Solution to 'sunset' diagram problem

The sunset diagram provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence $Z - 1$, where I am simplifying notation by using $Z \equiv Z_\phi$.

1. First, use Feynman parameters to write the product of 3 propagators as (to separate the ϵ of dimensional regularization from the Feynman propagator $i\epsilon$, I use $i0$ for the latter)

$$\prod_{j=1}^{3} \frac{i}{q_j^2 - m^2 + i0} = \iiint dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{2i^3}{(\mathcal{D})^3} \tag{1}$$

where

$$\mathcal{D} = xq_1^2 + yq_2^2 + zq_3^2 - m^2 + i0. \tag{2}$$

Then substitute $q_3 = p - q_1 - q_2$. Expanding q_3^2 and collecting all the terms containing the q_1 momentum into a full square, we find

$$xq_1^2 + yq_2^2 + z(p - q_1 - q_2)^2 = (x+z)\left(q_1 + \frac{z}{x+z}(q_2 - p)\right)^2 + \frac{xz}{x+z}(q_2 - p)^2 + yq_2^2. \tag{3}$$

We identify the first term here as αk_1^2, where

$$\alpha = (x+z), \qquad k_1 = q_1 + \frac{z}{x+z}(q_2 - p). \tag{4}$$

For the other two terms on the right hand side of Eq. (3), we expand $(q_2 - p)^2$ and collect all terms containing the q_2 momentum into another full square, obtaining

$$\frac{xz}{x+z}(q_2 - p)^2 + yq_2^2 = \frac{xz + y(x+z)}{x+z}\left(q_2 - \frac{xz}{xz + y(x+z)}\,p\right)^2 + \frac{xzy}{xz + y(x+z)}\,p^2. \tag{5}$$

Consequently, we define

$$\beta = \frac{xy + xz + yz}{x+z}, \qquad \gamma = \frac{xyz}{xy + xz + yz}, \qquad k_2 = q_2 - \frac{xz}{xy + xz + yz}\,p, \tag{6}$$

after which the right hand side of Eq. (5) takes the form $\beta k_2^2 + \gamma p^2$. Altogether, we have

$$xq_1^2 + yq_2^2 + zq_3^2 = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 \tag{7}$$

We will shift to these new 'diagonal' momenta. The Jacobian for replacing the original independent loop momenta q_1 and q_2 with k_1 and k_2 is, given Eqs. (4) and (6),

$$\frac{\partial(k_1, k_2)}{\partial(q_1, q_2)} = \det\begin{pmatrix} 1 & \frac{z}{x+z} \\ 0 & 1 \end{pmatrix} = 1,$$

and therefore $dk_1\, dk_2 = dq_1\, dq_2$, dimension by dimension. In other words, for fixed Feynman parameters,

$$\int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} = \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4}. \tag{8}$$

And, our final denominator form is

$$\mathcal{D} \;=\; \alpha k_1^2 \;+\; \beta k_2^2 \;+\; \gamma p^2 \;-\; m^2 \;+\; i0\,. \tag{9}$$

Note, we must not set $p^2 = m^2$ at this stage.

2. Assembling all the factors of the two loop amplitude and making use of the above results, we have

$$\begin{aligned}
-i\Sigma(p^2) \quad &= \quad \frac{(-i\lambda)^2}{3!}\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\,\frac{i}{q_1^2-m^2+i0}\,\frac{i}{q_3^2-m^2+i0}\,\frac{i}{q_3^2-m^2+i0}\\
&= \quad \frac{i\lambda^2}{6}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{2}{\mathcal{D}^3}
\end{aligned} \tag{10}$$

where $\mathcal{D}$ is as in Eq. (9). In particular, the dependence on the external momentum p comes solely through the γp^2 term in $\mathcal{D}$. Hence,

$$\frac{d\Sigma}{dp^2} \;=\; -\frac{\lambda^2}{6}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,\frac{(-6\gamma)}{\mathcal{D}^4}\,. \tag{11}$$

Note that for large loop momenta k_1 and k_2, $\mathcal{D}$ grows like k^2. Consequently, the integrand of the 8–dimensional momentum integral Eq. (10) behaves like $1/k^6$, and the integral diverges quadratically. On the other hand, the integrand of Eq. (11) behaves like $1/k^8$, so the divergence of this integral is only logarithmic.

3. Rotating both loop momenta k_1 and k_2 into Euclidean momentum space, we have $d^4k_1 \to id^4k_1^E$, $d^4k_2 \to id^4k_2^E$, and

$$\mathcal{D} \;\to -\alpha(k_1^E)^2 \;-\; \beta(k_2^E)^2 \;+\; \gamma p^2 \;-\; m^2, \tag{12}$$

hence

$$\frac{d\Sigma}{dp^2} \;=\; +\frac{\lambda^2}{6}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4k_1^E}{(2\pi)^4}\int\frac{d^4k_2^E}{(2\pi)^4}\,\frac{6\times(-\gamma)}{[\alpha(k_1^E)^2\;+\;\beta(k_2^E)^2\;+\;m^2\;-\;\gamma p^2]^4}\,. \tag{13}$$

4. Next, we need dimensional regularization to actually perform the momentum integrals. Changing

$$\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4} \;\;\to\;\; \mu^{2(4-n)}\int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n} \tag{14}$$

(where we employ Euclidean signature for all dimensions and have introduced the μ dependence in a way equivalent to pulling out the μ from the coupling constant), we have

$$\mu^{8-2n}\int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\,\frac{6}{[\alpha(k_1^E)^2\;+\;\beta(k_2^E)^2\;+\;m^2\;-\;\gamma p^2]^4}$$

$\langle\!\langle$using eq. (16)$\rangle\!\rangle$

$$
\begin{aligned}
&= \mu^{8-2n}\int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\int_0^\infty dt\, t^3\exp\Bigl(-t[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]\Bigr)\\
&= \mu^{8-2n}\int_0^\infty dt\, t^3 e^{-t(m^2-\gamma p^2)}\int\frac{d^nk_1}{(2\pi)^n}\int\frac{d^nk_2}{(2\pi)^n}\, e^{-t\alpha k_1^2}e^{-t\beta k_2^2}\\
&\quad \langle\!\langle\text{using eq. (17)}\rangle\!\rangle\\
&= \mu^{8-2n}\int_0^\infty dt\, t^3 e^{-t(m^2-\gamma p^2)}\times(4\pi\alpha t)^{-n/2}(4\pi\beta t)^{-n/2}\\
&= \frac{\mu^{8-2n}}{(4\pi)^n(\alpha\beta)^{n/2}}\times\int_0^\infty dt\, t^{3-n}e^{-t(m^2-\gamma p^2)}\\
&= \frac{\mu^{8-2n}}{(4\pi)^n(\alpha\beta)^{n/2}}\times\Gamma(4-n)(m^2-\gamma p^2)^{n-4}. \qquad (15)
\end{aligned}
$$

Note the $\Gamma(4-n)$ factor: It has a pole at $n=4$ but no poles at $n<4$. This is dimensional regularization's way to show that the momentum integrals diverge, but only logarithmically.

In obtaining the above, I have used the following formulae, the first two of which provide an alternative way of doing the dimensionally regulated momentum integrals. (You should get the same result if you use the approach given in the notes.)

$$
\begin{aligned}
\frac{6}{A^4} &= \int_0^\infty dt\, t^3\, e^{-At}, &(16)\\
\int\frac{d^nk}{(2\pi)^n}\, e^{-ctk^2} &= (4\pi ct)^{-n/2}, &(17)\\
\Gamma(2\epsilon)X^\epsilon &= \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2}\log X. &(18)
\end{aligned}
$$

The last formula above is simply the standard Gamma function expansion to be used below.

At this point, we may take $n = 4-2\epsilon$ for an infinitesimally small ϵ. Hence, the last line of Eq. (15) becomes

$$
\frac{1}{(4\pi)^4(\alpha\beta)^2}\,\Gamma(2\epsilon)\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2-\gamma p^2}\right)^{2\epsilon} \overset{\epsilon\to 0}{\longrightarrow} \frac{1}{(4\pi)^4(\alpha\beta)^2}\times\left(\frac{1}{2\epsilon} - \gamma_E + \log\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2-\gamma p^2}\right). \qquad (19)
$$

Plugging this formula back into Eq. (13) and assembling all the factors, we finally arrive at

$$
\begin{aligned}
\frac{d\Sigma}{dp^2} &= -\frac{\lambda^2}{3072\pi^4}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{\gamma}{(\alpha\beta)^2}\\
&\qquad\times\left\{\frac{1}{\epsilon} + 2\log\frac{\mu^2}{m^2} + C + \log\frac{\alpha\beta}{[1-(p^2/m^2)\gamma]^2}\right\} \qquad (20)
\end{aligned}
$$

where $C = 2\log(4\pi) - 2\gamma_E$ is a numerical constant while $\alpha(x,y,z)$, $\beta(x,y,z)$ and $\gamma(x,y,z)$ depend on the Feynman parameters according to Eqs. (4) and (6).

5. We are left with one more task, namely integrating over the Feynman parameters. This looks like a daunting task, especially if one wants analytic dependence on the external momentum p^2, but fortunately we are only interested in the particular value of p^2 = physical mass^2. Since

we are working at the leading order of perturbation theory which contributes to the $d\Sigma/dp^2$, we may neglect the difference between the physical and the bare masses as a higher-order correction and set $p^2 = m^2$. Consequently, Eq. (20) simplifies to

$$\begin{aligned}
\left.\frac{d\Sigma}{dp^2}\right|_{p^2=m^2} &= -\frac{\lambda^2}{3072\pi^4}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{\gamma}{(\alpha\beta)^2}\times \\
&\qquad \left\{\frac{1}{\epsilon}\;+\;2\log\frac{\mu^2}{m^2}\;+\;C+\;\log\frac{\alpha\beta}{(1-\gamma)^2}\right\} \\
&= -\frac{\lambda^2}{3072\pi^4}\iiint dx\,dy\,dz\,\delta(x+y+z-1)\,\left(\frac{xyz}{(xy+xz+yz)^3}\right)\times \\
&\quad \times\left\{\frac{1}{\epsilon}\;+\;2\log\frac{\mu^2}{m^2}\;+\;C\;+\;\log\frac{(xy+xz+yz)^3)}{(xy+xz+yz-xyz)^2}\right\} \qquad (21)
\end{aligned}$$

where the second equality follows from Eqs. (4) and (6) for the $\alpha(x,y,z)$, $\beta(x,y,z)$, and $\gamma(x,y,z)$.

At this stage, you have arrived at the promised form:

$$\begin{aligned}
\frac{d\Sigma}{dp^2} &= \iiint dxdydz\,\delta(x+y+z-1)\,F(x,y,z)\times \\
&\quad \times\left\{\frac{1}{\epsilon}\;+\;\text{const}\;+\;\log G(x,y,z;p^2/m^2)\right\} \qquad (22)
\end{aligned}$$

for the rational functions F, G of the Feynman parameters (and in the case of G, also of $p^2/m^2 = 1$) that are given explicitly in Eq. (21).

The reason that we have set $p^2 = m^2$ (which to the present order is the same as m^2_{physical}), is that our goal is the field strength renormalization factor

$$Z \;=\; \left[1\;-\;\frac{d\Sigma}{dp^2}\right]^{-1} \qquad (23)$$

where the derivative is evaluated at $p^2 = m^2_{\text{physical}}$.

6. Despite the above simplification, Eq. (21) is a painful mess to evaluate simply by hand. However, if we turn to Mathematica it is fairly straightforward to find that

$$\begin{aligned}
\iiint dxdydz\,\delta(x+y+z-1)\,\frac{xyz}{(xy+xz+yz)^3} &= \frac{1}{2}, \\
\iiint dxdydz\,\delta(x+y+z-1)\,\frac{xyz}{(xy+xz+yz)^3}\,\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} &= -\frac{3}{4}. \qquad (24)
\end{aligned}$$

Using Eqs. (24), we find

$$\left.\frac{d\Sigma}{dp^2}\right|_{p^2=m^2} \;=\; -\frac{\lambda^2}{6144\pi^4}\left\{\frac{1}{\epsilon}\;+\;2\log\frac{\mu^2}{m^2}\;+\;C\;-\;\frac{3}{2}\right\} \qquad (25)$$

to the leading order in λ, and therefore

$$Z \;=\; \left.\frac{1}{1 \;-\; \frac{d\Sigma}{dp^2}}\right|_{p^2=M^2} \;=\; 1 \;-\; \frac{\lambda^2}{6144\pi^4}\left\{\frac{1}{\epsilon} \;+\; 2\log\frac{\mu^2}{m^2} \;+\; C \;-\; \frac{3}{2}\right\} \;+\; O(\lambda^3). \tag{26}$$

As stated, to get these results in Mathematica it is useful to replace the (x, y, z) variables with (w, ξ) according to $x = \xi w$, $y = (1-\xi)w$, $z = 1-w$, then integrate over the w variable first and over ξ second. The Mathematica program and results follow.

7. Finally, we use the $\frac{1}{\epsilon}$ term to identify c_1 in the expansion for Z_ϕ in our general formulae:

$$\begin{aligned}
\lambda_0 &= (\mu^2)^\epsilon\left[\lambda + \sum_{k=1}^{\infty}\frac{a_k(\lambda)}{\epsilon^k}\right]\\
m_0^2 &= m^2\left[1 + \sum_{k=1}^{\infty}\frac{b_k(\lambda)}{\epsilon^k}\right]\\
Z_\phi &= \left[1 + \sum_{k=1}^{\infty}\frac{c_k(\lambda)}{\epsilon^k}\right],
\end{aligned} \tag{27}$$

obtaining

$$c_1 = -\frac{\lambda^2}{6144\pi^4}\,. \tag{28}$$

From this, we can now compute the anomalous dimension

$$\gamma_d(\lambda) = -\lambda\frac{dc_1}{d\lambda}\,. \tag{29}$$

The result is

$$\gamma_d(\lambda) = \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + \mathcal{O}(\lambda^3)\,, \tag{30}$$

as promised in the lecture notes.

```
z = 1 - w
```

$1 - w$

```
x = xi * w
```

$w\,xi$

```
y = (1 - xi) * w
```

$w\,(1 - xi)$

```
jac = w
```

w

```
f = jac * x y z / (x y + x z + y z) ^3
```

$$\frac{(1-w)\,w^3\,(1-xi)\,xi}{((1-w)\,w\,(1-xi)+(1-w)\,w\,xi+w^2\,(1-xi)\,xi)^3}$$

```
f1 = Integrate[f, {xi, 0, 1}, {w, 0, 1}]
```

$$\frac{1}{2}$$

```
g = f * Log[(x y + x z + y z) ^3 / (x y + x z + y z - x y z) ^2]
```

$$\Big((1-w)\,w^3\,(1-xi)\,xi\,\mathrm{Log}\Big[((1-w)\,w\,(1-xi)+(1-w)\,w\,xi+w^2\,(1-xi)\,xi)^3 \Big/ ((1-w)\,w\,(1-xi)+(1-w)\,w\,xi+w^2\,(1-xi)\,xi-(1-w)\,w^2\,(1-xi)\,xi)^2\Big]\Big) \Big/ ((1-w)\,w\,(1-xi)+(1-w)\,w\,xi+w^2\,(1-xi)\,xi)^3$$

```
g1 = NIntegrate[g, {w, 0, 1}, {xi, 0, 1}]
```

NIntegrate::slwcon :
Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration being 0, oscillatory integrand, or insufficient WorkingPrecision. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate. More…

$-0.75 - 5.769671016140136\times 10^{-320}\,i$