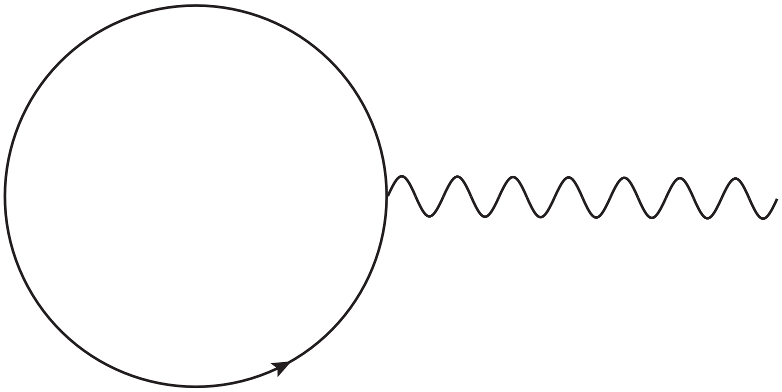
Solution to Peskin Problem 10.1

This problem is about verifying that indeed some of the superficially divergent diagrams that would destroy gauge invariance either converge or simply vanish. Here, we consider only QED and discuss the one-point, three-point and four-point diagrams in the following sections.

1 The 1-photon tadpole diagram

The diagram in question is given below. Superficially, it is cubically divergent.



Calling the loop momentum l and the external (outgoing) photon momentum q, the $\mathcal{M}$ for this diagram is:

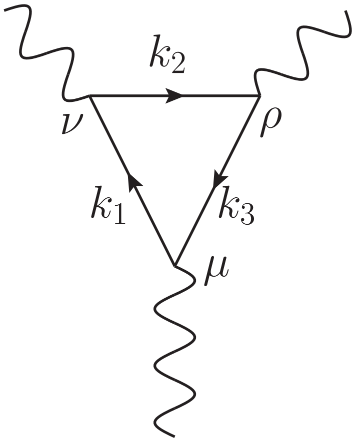
$$\begin{aligned}
i\mathcal{M} &= (-1)\epsilon^*_\mu(q)\int\frac{d^nl}{(2\pi)^n}\mathrm{Tr}\left[\frac{i(\not{l}+m)}{(l^2-m^2+i\epsilon)}(-ie\gamma^\mu)\right] \\
&= -e\epsilon^*_\mu(q)\int\frac{d^nl}{(2\pi)^n}\frac{\mathrm{Tr}\left[\not{l}\gamma^\mu+m\gamma^\mu\right]}{(l^2-m^2+i\epsilon)} \\
&= -4e\epsilon^*_\mu(q)\int\frac{d^nl}{(2\pi)^n}\frac{l^\mu}{(l^2-m^2+i\epsilon)} \\
&= 0\,, \qquad (1)
\end{aligned}$$

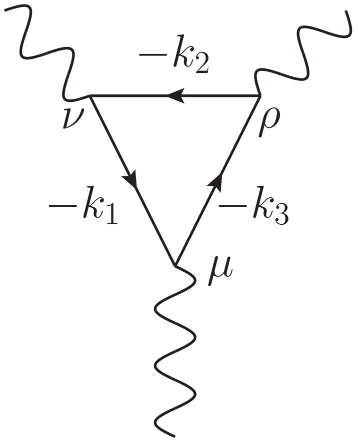
where the zero arises as a result of the fact that the integration and denominator are symmetric under $l \to -l$ whereas the numerator is antisymmetric.

Of course, this means that there is no contribution when this diagram is attached to anything anywhere, including a propagating fermion line.

2 The 3-photon vertex diagrams

There are two diagrams as given below. Naively, they are linearly divergent. As above, one might imagine that the actual divergence would be logarithmic as a result of antisymmetry of the loop integrand. But, in fact the sum of the two diagrams is simply zero. This is called 'Furry's Theorem'. We show this below.





The second diagram has been labelled with the same vertex indices as the first diagram, but the fermion line is reversed and we have used $-k_i$ in the second diagram for the momenta *in the direction of the fermion arrow* in the 2nd diagram, so that k_i is still the same as the momentum *in the direction of the fermion arrow* in the 1st diagram.

The first diagram is integrated over a numerator trace of the form (always working against the fermion arrow according to our rules):

$$\mathrm{Tr}\left[\gamma^\mu(\not{k}_3+m)\gamma^\rho(\not{k}_2+m)\gamma^\nu(\not{k}_1+m)\right] \tag{2}$$

Remembering that the trace of an odd number of gamma matrices vanishes, the above trace reduces to

$$\mathrm{Tr}\left[\gamma^\mu\not{k}_3\gamma^\rho\not{k}_2\gamma^\nu\not{k}_1\right]+m^2\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\gamma^\nu\not{k}_1\right]+m^2\mathrm{Tr}\left[\gamma^\mu\gamma^\rho\not{k}_2\gamma^\nu\right]+m^2\mathrm{Tr}\left[\gamma^\mu\not{k}_3\gamma^\rho\gamma^\nu\right] \tag{3}$$

Now consider the 2nd diagram. The numerator has a trace of the form:

$$\mathrm{Tr}\left[\gamma^\mu(-\not{k}_1+m)\gamma^\nu(-\not{k}_2+m)\gamma^\rho(-\not{k}_3+m)\right] \tag{4}$$

which reduces to the sum of terms

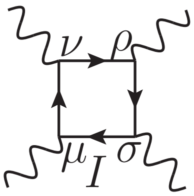
$$-\mathrm{Tr}\left[\gamma^\mu\not{k}_1\gamma^\nu\not{k}_2\gamma^\rho\not{k}_3\right]-m^2\mathrm{Tr}\left[\gamma^\mu\not{k}_1\gamma^\nu\gamma^\rho\right]-m^2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\not{k}_2\gamma^\rho\right]-m^2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\not{k}_3\right] \tag{5}$$

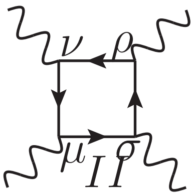
Now, we note that the fact that a trace is equal to the trace of the same objects in reverse order (the "trace reversal" theorem), each of the traces above is equal to the corresonding trace in the expression for the 1st diagram. But, every trace for the 2nd diagram comes with the opposite sign and so there is exact cancellation. Therefore, the two diagrams are actually equal in magnitude but opposite in sign and their sum is 0.

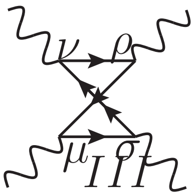
This persists to all orders and is called Furry's theorem.

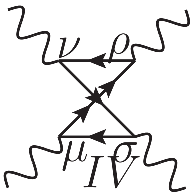
3 The 4-photon vertex diagrams

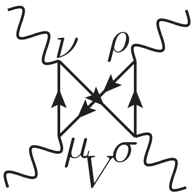
The full set of one-loop diagrams for the 4-photon vertex is given below.

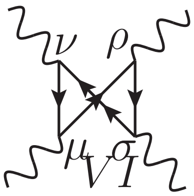




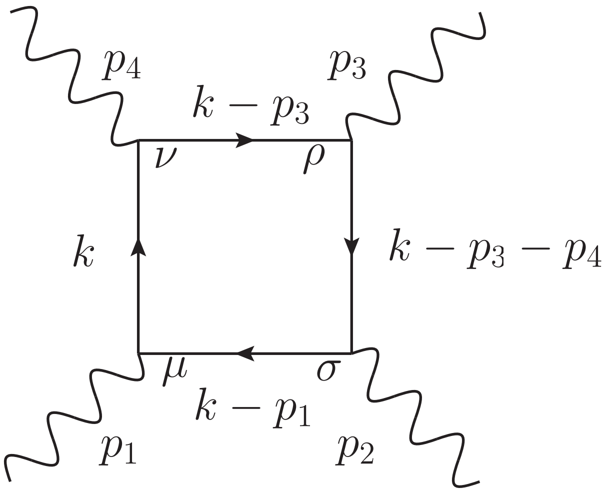








Let us begin by examining diagram I above in more detail. With explicit momentum choices we have



with explicit expression

$$\begin{aligned} &= -e^4 \int \frac{d^n k}{(2\pi)^n} \frac{\mathrm{Tr}\left[(\not{k} - \not{p}_1 + m)\gamma^\sigma(\not{k} - \not{p}_3 - \not{p}_4 + m)\gamma^\rho(\not{k} - \not{p}_3 + m)\gamma^\nu(\not{k} - m)\gamma^\mu\right]}{((k-p_1)^2 - m^2)((k-p_3-p_4)^2 - m^2))((k-p_3)^2 - m^2)(k^2 - m^2)} \\ &= -e^4 \int \frac{d^n k}{(2\pi)^n} \frac{\mathrm{Tr}\left[\not{k}\gamma^\sigma \not{k}\gamma^\rho \not{k}\gamma^\nu \not{k}\gamma^\mu\right]}{(k^2)^4} + finite\,. \qquad (6) \end{aligned}$$

This means that the divergences of the different diagrams depend only on the order of the γ matrices appearing in the trace. Further, it is easy to see that the divergence of I is the same as that of II and that similarly $III \sim IV$ and $V \sim VI$. This is because the relative change of sign for the loop momentum k, between each pair, will not change the divergence of the diagram because $k^4 = (-k)^4$. Second, the ordering of the vertices are precisely reversed for each pair and so they are equal according to the trace reversal theorem.

Let us continue with diagram I. Since the sum of all the diagrams will be finite, it is we can do the trace algebra in 4 dimensions and we will end up with the same result as if we use the dimensional regularization procedures for the trace algebra (equivalently, we will just be using the dimensional reduction conventions).

We then have the following sequence of steps.

$$\mathrm{Tr}\left[\not{k}\gamma^\sigma \not{k}\gamma^\rho \not{k}\gamma^\nu \not{k}\gamma^\mu\right] = k_\alpha k_\beta k_\gamma k_\delta \mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right]$$

$$
\begin{aligned}
&= \frac{1}{n(n+2)}(k^2)^2(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma})\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right] \\
&\propto \mathrm{Tr}[(-2\gamma^\mu)\gamma^\nu(-2\gamma^\rho)\gamma^\sigma]+\mathrm{Tr}[(-2\gamma^\nu\gamma^\beta\gamma^\mu)\gamma^\rho\gamma_\beta\gamma^\sigma]+\mathrm{Tr}[\gamma^\mu(-2\gamma^\nu)\gamma^\rho(-2\gamma^\sigma)] \\
&= 4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]-2\mathrm{Tr}[\gamma^\nu(4g^{\mu\rho})\gamma^\sigma]+4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \\
&= 8\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]-8g^{\mu\rho}\mathrm{Tr}[\gamma^\nu\gamma^\sigma] \\
&= 32(g^{\rho\sigma}g^{\mu\nu}-g^{\nu\sigma}g^{\mu\rho}+g^{\mu\sigma}g^{\nu\rho})-32g^{\mu\rho}g^{\nu\sigma} \\
&= 32(g^{\rho\sigma}g^{\mu\nu}-2g^{\nu\sigma}g^{\mu\rho}+g^{\mu\sigma}g^{\nu\rho}) \qquad (7)
\end{aligned}
$$

The final step of the argument is to note that the non equivalent diagrams I, III and V differ only by the order in which the gamma matrix indices appear. We thus have for the sum

$$32(g^{\rho\sigma}g^{\mu\nu}-2g^{\nu\sigma}g^{\mu\rho}+g^{\mu\sigma}g^{\nu\rho})+32(g^{\nu\sigma}g^{\mu\rho}-2g^{\rho\sigma}g^{\mu\nu}+g^{\mu\sigma}g^{\nu\rho})+32(g^{\rho\sigma}g^{\mu\nu}-2g^{\nu\rho}g^{\mu\sigma}+g^{\mu\rho}g^{\nu\sigma}) \qquad (8)$$

in which the coefficient of each gg product has the structure $(32-64+32)=0$, or some permutation thereof, and so the log divergent terms cancel with the result that the sum of all diagrams is finite.