = -2E^2(1-\cos\theta)$ we have $dt = 2E^2 d\cos\theta = (E^2/\pi)d\Omega = \frac{1}{4}(s/\pi)d\Omega$. With these substitutions, our differential cross

section can be expressed in the form

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{4\pi}{s}\frac{1}{8\pi^2 s}(4\pi\alpha)^2\left[\frac{t^2}{4}+\frac{u^2}{4}\right]\frac{1}{s^2} \\ &= \frac{2\pi\alpha^2}{s^2}\left[\left(\frac{t}{s}\right)^2+\left(\frac{u}{s}\right)^2\right]. \qquad (151)\end{aligned}$$

This is a Lorentz invariant form that actually holds in any frame, since it is expressed entirely in terms of Lorentz invariants.

A note on dimensions: $d\sigma/dt$ has 'energy' dimensions of $1/E^4$ in the $\hbar = c = 1$ units. In the massless limit, we see that all these dimensions are provided by the energy scale of the process, as encoded in s. This is always the case when the underlying theory has a dimensionless coupling constant such as e, which in turn is always the case if we generate interactions using the minimal substitution rule. In minimal substitution, $\partial_\mu \to \partial_\mu + iqA_\mu$. Since ∂_μ and A_μ have the same dimension, q must be dimensionless. To check the dimensions of A_μ remember that $\int d^4x\mathcal{L}$ must be a dimensionless action. With $\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$, it is clear that A_μ must have the same dimensions as ∂_μ in order to cancel the dimensions of d^4x.

Photon polarization sums for a cross section

- Write
$$\mathcal{M} = \epsilon^{\alpha}_{r_1}(\vec{k}_1)\epsilon^{\beta}_{r_2}(\vec{k}_2)\ldots\mathcal{M}_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots)\,. \quad (152)$$
The $\epsilon^\mu(\vec{k})$ are gauge dependent objects. In particular, if we write (we keep to a basis in which our initial choices — you will see why I use the word "initial" in a moment — for the polarization ϵ's are real)
$$\begin{aligned} A^{\mu\,+}(x) &= \sum_{\vec{k}}\sum_r \frac{1}{\sqrt{2V\omega(\vec{k})}}\epsilon^\mu_r(\vec{k})e^{-ik\cdot x}a_r(\vec{k}) \\ A^{\mu\,-}(x) &= \sum_{\vec{k}}\sum_r \frac{1}{\sqrt{2V\omega(\vec{k})}}\epsilon^\mu_r(\vec{k})e^{+ik\cdot x}a^\dagger_r(\vec{k}) \end{aligned} \quad (153)$$
and perform the gauge transformation $A^\mu(x) \to A^\mu(x) + \partial^\mu f(x)$ with $f(x) = \tilde{f}(k)\left(a_r(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x}\right)$, then the corresponding change in $\epsilon^\mu(\vec{k})$ is
$$\epsilon^\mu_r \to \epsilon^\mu_r(\vec{k}) - ik^\mu\tilde{f}(k)\,. \quad (154)$$

(Note that this choice of gauge transformation keeps the A^μ field hermitian). In QED, we can apply this for each of the external ϵ's, each with their own $\vec{k}_i$. Thus, if $\mathcal{M}$ is GI, we must have

$$\begin{aligned} 0 &= k_1^\alpha \epsilon_{r_2}^\beta(\vec{k}_2)\ldots\mathcal{M}_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) \\ &= \epsilon_{r_1}^\alpha(\vec{k}_1)k_2^\beta\ldots\mathcal{M}_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) \\ &= \ldots . \end{aligned} \tag{155}$$

MS extrapolates this to say that

$$k_1^\alpha\mathcal{M}_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) = k_2^\beta\mathcal{M}_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) = \ldots = 0 . \tag{156}$$

This is true only in the case of Abelian gauge theory (which QED is). In a non-abelian gauge theory, this final form is not true and only a single photon can be “removed” at a time — all the other polarizations must be kept in place for the identity to hold.

- Another point of view that gives this same result is current conservation.

  Let us focus on just one of the photons and write $\mathcal{M} = \epsilon^\mu(\vec{k})\mathcal{M}_\mu(k)$. We know that the photon is seeing an interaction of the form $\mathcal{L} =$

$\int d^4x e j_\mu(x) A_\mu(x)$, where $j_\mu = \overline{\psi}\gamma_\mu\psi$. Therefore,

$$\mathcal{M}_\mu(k) = \int d^4x e^{ik\cdot x} \langle f | j_\mu(x) | i \rangle \,. \tag{157}$$

Then

$$\begin{aligned} k^\mu \mathcal{M}_\mu(k) &= \int d^4x \left[(-i\partial^\mu) e^{ik\cdot x}\right] \langle f | j_\mu(x) | i \rangle \\ &= -\int d^4x e^{ik\cdot x} [-i\partial^\mu \langle f | j_\mu(x) | i \rangle] \\ &= 0 \end{aligned} \tag{158}$$

by virtue of current conservation, $\partial^\mu \langle f | j_\mu(x) | i \rangle = 0$. (The current is not necessarily conserved until placed in the context of definite on-shell states for the other particles involved in the process, as encapsulated in the $\langle i|$ and $|f\rangle$ notation.) We know that such current conservation applies for the free fields. In the interacting case, this current conservation is a consequence of gauge invariance. The identity $k^\mu \mathcal{M}_\mu(k) = 0$ is called the *Ward Identity*.

- With this result, we can now simply perform polarization sums for the cross section.

Writing $\mathcal{M}_r(k) = \epsilon_r^\mu(\vec{k})\mathcal{M}_\mu(\vec{k})$, the cross section will be proportional to

$$X = \sum_{r=1,2} \left|\mathcal{M}_r(\vec{k})\right|^2 = \mathcal{M}_\mu(\vec{k})\mathcal{M}_\nu^*(\vec{k}) \sum_{r=1,2} \epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k})\,. \tag{159}$$

For simplicity, we go to a Lorentz frame ($\mathcal{M}$ is a Lorentz invariant so we can choose any frame we like) where $k^\mu = (k, 0, 0, k)$ has only a direction 3 vector component. Then, in the Lorentz gauge we know that $\epsilon_1(\vec{k}) = (0, 1, 0, 0)$ and $\epsilon_2(\vec{k}) = (0, 0, 1, 0)$, *i.e.* $\epsilon_1^1 = 1$ and $\epsilon_2^2 = 1$, all others zero. So,

$$X = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2\,, \tag{160}$$

where the subscripts are the Lorentz indices. Now, for the above k, the Ward identity reduces to

$$k(\mathcal{M}_0 + \mathcal{M}_3) = 0\,, \quad \Rightarrow \quad \mathcal{M}_0 = -\mathcal{M}_3\,. \tag{161}$$

Then we can write equally well

$$X = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2 - |\mathcal{M}_0|^2 = -g^{\mu\nu}\mathcal{M}_\mu\mathcal{M}_\nu^* = -\mathcal{M}^\nu\mathcal{M}_\nu^*\,. \tag{162}$$

So, effectively, we have made the replacement

$$\sum_{r=1,2} \epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k}) \to -g^{\mu\nu}\,. \tag{163}$$

This is not an actual equality. In fact (for $k^2 = 0$)

$$\sum_{r=1,2} \epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k}) \to -g^{\mu\nu} - \frac{1}{(k\cdot n)^2}[k^\nu k^\mu - (k\cdot n)(k^\mu n^\nu + k^\nu n^\mu)]\,, \tag{164}$$

where $n^\mu = (1,0,0,0)$, but the other terms don't contribute because of the Ward identity.

- Let us now use this technology for the case of Compton scattering. This was discussed beginning with Eq. (61) in some detail with the result that we developed the two contributing amplitudes

$$\mathcal{M}_a = -e^2\overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}') \left[i\frac{\not q + m}{q^2 - m^2}\right]_{q=p+k} \not\epsilon_s(\vec{k})u_r(\vec{p}) \tag{165}$$

$$\mathcal{M}_b = -e^2 \overline{u}_{r'}(\vec{p}\,')\not{\epsilon}_{s}(\vec{k}) \left[i\frac{\not{q} + m}{q^2 - m^2} \right]_{q=p-k'} \not{\epsilon}_{s'}(\vec{k}')u_r(\vec{p}) \tag{166}$$

corresponding to the Feynman diagrams

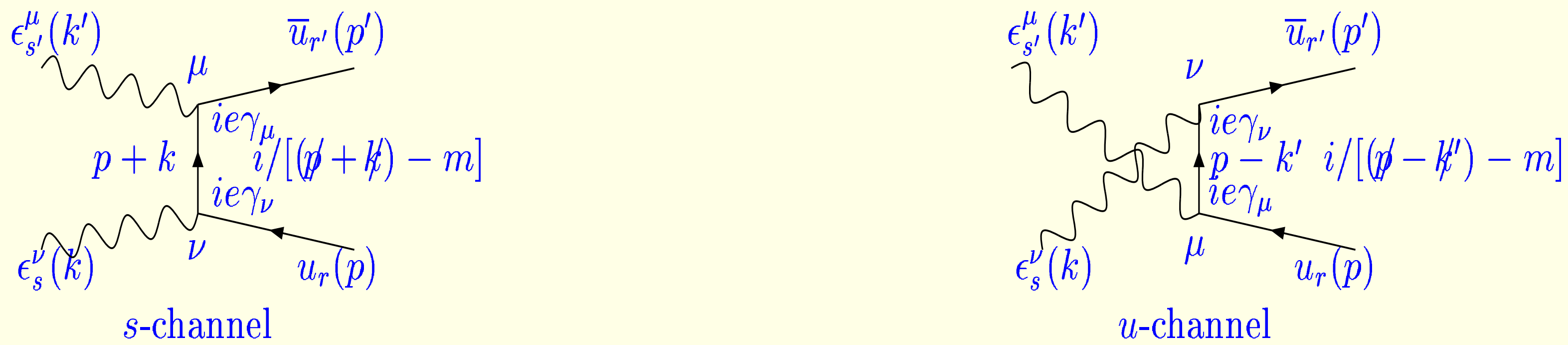

Figure 6: The two Feynman diagrams contributing to $e^-\gamma \to e^-\gamma$.

We need

$$\frac{1}{4}\sum_{\text{pol, s,s}'}\sum_{\text{spin, r,r}'} |\mathcal{M}_{rsr's'}|^2 = \frac{1}{4}\sum_{\text{spin}} \mathcal{M}^{\alpha\beta}\mathcal{M}^*_{\alpha\beta} \tag{167}$$

where we have written (dropping the r and r' indices for the moment)

$$\mathcal{M}_{ss'} = \epsilon^{\alpha}_{s}(\vec{k})\epsilon^{\beta}_{s'}(\vec{k}\,')\mathcal{M}_{\alpha\beta} \tag{168}$$

and used the following procedure:

$$\begin{aligned}
\sum_{s,s'} |\mathcal{M}_{ss'}|^2 &= \sum_{s,s'} \left[\epsilon_s^{\alpha}(\vec{k})\epsilon_{s'}^{\beta}(\vec{k}\,')\mathcal{M}_{\alpha\beta}\right]\left[\epsilon_s^{\alpha'}(\vec{k})\epsilon_{s'}^{\beta'}(\vec{k}\,')\mathcal{M}^*_{\alpha'\beta'}\right] \\
&= \left(\sum_s \epsilon_s^{\alpha}(\vec{k})\epsilon_s^{\alpha'}(\vec{k})\right)\left(\sum_{s'} \epsilon_{s'}^{\beta}(\vec{k}\,')\epsilon_{s'}^{\beta'}(\vec{k}\,')\right)\mathcal{M}_{\alpha\beta}\mathcal{M}^*_{\alpha'\beta'} \\
&= (-g^{\alpha\alpha'})(-g^{\beta\beta'})\mathcal{M}_{\alpha\beta}\mathcal{M}^*_{\alpha'\beta'} \\
&= \mathcal{M}^{\alpha\beta}\mathcal{M}^*_{\alpha\beta}\,. \qquad (169)
\end{aligned}$$

In the above, we have

$$\mathcal{M}_{\alpha\beta} = \mathcal{M}^a_{\alpha\beta} + \mathcal{M}^b_{\alpha\beta} \qquad (170)$$

with

$$\begin{aligned}
\mathcal{M}^a_{\alpha\beta} &= -ie^2\overline{u}_{r'}(\vec{p}\,')\gamma_\beta\left[\frac{\not p + \not k + m}{(p+k)^2 - m^2}\right]\gamma_\alpha u_r(\vec{p}) \\
\mathcal{M}^b_{\alpha\beta} &= -ie^2\overline{u}_{r'}(\vec{p}\,')\gamma_\alpha\left[\frac{\not p - \not k\,' + m}{(p-k')^2 - m^2}\right]\gamma_\beta u_r(\vec{p}) \qquad (171)
\end{aligned}$$

Using $p^2 = p'^2 = m^2$ and $k^2 = k'^2 = 0$, the denominators above reduce to $2p \cdot k$ and $-2p \cdot k'$, respectively, and we find

$$\begin{aligned}
\frac{1}{4} \sum_{\text{pol, s,s}'} \sum_{\text{spin, r,r}'} |\mathcal{M}|^2 &= \frac{1}{4} \sum_{\text{spin}} \mathcal{M}^{\alpha\beta} \mathcal{M}^*_{\alpha\beta} \\
&= \frac{e^4}{16} \left(\frac{X_{aa}}{(pk)^2} + \frac{X_{bb}}{(pk')^2} - \frac{X_{ab} + X_{ba}}{(pk)(pk')} \right) \quad (172)
\end{aligned}$$

where

$$\begin{aligned}
X_{aa} &= \sum_{spin} [\overline{u}_{r'}(\vec{p}\,')\gamma^\beta(\not{f}_1 + m)\gamma^\alpha u_r(\vec{p})][\overline{u}_{r'}(\vec{p}\,')\gamma_\beta(\not{f}_1 + m)\gamma_\alpha u_r(\vec{p})]^* \\
&= \sum_{spin} [\overline{u}_{r'}(\vec{p}\,')\gamma^\beta(\not{f}_1 + m)\gamma^\alpha u_r(\vec{p})][\overline{u}_r(\vec{p})\gamma_\alpha(\not{f}_1 + m)\gamma_\beta u_{r'}(\vec{p}\,')] \\
&= \mathrm{Tr}\left[\gamma^\beta(\not{f}_1 + m)\gamma^\alpha \left(\sum_r u_r(\vec{p})\overline{u}_r(\vec{p})\right) \gamma_\alpha(\not{f}_1 + m)\gamma_\beta \left(\sum_{r'} u_{r'}(\vec{p}\,')\overline{u}_{r'}(\vec{p}\,')\right)\right] \\
&= \mathrm{Tr}\left[\gamma^\beta(\not{f}_1 + m)\gamma^\alpha(\not{p} + m)\gamma_\alpha(\not{f}_1 + m)\gamma_\beta(\not{p}\,' + m)\right] \quad (173) \\
X_{bb} &= \sum_{spin} [\overline{u}_{r'}(\vec{p}\,')\gamma^\alpha(\not{f}_2 + m)\gamma^\beta u_r(\vec{p})][\overline{u}_{r'}(\vec{p}\,')\gamma_\alpha(\not{f}_2 + m)\gamma_\beta u_r(\vec{p})]^* \\
&= \mathrm{Tr}\left[\gamma^\alpha(\not{f}_2 + m)\gamma^\beta(\not{p} + m)\gamma_\beta(\not{f}_2 + m)\gamma_\alpha(\not{p}\,' + m)\right] \quad (174)
\end{aligned}$$

$$X_{ab} = \sum_{spin} [\overline{u}_{r'}(\vec{p}\,')\gamma^\beta(\not f_1 + m)\gamma^\alpha u_r(\vec{p})][\overline{u}_{r'}(\vec{p}\,')\gamma_\alpha(\not f_2 + m)\gamma_\beta u_r(\vec{p})]^*$$

$$= \mathrm{Tr}\left[\gamma^\beta(\not f_1 + m)\gamma^\alpha(\not p + m)\gamma_\beta(\not f_2 + m)\gamma_\alpha(\not p\,' + m)\right] \quad (175)$$

$$X_{ba} = \sum_{spin} [\overline{u}_{r'}(\vec{p}\,')\gamma^\alpha(\not f_2 + m)\gamma^\beta u_r(\vec{p})][\overline{u}_{r'}(\vec{p}\,')\gamma_\beta(\not f_1 + m)\gamma_\alpha u_r(\vec{p})]^*$$

$$= \mathrm{Tr}\left[\gamma^\alpha(\not f_2 + m)\gamma^\beta(\not p + m)\gamma_\alpha(\not f_1 + m)\gamma_\beta(\not p\,' + m)\right] \quad (176)$$

where $f_1 = p + k$ and $f_2 = p - k'$.

Under the substitutions $k \leftrightarrow -k'$, $\alpha \leftrightarrow \beta$ we find $f_1 \leftrightarrow f_2$ and hence $X_{aa} \leftrightarrow X_{bb}$ and $X_{ba} \leftrightarrow X_{ab}$. Thus, we need only compute X_{aa} and X_{ab} and then use the substitution rules to get X_{bb} and X_{ba}.

If we look closely at X_{ab} as compared to X_{ba}, we see that they are related by *trace reversal*. In the Appendix of MS, we see that the trace of a product of γ matrices is equal to the trace of the reverse product of the γ matrices by virtue of the fact that there is a matrix C such that $C\gamma^\mu C^{-1} = -[\gamma^\mu]^T$. Assuming an even number of γ matrices, the result is that inserting $CC^{-1} = 1$ throughout the trace converts the original trace to a trace such that the transpose of each γ matrix appears inside the trace and then we employ $\mathrm{Tr}[A^T B^T C^T \ldots Z^T] = \mathrm{Tr}[Z \ldots CBA]$.

- **To simplify these traces, we need some more trace identities.**

1. **First, we have**
$$\gamma^\alpha\gamma_\alpha = g_{\alpha\beta}\gamma^\alpha\gamma^\beta = \frac{1}{2}g_{\alpha\beta}[\gamma^\alpha,\gamma^\beta]_+ = \frac{1}{2}g_{\alpha\beta}2g^{\alpha\beta} = 4\,. \tag{177}$$
2. **Next, we have**
$$\gamma^\alpha\gamma^\nu\gamma_\alpha = (2g^{\alpha\nu} - \gamma^\nu\gamma^\alpha)\gamma_\alpha = (2-4)\gamma^\nu = -2\gamma^\nu \tag{178}$$
3. **Thirdly, we have**
$$\begin{aligned}\gamma_\alpha\gamma^\nu\gamma^\mu\gamma^\alpha &= (2g^\nu_\alpha - \gamma^\nu\gamma_\alpha)\gamma^\mu\gamma^\alpha\\ &= 2\gamma^\mu\gamma^\nu - \gamma^\nu(-2\gamma^\mu)\\ &= 2[\gamma^\mu,\gamma^\nu]_+\\ &= 4g^{\mu\nu}\end{aligned} \tag{179}$$
4. **Finally, we need**
$$\begin{aligned}\gamma_\alpha\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\alpha &= (2g^\nu_\alpha - \gamma^\nu\gamma_\alpha)\gamma^\mu\gamma^\sigma\gamma^\alpha\\ &= 2\gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\nu\gamma_\alpha\gamma^\mu\gamma^\sigma\gamma^\alpha\\ &= 2\gamma^\mu\gamma^\sigma\gamma^\nu - 4g^{\mu\sigma}\gamma^\nu\\ &= 2\gamma^\mu\gamma^\sigma\gamma^\nu - 2(\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu)\gamma^\nu\\ &= -2\gamma^\sigma\gamma^\mu\gamma^\nu\,.\end{aligned} \tag{180}$$

- **Using these identities we may now evaluate our X's.**

$$
\begin{aligned}
X_{aa} &= \mathrm{Tr}\left[\gamma^\beta(\not{f}_1+m)\gamma^\alpha(\not{p}+m)\gamma_\alpha(\not{f}_1+m)\gamma_\beta(\not{p}'+m)\right] \\
&= \mathrm{Tr}\left[(\not{f}_1+m)\gamma^\alpha(\not{p}+m)\gamma_\alpha(\not{f}_1+m)\gamma_\beta(\not{p}'+m)\gamma^\beta\right] \\
&= \mathrm{Tr}\left[(\not{f}_1+m)(-2\not{p}+4m)(\not{f}_1+m)(-2\not{p}'+4m)\right] \\
&= 4\mathrm{Tr}\left[(\not{f}_1+m)(\not{p}-2m)(\not{f}_1+m)(\not{p}'-2m)\right] \\
&= 4\left\{\mathrm{Tr}[\not{f}_1\not{p}\not{f}_1\not{p}'] + m^2\left[-4\mathrm{Tr}[\not{f}_1\not{p}] - 4\mathrm{Tr}[\not{f}_1\not{p}'] + \mathrm{Tr}[\not{p}\not{p}'] + 4\mathrm{Tr}[\not{f}_1\not{f}_1]\right] + 4m^4\mathrm{Tr}[1]\right\} \\
&= 16\left\{[2(f_1p)(f_1p') - (pp')(f_1f_1)] + m^2[-4(f_1p) - 4(f_1p')\right. \\
&\qquad \left. +4(f_1f_1) + (pp')] + 4m^4\right\} \\
&= 32[(pk)(pk') + m^2(pk) + m^4]\,, \qquad (181)
\end{aligned}
$$

where in the last step we used

$$
\begin{aligned}
(f_1f_1) &= (p+k)^2 = m^2 + 2(pk) \\
(f_1p) &= (p+k)\cdot p = m^2 + (pk) \\
(f_1p') &= (p'+k')\cdot p' = m^2 + (k'p') = m^2 + (kp) \\
(pp') &= p\cdot(p+k-k') = m^2 + (pk) - (pk') \qquad (182)
\end{aligned}
$$

as follow from $p^2 = p'^2 = m^2$, $(pk) = (p'k')$ and $(pk') = (p'k)$.

From the substitution rule, we immediately obtain

$$X_{bb} = 32[(pk)(pk') - m^2(pk') + m^4]\,. \tag{183}$$

For X_{ab} we find

$$\begin{aligned}
X_{ab} &= \mathrm{Tr}\left[\gamma^\beta(\not f_1 + m)\gamma^\alpha(\not p + m)\gamma_\beta(\not f_2 + m)\gamma_\alpha(\not p\,' + m)\right] \\
&= \mathrm{Tr}\left[\gamma^\beta(\not f_1 + m)(-2\not f_2\gamma_\beta\not p + 4mf_{2\beta} + 4mp_\beta - m^2 2\gamma_\beta)(\not p\,' + m)\right] \\
&= \mathrm{Tr}\Bigg[\Big\{-8(f_1f_2)\not p + 4m\not f_2\not p + 4m\not f_2(\not f_1 + m) + 4m\not p(\not f_1 + m) \\
&\qquad -2m^2(-2\not f_1 + 4m)\Big\}(\not p\,' + m)\Bigg] \\
&= 4\Bigg[\Big\{-8(f_1f_2)(pp') + 4m^2(f_2p') + 4m^2(pp') + 4m^2(f_1p')\Big\} + \Big\{4m^2(f_2p) \\
&\qquad +4m^2(f_2f_1) + 4m^2(f_1p) - 8m^4\Big\}\Bigg] \quad \text{1st from } \not p\,'\text{, 2nd from } m \\
&= 4\Big[-8m^2(pp') + 4m^2[(pp') - (pk)] + 4m^2(pp') + 4m^2[m^2 + (pk)] + 4m^2[m^2 - (pk')]
\end{aligned}$$

$$
\begin{aligned}
& +4m^2m^2 + 4m^2[m^2 + (pk)] - 8m^4\Big] \\
= \quad & 16[2m^4 + m^2(pk) - m^2(pk')]\,, \qquad (184)
\end{aligned}
$$

where we had to use

$$
\begin{aligned}
(f_1 f_2) &= (p+k)\cdot(p-k') = (p+k)\cdot p - (p'+k')\cdot k' = m^2 + (pk) - (p'k') = m^2 \\
(f_2 p') &= (p-k')\cdot p' = (pp') - (k'p') = (pp') - (pk) \\
(f_1 p') &= (p+k)\cdot p' = (p'+k')\cdot p' = m^2 + (k'p') = m^2 + (kp) \\
(f_2 p) &= (p-k')\cdot p = m^2 - (pk') \\
(f_1 p) &= (p+k)\cdot p = m^2 + (pk)\,.
\end{aligned}
$$

Using the substitution rule $(k \leftrightarrow -k')$ to get X_{ba}, we verify that $X_{ba} = X_{ab}$ as expected.

- Using these results in

$$
\begin{aligned}
\frac{1}{4}\sum_{\mathrm{pol,\,s,s'}}\sum_{\mathrm{spin,\,r,r'}} |\mathcal{M}|^2 &= \frac{1}{4}\sum_{\mathrm{spin}} \mathcal{M}^{\alpha\beta}\mathcal{M}_{\alpha\beta} \\
&= \frac{e^4}{16}\left(\frac{X_{aa}}{(pk)^2} + \frac{X_{bb}}{(pk')^2} - \frac{X_{ab}+X_{ba}}{(pk)(pk')}\right) \qquad (185)
\end{aligned}
$$

we obtain

$$\frac{1}{4}\sum_{\mathrm{pol,\,s,s'}}\sum_{\mathrm{spin,\,r,r'}}|\mathcal{M}|^2 = 2e^4\left\{\left(\frac{p\cdot k}{p\cdot k'}+\frac{p\cdot k'}{p\cdot k}\right)+2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)^2\right\} \tag{186}$$

This takes a very simple form in the laboratory system defined by $p = (m,0,0,0)$ for which we can write $k = (\omega, \vec{k})$, $k' = (\omega', \vec{k}\,')$ and $p' = (E', \vec{p}\,')$. For these definitions, we have

$$\frac{1}{4}\sum_{\mathrm{pol,\,s,s'}}\sum_{\mathrm{spin,\,r,r'}}|\mathcal{M}|^2 = 2e^4\left\{\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right)+2m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)+m^2\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)^2\right\}. \tag{187}$$

This can be further simplified by using the Compton scattering relation

$$\frac{1}{\omega}-\frac{1}{\omega'} = \frac{1}{m}(\cos\theta-1)\,, \tag{188}$$

where θ is the angle between $\vec{k}$ and $\vec{k}'$, which gives

$$\frac{1}{4}\sum_{\mathrm{pol,\,s,s'}}\sum_{\mathrm{spin,\,r,r'}}|\mathcal{M}|^2 = 2e^4\left\{\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right\}. \tag{189}$$

To derive Eq. (188) we note that since $p = p' + k' - k$ we also have

$$p \cdot k = (p' + k' - k) \cdot k = p' \cdot k + k' \cdot k = p \cdot k' + k' \cdot k \,, \qquad (190)$$

where we used $(p - k')^2 = m^2 - 2p \cdot k' = (p' - k)^2 = m^2 - 2p' \cdot k$, which in turn implies that $p \cdot k' = p' \cdot k$, for the 2nd equality.

If we take $\vec{k}$ along the $\hat{z}$ axis and $\vec{k}'$ at an angle θ with respect to the $\hat{z}$ axis, then $\vec{k}' \cdot \vec{k} = \omega\omega' \cos\theta$ and the above equation reduces to

$$\omega m = \omega' m + \omega\omega'(1 - \cos\theta) \qquad (191)$$

which, in turn, gives Eq. (188). For the cross section, we return to the expression

$$d\sigma = (2\pi)^4 \delta^4(\sum p'_f - \sum p_i) \frac{1}{4E_1 E_2 v_{rel}} \left(\prod_f \frac{d^3\vec{p}'_f}{(2\pi)^3 2E'_f} \right) |\mathcal{M}|^2 \qquad (192)$$

with $E_1 E_2 v_{rel} = [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}$. In our case we can identify particle 1 as a photon with $m_1 = 0$ and particle 2 as the target proton. Then we

get

$$
\begin{aligned}
d\sigma &= (2\pi)^4\delta^4(p'+k'-p-k)\frac{1}{4\omega m}\left(\frac{d^3\vec{p}'}{(2\pi)^3 2E'}\right)\left(\frac{d^3\vec{k}'}{(2\pi)^3 2\omega'}\right)|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2\omega\omega' m E'}\delta(\omega'+E'-m-\omega){\omega'}^2 d\omega' d\Omega\,|\mathcal{M}|^2 \\
&= \frac{\omega'}{64\pi^2\omega m E'}\left[\frac{\partial(\omega'+E')}{\partial\omega'}\right]^{-1} d\Omega\,|\mathcal{M}|^2
\end{aligned}
\tag{193}
$$

where we have used the notation $d\Omega$ for the $\vec{k}'$ solid angle. To compute the required derivative, we note that

$$
E' = [m^2 + (\vec{k}-\vec{k}')^2]^{1/2} = [m^2+\omega^2+\omega'^2-2\omega\omega'\cos\theta]^{1/2} \tag{194}
$$

so that

$$
\frac{\partial E'}{\partial\omega'} = \frac{\omega'-\omega\cos\theta}{E'} \tag{195}
$$

implying that

$$\begin{aligned}\frac{\partial(E'+\omega')}{\partial\omega'} &= \frac{\omega'-\omega\cos\theta}{E'}+1 \\ &= \frac{\omega'+E'-\omega\cos\theta}{E'} \\ &= \frac{\omega+m-\omega\cos\theta}{E'} \\ &= \frac{m\omega}{E'\omega'}\,,\end{aligned} \tag{196}$$

where for the next-to-last step we used energy conservation, $\omega'+E'=\omega+E$, and for the last step we used Eq. (188). Altogether, we get

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\omega'}{64\pi^2\omega m E'}\frac{E'\omega'}{m\omega}2e^4\left\{\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right\} \\ &= \frac{e^4}{32\pi^2m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right\}\end{aligned}$$

$$= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right\}. \tag{197}$$

In the non-relativistic limit, $\omega \ll m$, which means $\omega \sim \omega'$ (see Eq. (188)), and we get

$$\frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{2m^2}(1+\cos^2\theta), \tag{198}$$

which is the Thomson cross section. Obviously, the full relativistic result of Eq. (197) has been tested against experimental data in a highly detailed way and excellent agreement has been found (after including radiative, i.e. higher order, corrections).

- Finally, we wish to return to the gauge invariance issue. You will recall that we had the two amplitudes:

$$\mathcal{M}_a = -e^2\overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s'}(\vec{k}')\left[i\frac{\not q+m}{q^2-m^2}\right]_{q=p+k}\not\epsilon_s(\vec{k})u_r(\vec{p}) \tag{199}$$

$$\mathcal{M}_b = -e^2\overline{u}_{r'}(\vec{p}\,')\not\epsilon_{s}(\vec{k})\left[i\frac{\not q+m}{q^2-m^2}\right]_{q=p'-k}\not\epsilon_{s'}(\vec{k}')u_r(\vec{p}), \tag{200}$$

where for q in $\mathcal{M}_b$ we write $q = p' - k$ instead of the equivalent (by momentum conservation) form $q = p - k'$ employed earlier.

The gauge invariance claim is that if we replace $\epsilon(\vec{k}')$ by k' or $\epsilon(\vec{k})$ by k, then $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b \to 0$. Let's check this for the case of $\epsilon(\vec{k}) \to k$. We get, dropping spin indices for convenience, and writing in the explicit q values for each amplitude

$$\mathcal{M}_a \to -e^2 \overline{u}(\vec{p}\,')\not\epsilon\,(\vec{k}')\left[i\frac{\not p + \not k + m}{(p+k)^2 - m^2}\right]\not k\, u(\vec{p}) \tag{201}$$

$$\mathcal{M}_b \to -e^2 \overline{u}(\vec{p}\,')\not k\left[i\frac{\not p\,' - \not k + m}{(p'-k)^2 - m^2}\right]\not\epsilon\,(\vec{k}')u(\vec{p}) \tag{202}$$

In $\mathcal{M}_a$, we write

$$\begin{aligned}
\frac{\not p + \not k + m}{(p+k)^2 - m^2}\not k\, u(\vec{p}) &= \frac{\not p + \not k + m}{(p+k)^2 - m^2}(\not k + \not p - \not p\,)u(\vec{p}) \\
&= \frac{\not p + \not k + m}{(p+k)^2 - m^2}(\not k + \not p - m)u(\vec{p})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(p+k)^2 - m^2}{(p+k)^2 - m^2} u(\vec{p}) \\
&= u(\vec{p})\,, \qquad (203)
\end{aligned}
$$

so that

$$
\mathcal{M}_a \to -ie^2 \overline{u}(\vec{p}\,') \not{\epsilon}\,(\vec{k}') u(\vec{p})\,. \qquad (204)
$$

In $\mathcal{M}_b$, we write

$$
\begin{aligned}
\overline{u}(\vec{p}\,') \not{k} \frac{\not{p}\,' - \not{k} + m}{(p'-k)^2 - m^2} &= \overline{u}(\vec{p}\,')(\not{k} - \not{p}\,' + \not{p}\,') \frac{\not{p}\,' - \not{k} + m}{(p'-k)^2 - m^2} \\
&= \overline{u}(\vec{p}\,')(\not{k} - \not{p}\,' + m) \frac{\not{p}\,' - \not{k} + m}{(p'-k)^2 - m^2} \\
&= \overline{u}(\vec{p}\,') \frac{-(p'-k)^2 + m^2}{(p'-k)^2 - m^2} \\
&= -\overline{u}(\vec{p}\,')\,, \qquad (205)
\end{aligned}
$$

so that

$$
\mathcal{M}_b \to +ie^2 \overline{u}(\vec{p}\,') \not{\epsilon}\,(\vec{k}') u(\vec{p})\,. \qquad (206)
$$

Obviously, these two results (after the substitution) for $\mathcal{M}_a$ and $\mathcal{M}_b$ cancel to give zero, the requirement of gauge invariance.