Class Notes on Renormalization

Basic concepts of renormalization, Reduction Formalism, Coupling Constant and Wave
 Function Renormalization, BPH Renormalization, Power Counting, Renormalization
 Schemes, φ⁴ Theory, Renormalization Group, Gauge Theories, Explicit QED Calculations,
 Explicit QCD Calculations, Explicit Electroweak Results and Grand Unification
 Useful references include Peskin and Schroeder,
 Ryder, Itzykson and Zuber, and Cheng and Li.

Jack Gunion U.C. Davis

230C, U.C. Davis

Renormalization is not unique to QFT. For example, recall the difference between m and m^* (the effective mass) for an electron in a solid. $m \rightarrow m^*$ as a result of the electron's interaction with the lattice ions as it moves through the lattice.

However, there is an important difference relative to QFT. This is the fact that both m and m^* are, in principle, measurable. In particular, the bare mass m (i.e. outside the lattice) can be measured in the usual way. In QFT, we will find that:

1. The bare mass cannot be directly measured — only the renormalized mass is measurable.

There is no way to switch off the interactions. The electron is always exposed to the QED interactions through virtual processes involving virtual photons and virtual electron-positron loops.

2. $m - m^*$ is naively infinite until some cutoff due to the physics of very large momentum (small distances) is brought into the picture.

If we eventually acquire a full understanding of physics at all momentum scales, we will be able to *compute* the bare mass in terms of the measured renormalized mass m^* .

Thus, in QFT, the process of renormalization consists of:

- 1. Imagining that we will ultimately understand the high scale physics and in particular taking seriously the idea that there will be some way of cutting off the divergent integrals at high momentum.
- 2. Understanding that the exact form of this cutoff is not important when dealing with observations at energies well below the high scale of the cutoff physics, so long as one can rephrase all low energy predictions in terms of a small number of quantities measured at low energy.

Theories with this property are called *renormalizable* **theories.**

3. Rewriting all predictions in terms of a few measured quantities (e.g. in QED the mass and charge of the electron as measured in some particular experiment). Since the measured quantities are finite, all our predictions will be finite and well defined in terms of these few inputs.

Renormalization for $\lambda \phi^4$

By considering this relatively simple theory we will be able to understand essentially all the important basic procedures and ideas without the complexity of spinors and gauge fields.

The theory is defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{1}$$

with

$$\mathcal{L}_{0} = \frac{1}{2} [(\partial_{\mu} \phi_{0}) (\partial^{\mu} \phi_{0}) - \mu_{0}^{2} \phi_{0}^{2}]$$

$$\mathcal{L}_{I} = -\frac{\lambda_{0}}{4!} \phi_{0}^{4}.$$
 (2)

Here, I have been careful to use 0 subscripts to denote the "bare" field, mass and coupling.

The bare propagator and four-point Feynman interaction are, respectively,

$$i\Delta_0(p) = \frac{i}{p^2 - \mu_0^2 + i\epsilon}, \quad -i\lambda_0.$$
(3)

1PI Diagrams

An important concept is that of one-particle irreducible diagrams. 1PI means that you start with a single line and end with a single line and draw all diagrammatic structure in between which cannot be cut apart by cutting just a single line.

The complete set of insertions into a single line is then given by iterations of the 1PI diagram set with intermediate single line propagators.

1 particle propagator = bare line + bare line $(-i\Sigma(p))$ bare line + bl $(-i\Sigma(p))$ bl $(-i\Sigma(p))$ bl + ... (4)

Here, $-i\Sigma(p)$ is the algebraic expression for the 1PI diagram set. This can be written algebraically as:

$$egin{array}{rll} i\Delta(p) &=& i\Delta_0(p)+i\Delta_0(p)[-i\Sigma(p)]i\Delta_0(p)+\dots \ &=& rac{i}{p^2-\mu_0^2+i\epsilon}+rac{i}{p^2-\mu_0^2+i\epsilon}[-i\Sigma(p)]rac{i}{p^2-\mu_0^2+i\epsilon}+\dots \end{array}$$

$$= \frac{i}{p^{2} - \mu_{0}^{2} + i\epsilon} \left[\frac{1}{1 + i\Sigma(p^{2})\frac{i}{p^{2} - \mu_{0}^{2} + i\epsilon}} \right]$$
$$= \frac{i}{p^{2} - \mu_{0}^{2} - \Sigma(p^{2}) + i\epsilon}, \qquad (5)$$

where, midway through, I used the fact that Lorentz invariance and analyticity imply the Σ can really only be a function of p^2 .

Now, if we just simply write down an expression using Feynman rules of the lowest order contribution to $-i\Sigma(p^2)$ (coming from the bubble tadpole correction diagram), the expression is infinite (as we shall dwell on later). The ultimate high scale physics theory will make this expression finite. Effectively the divergence will be regulated by some high scale, but for the moment we can employ any regularization procedure that renders the diagram finite.

There are also divergent 1-loop corrections to the ϕ^4 interaction. All the relevant 1-loop diagrams appear below.



Figure 1: The one-loop corrections to the propagator and vertex. The algebraic expressions for these are easily given. For the propagator correction, we get

$$-i\Sigma(p^2) = -i\frac{\lambda_0}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2 - \mu_0^2 + i\epsilon}$$
(6)

Note that the integral is quadratically divergent. The first of the three vertex diagrams gives

$$\Gamma_a = \Gamma(p) = \Gamma(s) = rac{(-i\lambda_0)^2}{2} \int rac{d^4l}{(2\pi)^4} \left[rac{i}{l^2 - \mu_0^2 + i\epsilon}
ight] \left[rac{i}{(l+p_1+p_2)^2 - \mu_0^2 + i\epsilon}
ight]_{(7)}$$

and it is obvious that

$$\Gamma_b = \Gamma(t), \quad \Gamma_c = \Gamma(u),$$
(8)

where we have defined

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2.$$
 (9)

The integral defining Γ is only logarithmically divergent.

There are also vertex corrections in which the bare vertex is corrected by inserting the 1PI correction into one of the external lines. These will be handled in a special manner later on.

We will typically try to separate "standard" divergent pieces from finite pieces according to some scheme.

For example, if we write

$$\Gamma(p^{2}) = a_{0} + a_{1}p^{2} + \ldots + \frac{1}{n!}a_{n}(p^{2})^{n} + \ldots, \quad a_{n} = \frac{\partial^{n}}{\partial^{n}p^{2}}\Gamma(p^{2})\Big|_{p^{2}=0}, \quad (10)$$

then only a_0 is divergent. Thus, we can write

$$\Gamma(s) = \Gamma(0) + \widetilde{\Gamma}(s) \tag{11}$$

where $\widetilde{\Gamma}(s)$ is finite and $\widetilde{\Gamma}(0) = 0$.

So, let us now turn to

Mass and wave-function renormalization

which involves the 1 particle propagator corrections.

In ϕ^4 theory at one-loop, there is a special feature that is not typical of other field theories or higher orders. This is that $\Sigma(p^2) = \Sigma(0)$ because of the fact that at one loop p^2 does not enter into the closed loop. (See earlier figure.)

More generally, we must write

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'(\mu^2) + \widetilde{\Sigma}(p^2)$$
(12)

where $\Sigma(\mu^2)$ is quadratically divergent and $\Sigma'(\mu^2)$ is logarithmically divergent, before regularization. Further, from the above definition, $\tilde{\Sigma}(\mu^2) = 0$ and $\tilde{\Sigma}'(\mu^2) = 0$ (i.e. $\tilde{\Sigma} \propto (p^2 - \mu^2)^2 + \ldots$ — something that will be important later).

(At one-loop, $\Sigma'(p^2) = \widetilde{\Sigma}'(p^2) = 0$, but this is not the case once higher orders are considered.)

(Note that there is no term linear in p, which would be a linearly divergent term, since a term $\propto p^{\mu}$ would not be Lorentz invariant and since analyticity forbids Σ being a function of $\sqrt{p^2}$.)

Let us substitute the result Eq. (12) into our earlier expression for $\Delta(p^2)$, Eq. (5). We obtain

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'(\mu^2) - \widetilde{\Sigma}(p^2) + i\epsilon}.$$
 (13)

We next define μ^2 , the physical pole mass squared by the consistency requirement

$$\mu^2 = \mu_0^2 + \Sigma(\mu^2) \,. \tag{14}$$

Then, $i\Delta(p^2)$ has a pole at $p^2 = \mu^2$ by construction, and it is the pole or singularity that is the physical definition of the physical mass of the particle.

We should notice that if $\Sigma(\mu^2)$ should one day be computable given a theory with full ultraviolet (large momentum) information, it would be possible to figure out what μ_0^2 corresponds to the observed physical pole mass-squared μ^2 . But, even if this is not possible, we can still carry on in working out low-energy predictions of the theory so long as every such prediction can be rephrased in terms of the experimentally observed μ^2 . Theories for which the latter is true define what we mean by a renormalizable theory or model.

Anyway, inserting the consistency requirement Eq. (14), we obtain

$$i\Delta(p^2) = rac{i}{(p^2 - \mu^2)[1 - \Sigma'(\mu^2)] - \widetilde{\Sigma}(p^2) + i\epsilon}$$
 (15)

the infinity of $\Sigma(\mu^2)$ has been absorbed into the definition of the mass. Now we must turn to how to deal with $\Sigma'(\mu^2)$. We will discuss the procedure at order λ_0 (i.e. one-loop level). Both $\Sigma'(\mu^2)$ and $\widetilde{\Sigma}(p^2)$ are $\mathcal{O}(\lambda_0)$ (in fact, for the simple ϕ^4 theory, they are both 0 — the current discussion is meant to apply to any theory.) Thus, perturbatively (we of course imagine that we have regulated any infinities, or, equivalently, that we know the full theory at high momentum and have simply input the true finite values) we can write

$$\widetilde{\Sigma}(p^2) \simeq [1 - \Sigma'(\mu^2)]\widetilde{\Sigma}(p^2) + \mathcal{O}(\lambda_0^2),$$
 (16)

in which case we can (to order λ_0) write

$$i\Delta(p^2) = \frac{iZ_{\phi}}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + i\epsilon},$$
(17)

where

$$Z_{\phi} = \frac{1}{1 - \Sigma'(\mu^2)} \simeq 1 + \Sigma'(\mu^2) + \mathcal{O}(\lambda_0^2).$$
(18)

In this form, the remaining naively divergent component is a multiplicative factor and can be removed by redefining the field operator ϕ :

$$\phi = Z_{\phi}^{-1/2} \phi_0 \,. \tag{19}$$

We would then have the renormalized propagator

$$\begin{split} i\Delta_{R}(p^{2}) &\equiv \int d^{4}x e^{-ip \cdot x} \langle 0 | T\{\phi(x)\phi(0)\} | 0 \rangle \\ &= Z_{\phi}^{-1} \int d^{4}x e^{-ip \cdot x} \langle 0 | T\{\phi_{0}(x)\phi_{0}(0)\} | 0 \rangle \\ &= \frac{i}{p^{2} - \mu^{2} - \widetilde{\Sigma}(p^{2}) + i\epsilon} \\ &= i Z_{\phi}^{-1} \Delta(p^{2}) \,, \end{split}$$
(20

which is now completely finite.

The quantity Z_{ϕ} is called the wave function renormalization constant. Renormalized Green's functions are more generally defined by

$$G_{R}^{(n)}(x_{1},...) = \langle 0|T\{\phi(x_{1})...\phi(x_{n})\}|0\rangle$$

$$= Z_{\phi}^{-n/2} \langle 0|T\{\phi_{0}(x_{1})...\phi_{0}(x_{n})\}|0\rangle$$

$$= Z_{\phi}^{-n/2} G_{0}^{(n)}(x_{1},...).$$
(21)

One finds that the amplitudes for scattering of physical particles are given in terms of an appropriate projection operator operating on the appropriate G_R renormalized Green's function, and will thus be finite.

In momentum space, the same type of relationship between renormalized and bare Green's functions applies.

$$G_R^{(n)}(p_1,\ldots,p_n) = Z_{\phi}^{-n/2} G_0^{(n)}(p_1,\ldots,p_n)$$
(22)

where

$$(2\pi)^4 \delta^4(p_1 + \ldots + p_n) G_R^{(n)}(p_1, \ldots) = \int \prod_{i=1}^n d^4 x_i e^{-ip_i \cdot x_i} G_R^{(n)}(x_1, \ldots) .$$
 (23)

To go from the connected Green's function to the 1PI (amputated) Green's function, Γ , we remove the 1P reducible diagrams, and also remove the propagators for the external lines, i.e. remove the $\Delta_R(p_i)$'s from $G_R^{(n)}(p_1 \dots p_n)$ and $\Delta(p_i)$'s from $G_0^{(n)}(p_1 \dots p_n)$. Algebraically, this means

$$G_{0}^{(n)}(p_{1},...) = \prod_{i=1}^{n} \Delta(p_{i})\Gamma_{0}^{(n)}(p_{1},...),$$

$$G_{R}^{(n)}(p_{1},...) = \prod_{i=1}^{n} \Delta_{R}(p_{i})\Gamma_{R}^{(n)}(p_{1},...).$$
(24)

Obviously, this gives us

$$\frac{\Gamma_R^{(n)}(p_1,\ldots)}{\Gamma_0^{(n)}(p_1,\ldots)} = \frac{G_R^{(n)}(p_1,\ldots)}{G_0^{(n)}(p_1,\ldots)} \frac{\prod_{i=1}^n \Delta(p_i)}{\prod_{i=1}^n \Delta_R(p_i)} = Z_{\phi}^{-n/2} Z_{\phi}^n = Z_{\phi}^{n/2} .$$
(25)

The quantity Z_{ϕ} is precisely the quantity Z, specialized to the ϕ^4 model, that we talked about earlier that relates (at asymptotic times) a field with full free-particle normalization to the fully interacting field that must also connect

to multiple particle states and thus only has part of the full free-particle normalization at asymptotic times.

To see this, let us recall the way we introduced Z before when discussing the reduction formalism. In fact, let us first briefly review the reduction formalism for the scalar field. Before doing that, we should talk about the S matrix a bit.

The *S* matrix

Formally, the S matrix is defined by

$$|out
angle = S^{-1}|in
angle = S^{\dagger}|in
angle, \text{ or } |in
angle = S|out
angle,$$
 (26)

and by the relation

$$\phi_{out} = S^{-1} \phi_{in} S \,. \tag{27}$$

Suppose that at time $t \to -\infty$ the system is in a definite state, the vacuum for instance, *i.e.* contains no physical particles. In general, the final state has some computable probability to contain zero, one, two, etc. particles. For example, in the case of photons (which are massless), the probability to remain the ground state as t goes from $-\infty$ to $+\infty$ is

$$\langle 0 \ out | 0 \ in \rangle = \langle 0 \ in | S | 0 \ in \rangle = \langle 0 \ out | S | 0 \ out \rangle$$
(28)

and is not just equal to a phase factor but actually has normalization

$$p_0 = \left| \langle 0 \ out | 0 \ in \rangle \right|^2 < 1 \tag{29}$$

If the theory is such that the there is a mass gap, then life is simpler and $p_0 = 1$ and the in and out vacuums are the same up to a phase, as are the 1-particle in and out states:

$$|0 \ in\rangle = |0 \ out\rangle = |0\rangle$$
 (we set a possible phase to 1), (30)
 $|1 \ in\rangle = |1 \ out\rangle$. (31)

Since $\langle 0|\phi(x)|1\rangle$ has the same dependence on x as the corresponding matrix elements of ϕ_{in} or ϕ_{out} (all being $\propto e^{-ip_1 \cdot x}$), the normalization of the latter being fixed by their free-field character, we necessarily have

$$\langle 0|\phi_0(x)|1\rangle = Z^{1/2}\langle 0|\phi_{in}(x)|1\rangle = Z^{1/2}\langle 0|\phi_{out}(x)|1\rangle.$$
 (32)

This provides a very precise definition of the famous Z factor. We will employ this definition in what follows in many different ways.

Because the vacuum and one-particle states are stable, it must be that S cannot change them. That is, the S matrix in this case satisfies

$$\langle 0|S|0\rangle = \langle 0|0\rangle = 1, \quad \langle 1|S|1'\rangle = \langle 1|1'\rangle$$
 (33)

where indices have been omitted from the one-particle states and we have only translated the stability of these states.

However, by the time we go to 2-particle and higher states (as necessarily considered in the scattering situation) or to unstable single-particle resonance states, these will intrinsically have knowledge of the $t \to \infty$ or $t \to -\infty$ boundary conditions through the fact that interactions can (and will, in general) take place.

The Reduction Formalism

Recall that we want to compute

$$_{out}\langle p_1, p_2, \dots | k_A, k_B \rangle_{in} \tag{34}$$

for a $2 \to n$ process, where the in and out states are eigenstates of the full H at times when we have highly separated wave packets. In such a distant

past or future, the operator for creating a one particle wave packet should be free-particle like. Thus, we would, for instance, like to write

$$|\vec{k}\rangle_{in} = a_{\vec{k}}^{in\,\dagger} |\Omega\rangle \tag{35}$$

where $|\Omega\rangle$ represents the fully interacting vacuum. Recall that the state created should be a free particle state but with the actually measured physical mass μ .

Next, we recall that for a normalization in which we write a free particle ϕ as

$$\phi_{non-interacting} = \sum_{\vec{k}} \frac{1}{\sqrt{2VE_{\vec{k}}}} \left[a_{\vec{k}} e^{-ik \cdot x} + a_{\vec{k}}^{\dagger} e^{+ik \cdot x} \right]$$
(36)

we have

$$a_{\vec{k}}^{\dagger} = -i \int \frac{d^{3}\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{-ik \cdot x} \overleftrightarrow{\partial_{x_{0}}}^{\leftrightarrow} \phi_{non-int}(x^{0}, \vec{x}) \right]$$

$$a_{\vec{k}} = +i \int \frac{d^{3}\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{ik \cdot x} \overleftrightarrow{\partial_{x_{0}}}^{\leftrightarrow} \phi_{non-int}(x^{0}, \vec{x}) \right], \qquad (37)$$

which would apply for either the in or the out states. In the distant past or distant future there is only a normalization difference between the fully interacting ϕ_0 and ϕ^{in} or ϕ^{out} . Thus, we can write

$$a_{\vec{k}}^{in\,\dagger}|\Omega
angle = |\vec{k}
angle_{in} = -iZ^{-1/2} \lim_{x^0 \to -\infty} \int \frac{d^3\vec{x}}{\sqrt{2VE_{\vec{k}}}} \left[e^{-ik \cdot x} \overleftrightarrow{\partial_{x_0}} \phi_0(x)\right] |\Omega
angle \,,$$
(38)

where $Z^{-1/2}$ is the renormalization factor necessary to boost the interacting ϕ_0 to free-particle-like normalization, see Eq. (32). Another way of saying this is that

$$\lim_{x^0 \to -\infty} \phi_0(x) = \sqrt{Z} \phi_{in}(x) \,. \tag{39}$$

Once again, Z is required since $\phi_0(x)|\Omega\rangle$ in the interacting theory $\Rightarrow 1$ particle, 2 particle, . . . states (i.e. it does much more than the ϕ_{in} operator which only makes 1 particle states). In fact though, Eq. (39) cannot really be true in a strict operator sense. The reason is that both ϕ_{in} and ϕ_0 are supposed to obey the same equal-time commutation relations

$$\begin{split} [\phi_{in}(x), \dot{\phi}_{in}(y)]_{[x^0=y^0]} &= i\delta^3(\vec{x}-\vec{y})\,, \quad [\phi_0(x), \dot{\phi}_0(y)]_{[x^0=y^0]} = i\delta^3(\vec{x}-\vec{y})\,, \end{split} \tag{40}$$

whereas the above relation would imply a factor of Z in front of the ϕ_0 commutation relation if the ϕ_{in} commutation relation holds. Fortunately, the

real requirement is not so strict. What we need is only that

$$\lim_{x^0 \to -\infty} \phi_0^-(x) | \dots \rangle_{in} = \sqrt{Z} \phi_{in}^-(x) | \dots \rangle_{in},$$
$$\lim_{x^0 \to +\infty} {}_{out} \langle \dots | \phi_0^+(x) = \sqrt{Z} {}_{out} \langle \dots | \phi_{out}^+(x), \qquad (41)$$

where the +, - superscripts are the positive and negative frequency components of the fields containing the $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$ operators respectively. This is fully consistent with Eq. (32). The above is all that is required since all we really need to do at very early or very late times is to be able to create appropriate stuff in the incoming state or outgoing state (the latter having been written in the conjugate form appropriate to an S matrix calculation). Since no statement is made above about the positive frequency components for ϕ_{in} or negative frequency components for ϕ_{out} , there is no contradiction with relations between commutators which require knowing how + and frequency components commute. We will return later to see that this \sqrt{Z} factor must be the same as that which appeared in our 1PI etc. sum to get the full propagator.

Now let us look at the reduction process. We have

$$_{out}\langle p_1,p_2,\ldots|k_A,k_B
angle_{in} = _{out}\langle p_1,p_2,\ldots|a_{ec k_A}^{in\,\dagger}|k_B
angle_{in}$$

$$=-iZ^{-1/2}\lim_{x_A^0\to-\infty}\int_{x_A^0}\frac{d^3\vec{x}_A}{\sqrt{2VE_{\vec{k}_A}}}\left[e^{-ik_A\cdot x_A}\partial_{x_A^0}^{\leftrightarrow}_{out}\langle p_1\dots|\phi_0(x_A)|k_B\rangle_{in}\right] (42)$$

At this point, we bring in the trivial identity

$$\left[\lim_{t \to +\infty} -\lim_{t \to -\infty}\right] \int d^3 \vec{x} \psi(\vec{x}, t) = \lim_{t_f \to \infty, t_i \to -\infty} \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \int d^3 \vec{x} \psi(\vec{x}, t) \,.$$
(43)

We use this to substitute in the above equation, obtaining (where the 0 in the 2nd line comes from assuming that $\vec{k}_A \neq \vec{p}_i$ for any *i*)

$$\begin{array}{lll} {}_{out}\langle p_1, p_2, \dots | k_A, k_B \rangle_{in} &= & {}_{out}\langle p_1, p_2, \dots | a_{\vec{k}_A}^{out \dagger} | k_B \rangle_{in} \\ & + i Z^{-1/2} \int \frac{d^4 x_A}{\sqrt{2VE_{\vec{k}_A}}} \partial_{x_A^0} \left[e^{-ik_A \cdot x_A} \partial_{x_A^0}^{\leftrightarrow} \left[e^{-ik_A \cdot x_A} \partial_{x_A^0}^{\leftrightarrow} \left[e^{-ik_A \cdot x_A} \partial_{x_A^0}^{2} \left[e^{-ik_A \cdot x_A} \partial_{x_A^0} \left[e^{-ik_A \cdot x_A} \left[e^{-ik_A \cdot x_A} \partial_{x_A^0} \left[e^{-ik_A \cdot x_A} \left[e^{-ik_A \cdot$$

$$= iZ^{-1/2} \int \frac{d^4x_A}{\sqrt{2VE_{\vec{k}_A}}} e^{-ik_A \cdot x_A} (\Box_{x_A} + \mu^2) \quad out \langle p_1 \dots | \phi_0(x_A) | k_B \rangle_{in}$$

$$\tag{44}$$

where to obtain the last line we noted that

$$\partial_{x_A^0}^2 e^{-ik_A \cdot x_A} = \left(\vec{\nabla}_{\vec{x}_A}^2 - \mu^2\right) e^{-ik_A \cdot x_A},\tag{45}$$

substituted it in, and partial integrated twice.

Now, we reduce in a 2nd particle, let us say the p_1 outgoing particle. We have

$$out \langle p_{1} \dots | \phi_{0}(x_{A}) | k_{B} \rangle_{in} = out \langle p_{2} \dots | a_{out}(\vec{p}_{1})\phi_{0}(x_{A}) | k_{B} \rangle_{in}$$

$$= \lim_{y_{1}^{0} \to \infty} iZ^{-1/2} \int \frac{d^{3}y_{1}}{\sqrt{2VE_{\vec{p}_{1}}}} \left[e^{ip_{1} \cdot y_{1}} \partial_{y_{1}^{0}}^{\leftrightarrow} out \langle p_{2} \dots | \phi_{0}(y_{1})\phi_{0}(x_{A}) | k_{B} \rangle_{in} \right]$$

$$= \lim_{y_{1}^{0} \to \infty} iZ^{-1/2} \int \frac{d^{3}y_{1}}{\sqrt{2VE_{\vec{p}_{1}}}} \left[e^{ip_{1} \cdot y_{1}} \partial_{y_{1}^{0}}^{\leftrightarrow} out \langle p_{2} \dots | T\{\phi_{0}(y_{1})\phi_{0}(x_{A})\} | k_{B} \rangle_{in} \right]$$

$$= \lim_{y_{1}^{0} \to -\infty} iZ^{-1/2} \int \frac{d^{3}y_{1}}{\sqrt{2VE_{\vec{p}_{1}}}} \left[e^{ip_{1} \cdot y_{1}} \partial_{y_{1}^{0}}^{\leftrightarrow} out \langle p_{2} \dots | T\{\phi_{0}(y_{1})\phi_{0}(x_{A})\} | k_{B} \rangle_{in} \right] + correction$$

$$= out \langle p_{2} \dots | \phi_{0}(x_{A})a_{in}(\vec{p}_{1}) | k_{B} \rangle_{in} + correction$$

$$= 0 + correction, \qquad (46)$$

where we assumed that $p_1 \neq k_B$. Writing out the correction piece, we thus have

$$out \langle p_{1} \dots | \phi_{0}(x_{A}) | k_{B} \rangle_{in} = iZ^{-1/2} \int \frac{d^{4}y_{1}}{\sqrt{2VE_{\vec{p}_{1}}}} \partial_{y_{1}^{0}} \left[e^{ip_{1} \cdot y_{1}} \partial_{y_{1}^{0}} out \langle p_{2} \dots | T\{\phi_{0}(y_{1})\phi_{0}(x_{A})\} | k_{B} \rangle_{in} \right]$$
$$= iZ^{-1/2} \int \frac{d^{4}y_{1}}{\sqrt{2VE_{\vec{p}_{1}}}} e^{ip_{1} \cdot y_{1}} (\Box_{y_{1}} + \mu^{2}) out \langle p_{2} \dots | T\{\phi_{0}(y_{1})\phi_{0}(x_{A})\} | k_{B} \rangle_{in}$$

where we did the same manipulations as in the previous case, i.e. employed $(\Box_{y_1} + \mu^2)e^{-ip_1 \cdot y_1} = 0$ to replace

$$\partial_{y_1^0}^2 e^{-ip_1 \cdot y_1} = (\vec{\nabla}_{\vec{y}_1}^2 - \mu^2) e^{-ip_1 \cdot y_1} \tag{48}$$

and then partial integrated twice in \vec{y}_1 .

So, we would now insert this expression into Eq. (44). And, we would continue on to reduce in all the initial and final particles. I hope you can see that the result will be (for n final particles)

$$egin{aligned} {}_{out}\langle p_1,p_2,\dots|k_A,k_B
angle_{in}&=&[iZ^{-1/2}]^{n+2}\prod_{i=1}^{n+2}rac{1}{\sqrt{2VE_i}}\int d^4y_1\dots d^4y_n d^4x_A d^4x_B\ & imes\exp\left[i\sum_{k=1}^n p_k\cdot y_k-ik_A\cdot x_A-ik_B\cdot x_B
ight] \end{aligned}$$

 $\times (\Box_{\boldsymbol{y}_1} + \boldsymbol{\mu}^2) \dots (\Box_{\boldsymbol{x}_B} + \boldsymbol{\mu}^2) \langle \Omega | \boldsymbol{T} \{ \phi_0(\boldsymbol{y}_1) \dots \phi_0(\boldsymbol{x}_B) \} | \Omega \rangle .$ (49)

Before going back to renormalization, let's just check that we get the correct Feynman rule. For the basic connected vertex graph, our Feynman rules give (we set Z = 1 for this tree level case)

$$\langle \Omega | T \{ \phi_0(x_1) \dots \phi_0(x_4) \} | \Omega \rangle \equiv G_0^{(4)}(x_1, x_2, x_3, x_4)$$

$$= \left[\prod_i \int \frac{d^4 l_i}{(2\pi)^4} \frac{i}{l_i^2 - \mu^2 + i\epsilon} \right] (-i\lambda)(2\pi)^4 \delta^4(\sum_j l_j) e^{-i\sum_k l_k \cdot x_k}.$$
(50)

We now insert this into the above expression for the S matrix, but using the convention where all particles are outgoing. We find (all indexed sums and products go from 1 to 4):

$$\begin{split} out \langle p_1, p_2, p_3, p_4 | 0 \rangle_{in} &= [i]^4 \left[\prod_m \frac{1}{\sqrt{2VE_m}} \right] \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 e^{i \sum p_k \cdot x_k} \\ &\times (\Box_{x_1} + \mu^2) \dots (\Box_{x_4} + \mu^2) G_0^{(4)}(x_1, x_2, x_3, x_4) \\ &= [i]^4 \left[\prod_m \frac{1}{\sqrt{2VE_m}} \right] \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 e^{i \sum p_k \cdot x_k} \\ &\times \left[\prod_i \int \frac{d^4 l_i}{(2\pi)^4} (-l_i^2 + \mu^2) \frac{i}{l_i^2 - \mu^2 + i\epsilon} \right] (-i\lambda) (2\pi)^4 \delta^4 (\sum_j l_j) e^{-i \sum_k l_k \cdot x_k} \end{split}$$

$$= \left[\prod_{m} \frac{1}{\sqrt{2VE_{m}}}\right] \left[\prod_{i} \int \frac{d^{4}l_{i}}{(2\pi)^{4}} (-i)(2\pi)^{4} \delta^{4}(p_{i}-l_{i})\right] (-i\lambda)(2\pi)^{4} \delta^{4}(\sum_{j} l_{j})$$

$$= \left[\prod_{m} \frac{1}{\sqrt{2VE_{m}}}\right] (-i\lambda)(2\pi)^{4} \delta^{4}(\sum_{j} p_{j}).$$
(51)

As always, we remove the $\frac{1}{\sqrt{2VE}}$ factors and the $(2\pi)^4 \delta^4(\ldots)$ to define $i\mathcal{M}$ and obtain at tree-level

$$\mathcal{M} = -\lambda \,. \tag{52}$$

Let us generalize this a bit. We will write

$$G_0(x_1, \dots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} \exp[i \sum p_j \cdot x_j] \widetilde{G}_0(p_1, \dots) .$$
 (53)

Translational invariance implies that

$$\widetilde{G}_{0}(p_{1},...) = (2\pi)^{4} \delta^{4}(p_{1}+...+p_{n}) G_{0}(p_{1},...)$$
$$= \int d^{4}x_{1}...d^{4}x_{n} \exp[-i\sum p_{k}\cdot x_{k}] G_{0}(x_{1},...) \quad (54)$$

where the 2nd line gives the inverse Fourier transform.

Thus, the *S* matrix structure in the general case looks like (dropping the $\frac{1}{\sqrt{2EV}}$ factors that get dropped when we relate $_{out}\langle p_1, \dots p_n | \Omega \rangle_{in}$ to \mathcal{M})

$$\begin{aligned} out \langle p_{1}, \dots p_{n} | \Omega \rangle_{in} &= (iZ^{-1/2})^{n} \int d^{4}x_{1} \dots d^{4}x_{n} \exp[i\sum_{k} p_{k} \cdot x_{k}] (\Box_{x_{1}} + \mu^{2}) \dots (\Box_{x_{n}} + \mu^{2}) G_{0}(x_{1} \dots x_{n}) \\ &= (iZ^{-1/2})^{n} \int d^{4}x_{1} \dots d^{4}x_{n} \exp[i\sum_{k} p_{k} \cdot x_{k}] (\Box_{x_{1}} + \mu^{2}) \dots (\Box_{x_{n}} + \mu^{2}) \\ &\times \int \frac{d^{4}\bar{p}_{1}}{(2\pi)^{4}} \dots \exp[i\sum_{\bar{p}_{i}} \bar{p}_{i} \cdot x_{i}] (2\pi)^{4} \delta^{4}(\bar{p}_{1} + \dots + \bar{p}_{n}) G_{0}(\bar{p}_{1} \dots \bar{p}_{n}) \\ &= (iZ^{-1/2})^{n} \int d^{4}\bar{p}_{1} \dots d^{4}\bar{p}_{n} \delta^{4}(p_{1} + \bar{p}_{1}) \dots \delta^{4}(p_{n} + \bar{p}_{n}) (\mu^{2} - \bar{p}_{1}^{2}) \dots (\mu^{2} - \bar{p}_{n}^{2}) \\ &\times (2\pi)^{4} \delta^{4}(\bar{p}_{1} + \dots + \bar{p}_{n}) G_{0}(\bar{p}_{1}, \dots \bar{p}_{n}) \\ &= (iZ^{-1/2})^{n} \int d^{4}\bar{p}_{1} \dots d^{4}\bar{p}_{n} \delta^{4}(p_{1} + \bar{p}_{1}) \dots \delta^{4}(p_{n} + \bar{p}_{n}) (\mu^{2} - \bar{p}_{1}^{2}) \dots (\mu^{2} - \bar{p}_{n}^{2}) \\ &\times (2\pi)^{4} \delta^{4}(\bar{p}_{1} + \dots + \bar{p}_{n}) \frac{iZ}{\bar{p}_{1}^{2} - \mu^{2} - \tilde{\Sigma}(\bar{p}_{1}^{2})} \dots \frac{iZ}{\bar{p}_{n}^{2} - \mu^{2} - \tilde{\Sigma}(\bar{p}_{n}^{2})} \Gamma_{0}(\bar{p}_{1}, \dots \bar{p}_{n}) \end{aligned}$$

$$\tag{55}$$

where we wrote

$$G_0(\bar{p}_1\ldots) = \left[\prod_i i\Delta(\bar{p}_i^2)\right]\Gamma_0$$
(56)

with the $i\Delta(\bar{p}_i^2)$ factors coming from the full insertion set on each of the

external lines (see figure).



Figure 2: Graphical representation of Eq. (56).

Now, as we go on shell (which is what happens since $\bar{p}_i = -p_i$ via the δ^4 functions and since $p_i^2 = \mu^2$) we have

$$(\mu^2 - \bar{p}_i^2) \frac{iZ}{\bar{p}_i^2 - \mu^2 - \widetilde{\Sigma}(\bar{p}_i^2)} \to -iZ$$
(57)

for each i, leaving us with

$$out \langle p_1, \dots, p_n | \Omega \rangle_{in} = (iZ^{-1/2})^n (-iZ)^n (2\pi)^4 \delta^4 (p_1 + \dots + p_n) \Gamma_0 (p_1 \dots)$$

= $(Z^{1/2})^n (2\pi)^4 \delta^4 (p_1 + \dots + p_n) \Gamma_0 (p_1 \dots)$
= $(2\pi)^4 \delta^4 (p_1 + \dots + p_n) \Gamma_R (p_1 \dots)$, (58)

where we used Eq. (25).

Thus, you see that the S matrix elements are given in terms of the renormalized "amputated" momentum space n-particle interaction. This is the key result we need. We need only show now that Γ_R can be written as a finite expression in terms of a single (finite) measurement of the 4-point interaction at some conveniently chosen momentum setup.

The Z's

But first, we want to return to the claim that the Z_{ϕ} that appears in

$$i\Delta(p^2) = \frac{iZ_{\phi}}{p^2 - \mu^2 - \widetilde{\Sigma}(p^2) + i\epsilon}$$
(59)

is the same Z as that which appeared in our early and late time limits of the

 ϕ_0 field versus the $\phi_{in,out}$ fields. Obviously, the cancellation of all the Z's in the end will only occur if this is the case.

Without getting too formal about it (for more rigor, see later), that these two Z's are the same is certainly intuitively obvious. We wrote $Z^{-1/2}\phi_0^-|in\rangle \rightarrow \phi_{in}^-|in\rangle$ at very early times and $Z^{-1/2}\langle out|\phi_0^+ \rightarrow \langle out|\phi_{out}^+$ at very late times. Here, ϕ_0 is the fully interacting field while the out and in fields are those that are free particle like with full one-particle normalization at very early or very late times. It is the latter that are related to the a^{out} and a^{in} operators that define the S-matrix.

Consider the propagator for $p^2 \rightarrow \mu^2$. For the single particle propagator, the distinction between in and out states is not relevant since interactions cannot change the state in this limit (if there is a mass gap). We have

$$\int d^4(x-y)e^{-ip\cdot(x-y)}\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle = \frac{iZ_{\phi}}{p^2-\mu^2}.$$
 (60)

Intuitively, as $p^2 \to \mu^2$ what is happening is that the particle is propagating from $y^0 = -\infty$ to $x^0 = +\infty$ and it is that part of the integration above that will be dominant. Focusing on the $\langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle$ integrand, let us consider $x^0
ightarrow +\infty$ and $y^0
ightarrow -\infty$. Then,

$$\lim_{x^{0} \to \infty, y^{0} \to -\infty} \langle 0 | T \{ \phi_{0}(x) \phi_{0}(y) \} | 0 \rangle$$

$$= \lim_{x^{0} \to \infty, y^{0} \to -\infty} \langle 0 | \phi_{0}(x) \phi_{0}(y) | 0 \rangle$$

$$= \lim_{x^{0} \to \infty, y^{0} \to -\infty} \langle 0 | \phi_{0}^{+}(x) \phi_{0}^{-}(y) | 0 \rangle$$

$$= \langle 0 | Z^{1/2} \phi_{out}^{+}(x) Z^{1/2} \phi_{in}^{-}(y) | 0 \rangle$$

$$= Z \langle 0 | \phi_{out}^{+}(x) \phi_{in}^{-}(y) | 0 \rangle. \qquad (61)$$

Remembering that I insisted at the time that the in and out fields be freeparticle like, but with physical mass μ , the resulting contribution to the propagator from $\langle 0 | \phi_{out}^+ \phi_{in}^- | 0 \rangle$ will come only from 1-particle intermediate states (here we are again assuming the mass gap situation) and have unit normalization, yielding a propagator after Fourier transforming that is $\frac{i}{p^2 - \mu^2}$ as $p^2 \rightarrow \mu^2$. Inserting into Eq. (60) we obtain $Z = Z_{\phi}$. That is, the factor that appears in the asymptotic time relations must be the same as the Z_{ϕ} which appears in the propagator. Note that it is important in making the above switch to the Fourier transform to emphasize that the limit of $p^2 \rightarrow \mu^2$ means that we are really isolating just the free-particle-like component of the interacting field which connects a state very early in time to one very late in time. This means that even though I wrote things in the $\lim_{x^0\to\infty,y^0\to-\infty}$, this was not actually necessary in the $p^2 \to \mu^2$ limit. Further, there is no difference between ϕ_{in} and ϕ_{out} in the $p^2 \to \mu^2$ limit. As we discussed earlier, physically, all the above statements are equivalent to the statements that in the on-shell one particle and no particle environments:

- one-particle states are stable, i.e. $|1\rangle_{in} = |1\rangle_{out}$;
- the vacuum is unique, $|0\rangle_{in} = |0\rangle_{out}$ (up to a trivial phase that is taken to be 1 by convention).

Thus, the full Fourier transform in this limit will have the Z factor relative to the canonical free-particle $i/(p^2 - \mu^2)$ form. We now proceed to a more formal derivation of $Z = Z_{\phi}$.

Still more on Z: the spectral representation

We have already noted earlier that from the 2nd quantization expansion of ϕ_{in} and ϕ_0 , it is clear that $\langle 1|\phi_{in}|\Omega\rangle$ and $\langle 1|\phi_0|\Omega\rangle$ have the same functional dependence on x. The normalization factor $Z^{1/2}$ takes into account the fact that the content of the state $\phi_0(x)|\Omega\rangle$ is not exhausted by the matrix

elements $\langle 1|\phi_0|\Omega\rangle$ whereas the state $\phi_{in}(x)|\Omega\rangle$ is exhausted by $\langle 1|\phi_{in}|\Omega\rangle$. Once again, this intuitively means that $Z^{1/2}$ should obey $0 \leq Z^{1/2} < 1$.

For a real field, using translation invariance (i.e. $\phi(x) = e^{iP \cdot x}\phi(0)e^{-iP \cdot x}$), $P|\Omega\rangle = \langle \Omega|P = 0, P|\alpha\rangle = p_{\alpha}|\alpha\rangle$, etc., we can write

$$\begin{split} \langle \Omega | [\phi_0(x), \phi_0(y)] | \Omega \rangle &= \sum_{\alpha} [\langle \Omega | \phi_0(0) | \alpha \rangle e^{-ip_{\alpha} \cdot (x-y)} \langle \alpha | \phi_0(0) | \Omega \rangle - (x \leftrightarrow y)] \\ &= \sum_{\alpha} |\langle \Omega | \phi_0(0) | \alpha \rangle|^2 \left[e^{-ip_{\alpha} \cdot (x-y)} - e^{+ip_{\alpha} \cdot (x-y)} \right], \end{split}$$

where the sum runs over a complete set of positive energy states α . To compare this with the commutator of two free fields of mass μ ,

$$\Delta_{0}(x-y,\mu) \equiv \left[\phi_{0}^{free}(x),\phi_{0}^{free}(y)\right] = \int \frac{d^{3}l}{(2\pi)^{3}} \frac{1}{2E_{\vec{l}}} \left[e^{-il\cdot(x-y)} - e^{il\cdot(x-y)}\right]$$
$$= \int \frac{d^{4}l}{(2\pi)^{3}} \delta(l^{2} - \mu^{2}) \theta(l_{0}) \left[e^{-il\cdot(x-y)} - e^{+il\cdot(x-y)}\right], \qquad (63)$$

we insert into Eq. (62) the identity

$$1 = \int d^4 l \delta^4 (l - p_\alpha) \tag{64}$$

leading to

$$\langle \Omega | [\phi_0(x), \phi_0(y)] | \Omega \rangle = \int \frac{d^4l}{(2\pi)^3} \rho(l) \left(e^{-il \cdot (x-y)} - e^{+il \cdot (x-y)} \right)$$
(65)

with

$$ho(l) = (2\pi)^3 \sum_{lpha} \delta^4(l-p_{lpha}) \left| \langle \Omega | \phi_0(0) | lpha
angle \right|^2 \,.$$
 (66)

Clearly, $\rho(l) > 0$ and vanishes when l is not in the forward light cone, since all the physical states α have positive energy, $p_{\alpha}^{0} > 0$. Further, it is invariant under a Lorentz transformation, as required by the corresponding property of the field ϕ_{0} (which is to say that $\phi_{0}(x)$ is a scalar field and obeys $\phi'_{0}(x') = \phi_{0}(x)$, where the primes are after a Lorentz transform). Since l is the only available 4-vector, we thus have

$$\rho(l) = \sigma(l^2)\theta(l^0), \quad \text{with } \sigma(l^2) = 0 \text{ if } l^2 < 0.$$
(67)

In general, this is a positive measure where δ -function singularities might possibly occur. We shall also use the identity

$$\sigma(l^2) = \int d\mu'^2 \sigma(\mu'^2) \delta(l^2 - \mu'^2) \,. \tag{68}$$

Using the above, we have

$$\begin{split} &\langle \Omega | [\phi_{0}(x), \phi_{0}(y)] | \Omega \rangle \\ &= \int \frac{d^{4}l}{(2\pi)^{3}} \rho(l) \left(e^{-il \cdot (x-y)} - e^{+il \cdot (x-y)} \right) \\ &= \int \frac{d^{4}l}{(2\pi)^{3}} \sigma(l^{2}) \theta(l^{0}) \left(e^{-il \cdot (x-y)} - e^{+il \cdot (x-y)} \right) \\ &= \int d\mu'^{2} \sigma(\mu'^{2}) \int \frac{d^{4}l}{(2\pi)^{3}} \delta(l^{2} - \mu'^{2}) \theta(l^{0}) \left(e^{-il \cdot (x-y)} - e^{+il \cdot (x-y)} \right) \\ &= \int d\mu'^{2} \sigma(\mu'^{2}) \Delta_{0}(x-y;\mu') \end{split}$$
(69)

where we used the analogue of Eq. (63):

$$\Delta_0(x-y;\mu') = \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - {\mu'}^2) \theta(l^0) \left(e^{-il \cdot (x-y)} - e^{+il \cdot (x-y)} \right) (70)$$

where, to repeat, $\Delta_0(x - y; \mu')$ is our old way of writing the free-field commutator. We now wish to separate out the 1-particle state component

using

$$\langle 1|\phi_0|\Omega\rangle = Z^{1/2}\langle 1|\phi^{free}|\Omega\rangle = Z^{1/2}\langle 1|\phi^{in}|\Omega\rangle = Z^{1/2}\langle 1|\phi^{out}|\Omega\rangle.$$
(71)

For this, it is best to convert from finite volume normalization, for which $\sum_{\alpha}^{1 particle} = \sum_{\vec{k}}$, to continuum $V \to \infty$ normalization using our old friend

$$\sum_{\vec{k}} \to \frac{d^3 \vec{k} V}{(2\pi)^3} \tag{72}$$

and recall that for

$$\phi_0^{free} = \sum_{\vec{q}} \frac{1}{\sqrt{2VE_{\vec{q}}}} (a_{\vec{q}}^{free} e^{-iq \cdot x} + a_{\vec{q}}^{\dagger free} e^{iq \cdot x})$$
(73)

and one particle states (with full normalization) defined by $|\vec{k}\rangle = a_{\vec{k}}^{\dagger free} |\Omega\rangle$, we have

$$\langle \Omega | \phi_0^{free} | \vec{k} \rangle = \frac{1}{\sqrt{2VE_{\vec{k}}}} e^{-ik \cdot x} \,, \tag{74}$$

and, hence,

$$\langle \Omega | \phi_0^{interacting} | \vec{k} \rangle = Z^{1/2} \frac{1}{\sqrt{2VE_{\vec{k}}}} e^{-ik \cdot x} , \qquad (75)$$

where $E_{\vec{k}}$ is computed using μ^2 . As a result, the one-particle content of ρ is computed as

$$egin{aligned} &
ho(l) &= & (2\pi)^3 \sum_lpha \delta^4 (l-p_lpha) \left| \langle \Omega | \phi_0(0) | lpha
angle
ight|^2 \ &
onumber &
onumber \ &
onumber \ & (2\pi)^3 \sum_{ec k} \delta^4 (l-k) Z rac{1}{2V E_{ec k}} \ &
onumber \ &
on$$

(76)
from which we read off that

$$\sigma^{1 \ particle}(l^2) = Z\delta(l^2 - \mu^2), \qquad (77)$$

given that $\rho(l) \equiv \sigma(l^2)\theta(l^0)$.

We now use Eq. (69) to conclude that

$$\langle \Omega | [\phi_0(x), \phi_0(y)] | \Omega \rangle = Z \Delta_0(x - y; \mu) + \int_{\mu_1^2} d\mu'^2 \sigma(\mu'^2) \Delta_0(x - y; \mu'),$$
 (78)

where $\mu_1 > \mu$ is the threshold for multiparticle states (= 3μ in the ϕ^4 theory context — can't go from 1 to 2 particle state in ϕ^4). Now, assuming that the interaction Lagrangian does not involve field derivatives, $\dot{\phi}_0$ will be conjugate to ϕ_0 . By taking the time derivative in x^0 of both sides of the above equation and then taking the $x_0 = y_0$ equal time limit, if we identify the coefficients of $i\delta^3(\vec{x} - \vec{y})$, we find

$$1 = Z + \int_{\mu_1^2} d\mu'^2 \sigma(\mu'^2) , \qquad (79)$$

so that $\sigma > 0$ implies $0 \le Z \le 1$.

For the time-ordered product, we follow a very similar procedure. We start with

$$\langle \Omega | \phi_0(x) \phi_0(y) | \Omega \rangle = \sum_{\alpha} \langle \Omega | \phi_0(0) | \alpha \rangle e^{-ip_{\alpha} \cdot (x-y)} \langle \alpha | \phi_0(0) | \Omega \rangle ,$$
 (80)

and introduce

$$1 = \int d^4 l \delta^4 (l - p_\alpha) \tag{81}$$

leading to

$$\langle \Omega | \phi_0(x) \phi_0(y) | \Omega \rangle = \int \frac{d^4 l}{(2\pi)^3} \rho(l) e^{-il \cdot (x-y)}$$
 (82)

with $\rho(l)$ defined as before. Using this, we can write

$$\begin{split} \theta(x^0 - y^0) \langle \Omega | \phi_0(x) \phi_0(y) | \Omega \rangle &= \theta(x^0 - y^0) \int \frac{d^4l}{(2\pi)^3} \sigma(l^2) \theta(l^0) e^{-il \cdot (x-y)} \\ &= \theta(x^0 - y^0) \int d\mu'^2 \int \frac{d^4l}{(2\pi)^3} \delta(l^2 - \mu'^2) \theta(l^0) e^{-il \cdot (x-y)} \sigma(\mu'^2) \\ &= \theta(x^0 - y^0) \int d\mu'^2 \int \frac{d^3l}{(2\pi)^3} \frac{1}{2E_{\vec{l}}} e^{-il \cdot (x-y)} \sigma(\mu'^2) \,. \\ &= \theta(x^0 - y^0) \int d\mu'^2 \int \frac{d^4l}{(2\pi)^4} \frac{i}{l^2 - \mu'^2 + i\epsilon} e^{-il \cdot (x-y)} \sigma(\mu'^2) \,. \end{split}$$

(To derive the above, you need to recall that $\frac{i}{l^2 - \mu'^2 + i\epsilon} = \frac{i}{(l^0 - E_{\vec{l}} + i\epsilon)(l^0 + E_{\vec{l}} - i\epsilon)}$, where $E_{\vec{l}} = \sqrt{\vec{l}^2 + \mu'^2}$, and that for $x^0 > y^0$, the dl^0 contour is closed down and is clockwise, picking up the $l^0 = +E_{\vec{l}}$ pole in the lower 1/2 plane yielding a factor of $(-)\frac{2\pi i}{2E_{\vec{l}}}i = \frac{2\pi}{2E_{\vec{l}}}$.)

Similarly, if $y^0 > x^0$ we have

$$\theta(y^{0} - x^{0}) \langle \Omega | \phi_{0}(y) \phi_{0}(x) | \Omega \rangle = \theta(y^{0} - x^{0}) \int d\mu'^{2} \int \frac{d^{4}l}{(2\pi)^{4}} \frac{i}{l^{2} - \mu'^{2} + i\epsilon} e^{-il \cdot (x-y)} \sigma(\mu'^{2}).$$
(84)

In this case, the dl^0 contour is closed up and one picks up the pole at $-E_{\vec{l}}$ and we obtain factor of $(+)\frac{2\pi i}{-2E_{\vec{l}}}i=\frac{2\pi}{2E_{\vec{l}}}$, *i.e.* the same as for $x^0 > y^0$.

Combining these two results using $heta(x^0-y^0)+ heta(y^0-x^0)=1$ we find

$$\langle \Omega | T\{\phi_0(x)\phi_0(y)\} | \Omega \rangle = \int d\mu'^2 \Delta_F(x-y,\mu')\sigma(\mu'^2),$$
 (85)

where Δ_F is the usual Feynman propagator. We now take the Fourier

(83)

transform of this result

$$\int d^{4}x e^{-ip \cdot x} \langle \Omega | T\{\phi_{0}(x)\phi_{0}(0)\} | \Omega \rangle = \int d\mu'^{2} \frac{i}{p^{2} - \mu'^{2} + i\epsilon} \sigma(\mu'^{2})$$

$$= \frac{iZ}{p^{2} - \mu^{2} + i\epsilon} + \int_{9\mu^{2}} d\mu'^{2} \frac{i}{p^{2} - \mu'^{2} + i\epsilon} \sigma(\mu'^{2})$$

$$= \frac{iZ_{\phi}}{p^{2} - \mu^{2} - \widetilde{\Sigma}(p^{2}) + i\epsilon}$$
(86)

where for the 2nd equality we isolated the 1-particle portion of σ , which explicitly contains the Z factor defined by asymptotic state limits, and the 3rd equality is simply our definition of Z_{ϕ} as the pole residue for the interacting field propagator. Now, as $p^2 \rightarrow \mu^2$, the $\int_{9\mu^2}$ term above has no singularity and $\tilde{\Sigma}(p^2) \propto (p^2 - \mu^2)^2$ (by its definition) and so we conclude that $Z_{\phi} = Z$.

Coupling Constant Renormalization

We consider the 1PI 4-point function at one-loop. The expression for it is

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u).$$
(87)

Noting that $s + t + u = 4\mu^2$ we choose our "experimental" input observation

as that made at the (unphysical, but symmetric)¹ point

$$s_0 = t_0 = u_0 = \frac{4\mu^2}{3}.$$
 (88)

We define the renormalized coupling constant (the thing we measure) by

$$\Gamma_R^{(4)}(s_0, t_0, u_0) \equiv -i\lambda.$$
(89)

We next write

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + 3\Gamma(s_0) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u), \qquad (90)$$

where $\Gamma(s) \equiv \Gamma(s) - \Gamma(s_0) \rightarrow 0$ when $s = s_0$. Now define Z_{λ} through the equation

$$-iZ_{\lambda}^{-1}\lambda_0 \equiv -i\lambda_0 + 3\Gamma(s_0) \tag{91}$$

so that

$$\Gamma_0^{(4)}(s,t,u) = -iZ_{\lambda}^{-1}\lambda_0 + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)$$
(92)

¹We could choose a truly physical point but that is less convenient.

and

$$\Gamma_0^{(4)}(s_0, t_0, u_0) = -i Z_{\lambda}^{-1} \lambda_0.$$
(93)

 Z_{λ} carries the infinities.

Now, we use the relation

$$\Gamma_R^{(4)}(s,t,u) = Z_{\phi}^2 \Gamma_0^{(4)}(s,t,u)$$
(94)

discussed earlier and we define the physically observable coupling λ by $\Gamma_R^{(4)}(s_0, t_0, u_0) \equiv -i\lambda$. Then, the above equations yield

$$\lambda = Z_{\phi}^2 Z_{\lambda}^{-1} \lambda_0 \,. \tag{95}$$

Note that $-i\lambda$ is what would actually be measured at the symmetric point according to Eq. (58) (obtained via the reduction formalism) since $-i\lambda$ is directly related to $\Gamma_R^{(4)}$ and $\Gamma_R^{(4)}$ directly gives the scattering matrix element. To repeat Eq. (58) for the 4-point case:

$$_{out}\langle p_1, p_2, p_3, p_4 | 0 \rangle_{in} = (2\pi)^4 \delta^4 (p_1 + p_2 + p_3 + p_4) \Gamma_R(p_1, p_2, p_3, p_4).$$
 (96)

We can now show that the renormalized 1PI four-point function for any kinematic configuration will be finite up to order λ^2 when written in terms of

λ . This follows from

$$\Gamma_{R}^{(4)}(p_{1}, \dots p_{4}) = Z_{\phi}^{2}\Gamma_{0}^{(4)}(p_{1}\dots)$$

$$= -iZ_{\lambda}^{-1}Z_{\phi}^{2}\lambda_{0} + Z_{\phi}^{2}\left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right]$$

$$= -i\lambda + Z_{\phi}^{2}\left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right]$$

$$\simeq -i\lambda + \left[\widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u)\right] + \mathcal{O}\left(\lambda^{3}\right) \quad (97)$$

given that $Z_{\phi} = 1 + \mathcal{O}(\lambda_0)$ and $\widetilde{\Gamma} = \mathcal{O}(\lambda_0^2)$. Clearly, to the order considered, the final expression for $\Gamma_R^{(4)}$ above is completely finite when expressed as a function of the renormalized (i.e. measured) coupling λ .

Let us look at $G^{(4)}$ in very explicit fashion using this same approach. To order λ^2 , we need to consider the bubble insertions on external propagators as well as the 1PI fish diagrams. There are 4 such bubble insertions. We can write

$$G_0^{(4)}(p_1,\ldots p_4) = \prod_j \left(rac{i}{p_j^2-\mu_0^2+i\epsilon}
ight) \left[-i\lambda_0+3\Gamma(s_0)+\widetilde{\Gamma}(s)+\widetilde{\Gamma}(t)+\widetilde{\Gamma}(u)
ight]$$

$$+(-i\lambda_{0})\sum_{k}[-i\Sigma(p_{k}^{2})]\frac{i}{p_{k}^{2}-\mu_{0}^{2}+i\epsilon}$$
(98)

The first and last terms can be combined to order λ_0^2 in the form:

$$(-i\lambda_0)\left[\prod_j \left(\frac{i}{p_j^2 - \mu_0^2 + i\epsilon}\right)\right]\left[1 + \sum_k \Sigma(p_k^2) \frac{1}{p_k^2 - \mu_0^2 + i\epsilon}\right] = (-i\lambda_0)\left[\prod_j \left(\frac{i}{p_j^2 - \mu_0^2 - \Sigma(p_j^2) + i\epsilon}\right)\right] + \mathcal{O}(\lambda_0^3) \,.$$

Since $\Gamma \sim \mathcal{O}(\lambda_0^2)$, $\widetilde{\Gamma}_0 \sim \mathcal{O}(\lambda_0^2)$, and $\Sigma \sim \mathcal{O}(\lambda_0)$, we can also write to this order

$$\left[\prod_{j}\frac{i}{p_{j}^{2}-\mu_{0}^{2}+i\epsilon}\right]\left[3\Gamma(s_{0})+\widetilde{\Gamma}(s)+\ldots\right]=\left[\prod_{j}\frac{i}{p_{j}^{2}-\mu_{0}^{2}-\Sigma(p_{j}^{2})+i\epsilon}\right]\left[3\Gamma(s_{0})+\widetilde{\Gamma}(s)+\ldots\right]+\mathcal{O}(\lambda_{0}^{3})(100)$$

Altogether, we get

$$G_{0}^{(4)}(p_{1} \dots p_{4}) = \prod_{j} \left[\frac{i}{p_{j}^{2} - \mu_{0}^{2} - \Sigma(p_{j}^{2}) + i\epsilon} \right] \left[-i\lambda_{0} + 3\Gamma(s_{0}) + \widetilde{\Gamma}(s) + \widetilde{\Gamma}(t) + \widetilde{\Gamma}(u) \right]$$
$$= \left[\prod_{j} i\Delta(p_{j}^{2}) \right] \Gamma_{0}^{(4)}(p_{1} \dots p_{4}).$$
(101)

Recalling that

$$G_R^{(4)} = Z_{\phi}^{-2} G_0^{(4)}, \quad \Delta_R = Z_{\phi}^{-1} \Delta, \quad \Gamma_R^{(4)} = Z_{\phi}^2 \Gamma_0^{(4)}$$
(102)

we then find from the above equation that

$$G_{R}^{(4)} = Z_{\phi}^{-2} \left[Z_{\phi}^{4} \prod_{j} i \Delta_{R}(p_{j}^{2}) \right] Z_{\phi}^{-2} \Gamma_{R}^{(4)}$$

$$= \prod_{j} \left[i \Delta_{R}(p_{j}^{2}) \right] \Gamma_{R}^{(4)}, \qquad (103)$$

which is finite since Δ_R and $\Gamma_R^{(4)}$ are. Altogether, if we define

$$\phi = Z_{\phi}^{-1/2} \phi_0$$
, $\lambda = Z_{\lambda}^{-1} Z_{\phi}^2 \lambda_0$, $\mu^2 = \mu_0^2 + \delta \mu^2$ (with $\delta \mu^2 = \Sigma(\mu^2)$), (104)
then the renormalized quantities are finite expressions as functions of μ^2 and λ . This feature, namely that all divergences, after rewriting everything in terms of λ and μ , can be absorbed, is the hallmark of a renormalizable theory. And,

t

λ

as we have seen, the S matrix scattering amplitudes are given in terms of the renormalized Green's functions, more precisely the Γ_R for any given process.

BPH Renormalization

BPH=Bogoliubov, Parasink and Hepp (+Zimmerman)

The alternative (but equivalent) approach of BPH renormalization is often more convenient.

First, let us rephrase our result just obtained. We begin once again with

$$\mathcal{L}_{0} = \frac{1}{2} [(\partial_{\mu}\phi_{0})(\partial^{\mu}\phi_{0}) - \mu_{0}^{2}\phi_{0}^{2}] - \frac{\lambda_{0}}{4!}\phi_{0}^{4}.$$
(105)

(here, the 0 subscript refers to before renormalization rather than to just the free particle part of \mathcal{L}). If we substitute in terms of the renormalized field and mass and coupling constant, this becomes

$$\mathcal{L}_{0} = \frac{1}{2} Z_{\phi} \left[\partial_{\mu} \phi \partial^{\mu} \phi - (\mu^{2} - \delta \mu^{2}) \phi^{2} \right] - \frac{\left(\frac{Z_{\lambda} \lambda}{Z_{\phi}^{2}} \right)}{4!} Z_{\phi}^{2} \phi^{4}$$
$$= \frac{1}{2} Z_{\phi} \left[\partial_{\mu} \phi \partial^{\mu} \phi - (\mu^{2} - \delta \mu^{2}) \phi^{2} \right] - \frac{\lambda Z_{\lambda}}{4!} \phi^{4} .$$
(106)

We rewrite this in the form

$$\mathcal{L}_0 = \mathcal{L} + \Delta \mathcal{L} \tag{107}$$

where

$$\mathcal{L} = \frac{1}{2} \left[\partial^{\mu} \phi \partial_{\mu} \phi - \mu^{2} \phi^{2} \right] - \frac{\lambda}{4!} \phi^{4}$$

$$\Delta \mathcal{L} = (Z_{\phi} - 1) \frac{1}{2} \left[\partial^{\mu} \phi \partial_{\mu} \phi - \mu^{2} \phi^{2} \right] + Z_{\phi} \frac{1}{2} \delta \mu^{2} \phi^{2} - \frac{\lambda (Z_{\lambda} - 1)}{4!} \phi^{4} 108)$$

where

- *L* is the renormalized Lagrangian density
- $\Delta \mathcal{L}$ is the counter term Lagrangian and is explicitly of order $\lambda \mathcal{L}$.

The BPH scheme is the following:

- 1. Start with \mathcal{L} and compute propagators and vertices.
- 2. Isolate the divergent 1PI diagrams at order λ (i.e. at one loop).

- **3.** Choose $\Delta \mathcal{L}^{(1)}$ to cancel these divergences.
- 4. Use $\mathcal{L}^{(1)} = \mathcal{L} + \Delta \mathcal{L}^{(1)}$ to generate 2-loop diagrams.
- 5. Isolate the infinities of the 2-loop diagrams.
- 6. Choose $\Delta \mathcal{L}^{(2)}$ (order λ^3) to cancel these infinities.
- 7. and so forth.

Note that to any given order in λ , this procedure will give a finite expression in terms of the renormalized quantities, λ and μ . In the end, we will have

$$\Delta \mathcal{L} = \Delta \mathcal{L}^{(1)} + \Delta \mathcal{L}^{(2)} + \dots$$
 (109)

To analyze further, we need to do some general *power* counting. This will tell us what kinds of counter terms we will need.

A theory is renormalizable if only a small number of counter terms are required and if these counter terms have the same form as terms already in the bare Lagrangian. To analyze the divergent structure of any Feynman diagram, we introduce the term "superficial degree of divergence", D, which is the number of loop momenta in the numerator minus the number of loop momenta in the denominator. For example, in our fish diagrams of Fig. 1, we find the d^4l in the numerator and $1/(l^2)^2$ (large l) in the denominator, giving D = 4 - 4 = 0, implying a logarithmic divergence. In general, D can be computed once we know:

- B = number of external bosons;
- IB = number of internal boson lines;
- n = number of vertices.

The computation of D that we desire is as follows:

1. Since each vertex has 4 lines entering or exiting, and *both* ends of an internal line must terminate on vertices, but only *one* end of an external line must terminate on a vertex, we have

$$4n = 2(IB) + B \tag{110}$$

2. The number of *independent* loop moment that must be integrated over after employing all the δ functions from the vertices is given by

$$L = IB - n + 1 \tag{111}$$

where the +1 is because the 4-momenta delta functions at the *n* vertices are not all independent given that an overall $\delta(\sum p_i)$ must emerge.

3. The degree of divergence is

$$D = 4L - 2(IB) \tag{112}$$

where there is a 4 for each loop d^4l and a -2 for each internal propagator (since they fall off as $1/l^2$).

4. Using Eq. (111) for L in (112) and then using Eq. (110) to substitute for IB gives

D = 4(IB - n + 1) - 2IB = 2IB - 4n + 4 = 4n - B - 4n + 4 = 4 - B.(113)

The fact that D does not depend on the internal structure of the diagram is another hallmark of a renormalizable theory. To see how many divergences are possible, we need only look at external structures.

Since $\lambda \phi^4$ has a reflection symmetry under $\phi \rightarrow -\phi$, *B* must be an even number. Thus, only B = 2 and B = 4 diagrams are superficially divergent. We reemphasize the fact that this statement is valid to all orders in perturbation theory (i.e. for arbitrarily complicated internal diagram structures).

Of course, we know very well what the most relevant diagrams are (at least at one-loop order). The B = 2 diagram at one-loop is the propagator correction which has D = 2 and contains quadratically divergent and log divergent pieces. The crucial B = 4 diagrams are the 1PI vertex correction diagrams (the fish diagrams at one-loop).

Note however that one can draw diagrams with D = 0 or even D < 0 that are actually quadratically or logarithmically divergent. An example is the diagram with 6 external bosons connected by a propagator supplemented by a loop correction on any one of the external legs. This is an example of a diagram that superficially has D = -2 but is actually quadratically divergent because of the presence of a divergent (in this case, propagator correction) subdiagram. However, this subdiagram will be rendered finite once we have

included counterterm Feynman rules that make the basic propagator correction finite. Part of the proof of renormalizability is to show that all D < 0 diagrams are rendered finite once counterterms for the basic (1PI) D = 2 and D = 0 diagrams have been introduced in $\Delta \mathcal{L}$.

Before proceeding to analyze these basic D = 2 and D = 0 1PI diagrams and the required counterterms from the BPH perspective, let us first discuss the general case of a ϕ^N theory. We have

Nn = (N-4)n + 4n = 2IB + B L = IB - n + 1 D = 4L - 2IB = 4(IB - n + 1) - 2IB = 2IB - 4n + 4 = (N - 4)n + 4n - B - 4n + 4 = (4 - B) + (N - 4)n.(114)

• For N > 4, as we move to higher orders, n, we generate higher and higher

degrees of divergence. The nature of the divergence is not just a function of the external particle number, but depends also on the order of perturbation theory being considered.

The theory is called *non-renormalizable*.

• For N < 4, higher orders quickly lead to convergent diagrams. There are only a limited number of actually divergent diagrams.

For example, for N = 3, D = 4 - B - n.

1. For B = 2, only diagrams with n = 1 or 2 vertices are divergent. But, n = 1 is topologically impossible for B = 2. The smallest possible n is n = 2 and the associated diagram (the loop insertion in a line) is log divergent.

n = 3 is topologically impossible.

n = 4 gives two 1PI diagrams that are convergent; of course, the double iteration of the n = 2 loop diagram is handled if we deal with the n = 2 diagram in the first place.

2. For B = 3, there are no n = 1 loop diagrams and all higher n diagrams are convergent. (The simplest diagram is the triangle correction to the 3-boson vertex, which has 3 internal propagators and converges as $\int d^4l/(l^2)^3$.)

3. For $B \ge 4$, any diagram with at least one vertex (so that it is a non-trivial diagram) will be convergent, other than diagrams with a 1-loop correction to a propagator, which are handled by the B = 2 renormalization. The simplest example to think of here is to consider the tree-level diagram with one virtual boson exchange between two three-boson vertices with two external bosons connected to each. If you insert a loop into the exchanged boson, that will be divergent, but handled by the B = 2 renormalization. The other diagrams are: a) triangle diagram corrections to the two vertices, which are convergent as discussed for B = 3; and b) the box diagram, which converges as $\int d^4 l/(l^2)^4$.

Thus, there is only *one* divergent diagram and it arises only in the lowest order at which a one-loop diagram propagator correction can be drawn.

The theory is called *super-renormalizable*.

Let us now return to ϕ^4 theory.

We have seen that we should only have to deal with B = 2 (D = 2) and B = 4 (D = 0) for renormalization.

For the D = 2 case, we have divergences associated with the mass and with the field strength. In a Taylor expansion, we would write

$$\Sigma(p^2) = \Sigma(0) + p^2 \Sigma'(0) + \widetilde{\Sigma}(p^2)$$
(115)

where $\Sigma(0)$ and $\Sigma'(0)$ are divergent but $\Sigma(p^2)$ is finite. (It will usually be simpler in the BPH approach to expand about zero-momentum configurations.) This means we need two counterterms:

$$\frac{1}{2}\Sigma(0)\phi^2 + \frac{1}{2}\Sigma'(0)(\partial_\mu\phi)(\partial^\mu\phi)$$
(116)

to cancel the divergences. If we introduce these terms into $\Delta \mathcal{L}$ that means we introduce new Feynman rules associated with these Lagrangian structures. The Feynman rules are easily derived using our standard technique and correspond to point-like insertions onto a single particle line of weights $i\Sigma(0)$ and $i\Sigma'(0)p^2$, respectively — these are the desired signs since $-i\Sigma(p^2)$ is the result of the 1PI loop insertion in a single particle line and so the new $\Delta \mathcal{L}$ terms will have the correct sign to cancel the one-loop divergent expansion components.

Similarly, for D = 0, B = 4 we know that

$$\Gamma^4(p_1, p_2, p_3, p_4) = \Gamma^4(0) + \widetilde{\Gamma}^4(p_1, p_2, p_3, p_4)$$
(117)

where we find it convenient to expand about the point where all the $p_{i=1,2,3,4}$ are 0. $\Gamma^4(0)$ we know is log divergent. To cancel this log divergence we

introduce a counter term in $\Delta \mathcal{L}$ of the form

$$\Delta \mathcal{L} \ni \frac{i\Gamma^4(0)}{4!} \phi^4 \tag{118}$$

which gives rise to a new 4-point vertex in our set of Feynman rules:

$$-\Gamma^4(0)$$
. (119)

(To check sign and phase, recall that $\mathcal{L} \ni -\frac{\lambda}{4!}$ produces Feynman rule $-i\lambda$, i.e. multiply by i4!.) Thus, (118) will produce Feynman rule (119), which will therefore have the correct sign to cancel the $+\Gamma^4(0)$ coming from the one-loop calculations.

Altogether, we can rewrite $\Delta \mathcal{L}$ in the form

$$\Delta \mathcal{L} = \frac{1}{2} (Z_{\phi} - 1) [(\partial_{\mu} \phi) (\partial^{\mu} \phi) - \mu^{2} \phi^{2}] + \frac{1}{2} Z_{\phi} \delta \mu^{2} \phi^{2} - \frac{\lambda (Z_{\lambda} - 1)}{4!} \phi^{4}$$
(120)

if we make the following identifications:

$$\begin{split} \Sigma'(0) &= Z_{\phi} - 1 \\ \Sigma(0) &= -(Z_{\phi} - 1)\mu^{2} + Z_{\phi}\delta\mu^{2} = -\Sigma'(0)\mu^{2} + Z_{\phi}\delta\mu^{2} \\ \Gamma^{4}(0) &= -i\lambda(1 - Z_{\lambda}). \end{split}$$
(121)

These results are consistent with our earlier renormalization equations, up to some differences in the treatment of finite terms. In particular, our previous "old" approach gave the results:

$$Z_{\phi} = 1 + \Sigma'(\mu^{2}) + \mathcal{O}(\lambda_{0}^{2})$$

$$\mu^{2} = \mu_{0}^{2} + \delta\mu^{2} = \mu_{0}^{2} + \Sigma(\mu^{2})$$

$$-iZ_{\lambda}^{-1}\lambda_{0} = -i\lambda_{0} + 3\Gamma(s_{0} = 4\mu^{2}/3). \qquad (122)$$

It is easy to see that (121) is the same as (122), to the order that we are working and as regards the infinite terms. There are finite terms that are treated differently in the two approaches unless it happens that the physical mass $\mu^2 = 0$ so that the subtraction point employed in the BPH scheme and the physical particle mass squared are the same.

For physical mass squared $\mu^2 = 0$, we have the mapping:

$$old (at \ \mu^{2} = 0) \quad \leftrightarrow \quad new \ (BPH)$$

$$Z_{\phi} = 1 + \Sigma'(0) \quad \leftrightarrow \quad \Sigma'(0) = Z_{\phi} - 1 \tag{123}$$

$$\delta\mu^{2} = \Sigma(0) \quad \leftrightarrow \quad \Sigma(0) = -\Sigma'(0) \times (\mu^{2} = 0) + (1 + \Sigma'(0))\delta\mu^{2}$$

$$= -0 + \delta\mu^{2} + \mathcal{O}(\lambda_{0}^{2}) \tag{124}$$

$$-iZ_{\lambda}^{-1}\lambda_{0} = -i\lambda_{0} + 3\Gamma(0) \quad \leftrightarrow \quad \Gamma^{4}(0) = -i\lambda(1-Z_{\lambda}).$$
(125)

The equivalences (123) and (124) are trivially obvious. To see the equivalence (125), let us rewrite the left-hand side as

$$-i\lambda_0 \frac{1-Z_\lambda}{Z_\lambda} = 3\Gamma(0). \qquad (126)$$

Since $1 - Z_{\lambda}$ is of order λ already, neglecting terms of order λ^2 we may replace Z_{λ} in the denominator by 1 and λ_0 in the numerator by λ to obtain

$$-i\lambda(1-Z_{\lambda}) = 3\Gamma(0). \qquad (127)$$

This is the same as the right-hand side of (125) since $\Gamma^4(0) = 3\Gamma(0)$, i.e. the sum of the three fish diagrams evaluated in the $s_0 = t_0 = u_0 = 0$ (off-mass-shell) symmetric configuration.

The fact that the two schemes differ by finite terms if $\mu^2 \neq 0$ is perfectly ok. All that renormalization has to handle is the potentially infinite terms.

Schemes that differ as a result of the renormalization quantities Z_{ϕ} , Z_{λ} and $\delta \mu^2$ having different finite terms will give the same final answer after summing the full perturbation series.

However, it is true that some schemes have faster convergence for the perturbation series than others. We shall return to this point later.

It is important to again stress that the divergent graphs (B = 2 and B = 4) gave divergences that corresponded to 2 and 4 point interactions that were already present in \mathcal{L} . We did not have to introduce any new interactions in order to renormalize the theory.

All of this can be extended to higher orders, but this extension is definitely more complex than what we discussed above. One gets involved with divergent subgraphs, "primitively" divergent graphs, Weinberg's theorem, We don't have time for this material here. We must turn to the more practical problem at one loop of

Regularization Schemes

A good regularization scheme should maintain Lorentz invariance and the symmetries of the theory.

An old scheme was to employ a Covariant Cutoff.

We will focus on the more modern scheme that is very generally useful, even in the context of supersymmetry, of Dimensional Regularization. Some key ingredients in this scheme are:

- 1. Make integrals finite by using n < 4 dimensions.
- 2. Recognize that Feynman integrals are analytic functions of n.
- 3. Observe that ultraviolet (high-momentum) divergences \Rightarrow poles at n = 4.
- 4. Check that Ward identities are preserved this is important when discussing gauge theories.

We begin with the example of one of the fish diagrams.

$$\Gamma(p^2) = \frac{(-i\lambda)^2}{2} \int \frac{d^4l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu^2 + i\epsilon} \frac{i}{l^2 - \mu^2 + i\epsilon} \,. \tag{128}$$

To define this integral in n dimensions, take

$$l = (l_0, l_1, \dots l_{n-1}), \quad p = (p_0, p_1, p_2, p_3, 0, 0, \dots)$$
 (129)

and write

$$\Gamma(p^2) = \frac{(\lambda)^2}{2} \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l-p)^2 - \mu^2 + i\epsilon} \frac{1}{l^2 - \mu^2 + i\epsilon}, \quad (130)$$

which is convergent for n < 4. We wish to define this for non-integer n. To do so, we first combine the denominators using Feynman parameters and make a Wick rotation.

The Feynman trick is to write

$$\frac{1}{ab} = \int_0^1 \frac{d\alpha}{[\alpha a + (1 - \alpha)b]^2}.$$
 (131)

More generally,

$$\frac{1}{a_1 a_2 \dots a_n} = (n-1)! \int_0^1 \frac{dz_1 \dots dz_n}{(a_1 z_1 + \dots a_n z_n)^n} \delta(1 - \sum_{i=1}^n z_i) \,. \tag{132}$$

Returning to our $\Gamma(p^2)$, we write

$$rac{1}{(l^2-\mu^2+i\epsilon)}\,rac{1}{(l-p)^2-\mu^2+i\epsilon} \ = \ \int rac{dlpha}{[(1-lpha)l^2+lpha(l-p)^2-\mu^2+i\epsilon]^2}$$

$$= \int \frac{d\alpha}{[(l-\alpha p)^2 + \alpha(1-\alpha)p^2 - \mu^2 + i\epsilon]^2}$$
$$\equiv \int \frac{d\alpha}{[k^2 - a^2 + i\epsilon]^2}.$$
(133)

At this point we shift from l to k: $\int d^4l = \int d^4k$.

The Wick rotation, which is performed in the complex k^0 plane, takes the Minkowski metric to a Euclidean metric without passing the denominator poles.

The poles in question are determined by

$$k^{2} - a^{2} + i\epsilon = (k^{0})^{2} - \vec{k}^{2} - a^{2} + i\epsilon = (k^{0})^{2} - [(\vec{k}^{2} + a^{2})^{1/2} - i\epsilon]^{2},$$
 (134)

where we have rescaled ϵ without changing its sign, which gives poles at

$$k^{0} = (\vec{k}^{2} + a^{2})^{1/2} - i\epsilon$$

$$k^{0} = -(\vec{k}^{2} + a^{2})^{1/2} + i\epsilon.$$
(135)

The idea of the Wick rotation is to rotate the integral along the real k^0 axis to an integral along the imaginary k^0 axis without wrapping around either of the poles. This specifies a unique direction of rotation. The rotation is pictured in Fig. 3. Since no poles are contained within C, $\oint_C f(k^0) = 0$, and further the contours at ∞ give zero contribution (for well-behaved $f(k^0)$). As a result

$$\int_{real axis} f(k^0) dk^0 = \int_{imag.-axis} f(k^0) dk^0, \qquad (136)$$

where

$$\boldsymbol{f}(\boldsymbol{k}^{0}) = \frac{1}{\left[(\boldsymbol{k}^{0})^{2} - \left\{ (\vec{\boldsymbol{k}}^{2} + \boldsymbol{a}^{2})^{1/2} - \boldsymbol{i}\boldsymbol{\epsilon} \right\}^{2} \right]^{2}}.$$
(137)



Figure 3: Graphical illustration of the Wick rotation for k^0 , with pole locations indicated.

This can be written explicitly as

$$\int_{-\infty}^{\infty} dk^0 f(k^0) = \int_{-i\infty}^{+i\infty} dk^0 f(k^0)$$
$$= i \int_{-\infty}^{\infty} d\bar{k}^0 f(i\bar{k}^0) \quad \text{with } k^0 = i\bar{k}^0, \qquad (138)$$

and

$$f(i\bar{k}^{0}) = \frac{1}{\left[(\bar{k}^{0})^{2} + l_{1}^{2} + l_{2}^{2} + \dots + l_{n-1}^{2} + a^{2} - i\epsilon\right]^{2}}.$$
 (139)

So, now simply relabeling $ar{k}^0
ightarrow l^0$ in the Euclidean metric form, we obtain

$$\Gamma(p^2) = \frac{i\lambda^2}{2} \int_0^1 d\alpha \int \frac{d^n \vec{l}}{(2\pi)^n} \frac{1}{\left[|\vec{l}|^2 + a^2 - i\epsilon\right]^2}, \vec{l} = \text{Euclidean}$$
(140)

where $a^2 = \mu^2 - \alpha(1 - \alpha)p^2$. The integral is now independent of angles, which can be integrated out after defining $l = |\vec{l}|$:

$$\int d^n \vec{l} = \int_0^\infty l^{n-1} dl \int_0^{2\pi} d\theta_1 \int_0^\pi \sin \theta_2 d\theta_2 \int_0^\pi \sin^2 \theta_3 d\theta_3 \dots \int_0^\pi \sin^{n-2} \theta_{n-1} d\theta_{n-1}$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty l^{n-1} dl \,. \tag{141}$$

We employed

$$\int_{0}^{\pi} \sin^{m}(\theta) d\theta = \frac{\sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$
(142)

to obtain the final form above by noting:

Angle stuff =
$$2\pi \frac{\sqrt{\pi}\Gamma(1)}{\Gamma(3/2)} \frac{\sqrt{\pi}\Gamma(3/2)}{\Gamma(2)} \dots \frac{\sqrt{\pi}\Gamma(n-1)}{\Gamma(n/2)}$$

= $2\pi (\sqrt{\pi})^{n-2} \frac{\Gamma(1)}{\Gamma(n/2)}$
= $\frac{2\pi^{n/2}}{\Gamma(n/2)}$. (143)

So, the bottom line is that

$$\Gamma(p^{2},n) = \frac{\lambda^{2}}{2} \frac{2i\pi^{n/2}}{(2\pi)^{n}\Gamma(n/2)} \int d\alpha \int_{0}^{\infty} \frac{l^{n-1}dl}{\left[l^{2} + a^{2} - i\epsilon\right]^{2}}.$$
 (144)

An alternative derivation for the angular integrals is to note that

 $\int d\Omega_n = ext{area of unit sphere in } n ext{ dimensions} \equiv S_n = rac{2\pi^{n/2}}{\Gamma(n/2)}, \quad (145)$

where this latter result might also be derived by noting that $\sqrt{\pi} = \int_{-\infty}^{\infty} dx e^{-x^2}$ so that

$$(\sqrt{\pi})^{n} = \int dx_{1} \dots dx_{n} e^{-(x_{1}^{2} + \dots x_{n}^{2})}$$

$$= \int d\Omega_{n} \int_{0}^{\infty} d|\vec{x}| |\vec{x}|^{n-1} e^{-|\vec{x}|^{2}}$$

$$= \int d\Omega_{n} \frac{1}{2} \int_{0}^{\infty} dy y^{\frac{n}{2} - 1} e^{-y} \quad y = |\vec{x}|^{2}$$

$$= \int d\Omega_{n} \frac{1}{2} \Gamma\left(\frac{n}{2}\right).$$
(146)

Turning this around we get

$$\int d\Omega_n \equiv S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$
(147)

Now, we must do the final $\int dl$. It is well defined for 0 < Ren < 4. But, in fact, we can extend it to any Ren < 4 by multiple parts integration. Thus, only $n \rightarrow 4$ needs to be analyzed. First, to perform the $\int dl$ we use the identity:

$$\int_{0}^{\infty} \frac{t^{m-1} dt}{(t+a^{2})^{n}} = \frac{1}{(a^{2})^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} \,. \tag{148}$$

Our integral can be rewritten in this form by shifting to $t = l^2$, $dt = 2ldl = 2\sqrt{t}dl$:

$$egin{aligned} &\int rac{l^{n-1} dl}{\left[l^2+a^2
ight]^2} &= &\int rac{dt}{2\sqrt{t}} rac{t^{rac{n-1}{2}}}{\left[t+a^2
ight]^2} \ &= &rac{1}{2} \int rac{t^{rac{n}{2}-1} dt}{\left[t+a^2
ight]^2} \end{aligned}$$

$$= \frac{1}{2} \frac{1}{(a^2)^{2-\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}$$
$$= \frac{1}{2} \frac{1}{(a^2)^{2-\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \Gamma\left(2-\frac{n}{2}\right) .$$
(149)

Now use

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{2-\frac{n}{2}} \stackrel{n\to4}{\to} \frac{2}{4-n}$$
(150)

to see that the singularity is a simple pole. To be more precise, let's expand everything around n=4 by writing $n=4-2\epsilon^2$ and use the general result that

$$\Gamma(-k+\epsilon) = \frac{(-1)^k}{k!} \left[\frac{1}{\epsilon} + \psi(k+1) + \frac{1}{2}\epsilon \left\{ \frac{\pi^2}{3} + \psi^2(k+1) - \psi'(k+1) \right\} + \mathcal{O}(\epsilon^2) \right]$$
(151)

where $\psi(s) \equiv rac{d \ln \Gamma(s)}{ds}$. The specific values

$$egin{array}{rll} \psi(1) &=& -\gamma \ \psi(k+1) &=& 1+rac{1}{2}+\ldots+rac{1}{k}-\gamma \end{array}$$

²Ryder writes $n = 4 - \epsilon$, so be careful if you are looking at his book.

$$\psi'(1) = \frac{\pi^2}{6}$$

$$\psi'(k+1) = \frac{\pi^2}{6} - \sum_{l=1}^k \frac{1}{l^2}$$
 (152)

will be of particular use. In the immediate context, we write

$$\Gamma\left(2-\frac{n}{2}\right) = \Gamma\left(2-\frac{4-2\epsilon}{2}\right)$$
$$= \Gamma(\epsilon)$$
$$= \left[\frac{1}{\epsilon}-\gamma+\frac{1}{2}\epsilon\left\{\frac{\pi^2}{3}+\gamma^2-\frac{\pi^2}{6}\right\}+\mathcal{O}(\epsilon^2)\right]. \quad (153)$$

We need to do one more thing. We note that when $n \neq 4$, then λ is not dimensionless any longer. To relate always to a dimensionless coupling constant, we need to rescale with some mass:

$$\lambda_{old} = \lambda_{new} (\mu^2)^{2 - \frac{n}{2} = \epsilon}$$
(154)

where λ_{new} is dimensionless. That this is the appropriate scaling is argued as

follows:

- 1. $\int d^n x (\partial_\mu \phi)^2$ must be dimensionless.
- 2. Meanwhile $d^n x = d^{4-2\epsilon} x$ has mass dimension in the amount $M^{2\epsilon-4}$, while each ∂_{μ} has mass dimension M.
- 3. Thus, the net dimension of $M^{2\epsilon-2}$ must be compensated by the dimension of ϕ^2 , implying that ϕ should have dimension $M^{1-\epsilon}$.
- 4. Next, we examine the $d^n x \lambda \phi^4$ part of the Lagrangian which has mass dimension $M^{2\epsilon-4} M^{\dim[\lambda]} M^{4-4\epsilon}$.

To get M^0 requires that λ has mass dimension $M^{2\epsilon}$ which we have written in the conventional form $(\mu^2)^{\epsilon}$.

Apologies for the multiple use of μ , but both are conventional. The μ above is called the renormalization scale.

However, in what follows, the μ scalar mass will be called m.

Anyway, dropping the *new* subscript on the λ , our net result is

$$\Gamma(p^2) = rac{\lambda^2}{2}(\mu^2)^{2\epsilon}rac{i\pi^{rac{4-2\epsilon}{2}}}{(2\pi)^{4-2\epsilon}}\left[rac{1}{\epsilon}-\gamma+rac{1}{2}\epsilon\left\{rac{\pi^2}{3}+\gamma^2-rac{\pi^2}{6}
ight\}+\mathcal{O}(\epsilon^2)
ight]\int dlpharac{1}{\left[m^2-p^2lpha(1-lpha)
ight]^{\epsilon}}$$

$$= \frac{\lambda^{2}}{2} (\mu^{2})^{\epsilon} e^{2\epsilon \ln \mu} \frac{i\pi^{2}}{(2\pi)^{4}} e^{\epsilon \ln(4\pi)} \left[\frac{1}{\epsilon} - \gamma + \frac{1}{2} \epsilon \left\{ \frac{\pi^{2}}{3} + \gamma^{2} - \frac{\pi^{2}}{6} \right\} \right] \int_{0}^{1} d\alpha e^{-\epsilon \ln[m^{2} - p^{2}\alpha(1 - \alpha)]},$$
(155)

where we have isolated the power $(\mu^2)^{\epsilon}$, which is the dimension of λ_{old} which must also be the net dimension of Γ , as we shall see. The above can be rewritten in the form

$$\Gamma(p^{2}) = \frac{i\pi^{2}}{2(2\pi)^{4}}\lambda^{2}(\mu^{2})^{\epsilon} \left[\frac{1}{\epsilon} - \gamma + \frac{1}{2}\epsilon \left\{\frac{\pi^{2}}{3} + \gamma^{2} - \frac{\pi^{2}}{6}\right\}\right] \int d\alpha e^{-\epsilon \ln \frac{m^{2} - p^{2}\alpha(1-\alpha)}{4\pi\mu^{2}}},$$
(156)

which may be expanded up to power ϵ^0 in the form:

$$\frac{i\pi^2}{2(2\pi)^4}\lambda^2(\mu^2)^\epsilon \left[\frac{1}{\epsilon} - \gamma - \int d\alpha \ln \frac{m^2 - p^2\alpha(1-\alpha)}{4\pi\mu^2}\right].$$
 (157)

(Notice how the ϵ^1 from the exponential expansion canceled against the $1/\epsilon$ from the ultraviolet singularity. This is a typical thing that one must always be careful to keep.)

We write

$$-\ln\left\{\frac{m^{2}-p^{2}\alpha(1-\alpha)}{4\pi\mu^{2}}\right\} = -\ln\left\{\frac{m^{2}}{4\pi\mu^{2}}\left[1-\frac{p^{2}}{m^{2}}\alpha(1-\alpha)\right]\right\}$$
$$= \ln\frac{4\pi\mu^{2}}{m^{2}}-\ln\left[1+\frac{4}{b}\alpha(1-\alpha)\right]$$
(158)

with $b = -\frac{4m^2}{p^2}$. Of course, $\int d\alpha$ of the first term just gives $1 \times \ln \frac{4\pi\mu^2}{m^2}$. If we temporarily take $p^2 < 0$ so that b > 0, we can use the result below to evaluate the integral of the 2nd term, and then later continue to $p^2 > 0$.

$$\int d\alpha \ln\left[1 + \frac{4}{b}\alpha(1-\alpha)\right] \stackrel{b \ge 0}{=} -2 + \sqrt{1+b} \ln\left(\frac{\sqrt{1+b}+1}{\sqrt{1+b}-1}\right)$$
(159)

The result we obtain in this manner is

$$\Gamma(p^2) = \frac{i\lambda^2}{32\pi^2} (\mu^2)^{\epsilon} \left[\frac{1}{\epsilon} - \gamma + 2 + \ln\frac{4\pi\mu^2}{m^2} - \sqrt{1 - \frac{4m^2}{p^2}} \ln\left(\frac{\sqrt{1 - \frac{4m^2}{p^2}} + 1}{\sqrt{1 - \frac{4m^2}{p^2}} - 1}\right) \right] .$$
(160)

This equation will be applied for all three of our fish diagrams, that is for $p^2 = s$, t and u.
Next, we need to turn our attention to Σ . The diagram in question is the one-loop tadpole-like insertion.

$$\begin{split} \Sigma(p^2) &= \frac{-i\lambda(\mu^2)^{\epsilon}}{2} \int \frac{d^n l}{(2\pi)^n} \frac{i}{l^2 - m^2 + i\epsilon} \\ &= \frac{\lambda}{2} (\mu^2)^{\epsilon} \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 - m^2 + i\epsilon} \\ &= i\frac{\lambda}{2} (\mu^2)^{\epsilon} \int \frac{d^n \vec{l}}{(2\pi)^n} \left[\frac{1}{-|\vec{l}|^2 - m^2 + i\epsilon} \right] \\ &= -i\frac{\lambda}{2} (\mu^2)^{\epsilon} \frac{1}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} \frac{l^{n-1}dl}{l^2 + m^2} \\ &= -i\frac{\lambda}{2} (\mu^2)^{\epsilon} \frac{1}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{2} \int \frac{t^{\frac{n}{2} - 1}dt}{t + m^2} \\ &= -i\frac{\lambda}{2} (\mu^2)^{\epsilon} \frac{1}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1 - \frac{n}{2}\right)}{\Gamma(1)} \end{split}$$

$$= -\frac{i\lambda}{2} (\mu^2)^{\epsilon} \frac{2}{(4\pi)^{n/2}} \frac{1}{2} \frac{1}{(m^2)^{-1+\epsilon}} \Gamma(-1+\epsilon)$$

$$= -\frac{i\lambda m^2}{32\pi^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} \Gamma(-1+\epsilon).$$
(161)

We now do an expansion, using Eq. (151):

$$\Gamma(-1+\epsilon) = -\left[\frac{1}{\epsilon} + \psi(2) + \mathcal{O}(\epsilon)\right], \qquad (162)$$

where $\psi(2) = 1 - \gamma$. Substituting this in we find:

$$-i\Sigma(p^2) = \frac{i\lambda m^2}{32\pi^2} \left[\frac{1}{\epsilon} + 1 - \gamma + \ln\frac{4\pi\mu^2}{m^2} + \mathcal{O}(\epsilon)\right].$$
 (163)

Again, there was a $\mathcal{O}(\epsilon)$ term from the expansion of

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} = e^{\epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right)} \sim 1 + \epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right)$$
(164)

that multiplied the $\frac{1}{\epsilon}$ term to give a term of order $\mathcal{O}(\epsilon^0)$.

Also, we should note: (a) there is no p^2 dependence in this one-loop expression; (b) $\Gamma(-1 + \epsilon = 1 - \frac{n}{2})$ also has a pole at n = 2 reflecting the quadratic divergence.

We are now in a position to explicitly carry out the renormalization procedure. Define

$$\Delta \mathcal{L} = \frac{1}{2} \Sigma(0) \phi^2 + \frac{1}{2} \Sigma'(0) (\partial_\mu \phi) (\partial^\mu \phi) + \frac{i \Gamma^4(0)}{4!} \phi^4$$
(165)

as before, where $\Sigma'(0)$ is zero because of the lack of p^2 dependence at the present order λ at which we are working. For the non- ϕ^4 part of $\Delta \mathcal{L}$, the appropriate expression is

$$\Delta \mathcal{L} = -\frac{1}{2} \phi^2 \left[\frac{\lambda m^2}{32\pi^2} \right] \left[\frac{1}{\epsilon} + F(\epsilon, \frac{4\pi\mu^2}{m^2}) \right] , \qquad (166)$$

where F is an arbitrary function except for being analytic as $\epsilon \to 0$ and a function of the dimensionless ratio $\frac{\mu^2}{m^2}$. The counter-term Feynman rule

associated with this $\Delta \mathcal{L}$ is

$$\frac{-i\lambda m^2}{32\pi^2} \left[\frac{1}{\epsilon} + F\right] , \qquad (167)$$

so that

$$-i\Sigma(p^2) + C.T. = \frac{i\lambda m^2}{32\pi^2} \left[1 - \gamma + \ln\frac{4\pi\mu^2}{m^2} - F \right] .$$
 (168)

This can now be combined with the 0th order propagator as indicated in the figure to obtain:

 $bare + bare \times [1PI \ insertion + CT \ insertion] \times bare + iterations$

$$= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} \frac{i\lambda m^2}{32\pi^2} \left[1 - \gamma + \ln \frac{4\pi\mu^2}{m^2} - F \right] \frac{i}{p^2 - m^2} + \dots$$

$$= \frac{i}{p^2 - m^2 + \frac{\lambda m^2}{32\pi^2} \left[1 - \gamma + \ln \frac{4\pi\mu^2}{m^2} - F \right]}.$$
 (169)



Figure 4: Graphical illustration of counterterm approach for the propagator. We now need to play the same game for the ϕ^4 term. Recalling Eq. (160), the important singular part is

$$\Gamma(p^2) = \frac{i\lambda^2}{32\pi^2} (\mu^2)^{\epsilon} \frac{1}{\epsilon}.$$
(170)

and we must remember that since there are 3 fish diagrams this singular structure actually appears with a factor of 3. Thus, we will choose (in

Eq. (165))

$$\Gamma^{4}(0) = \frac{3i\lambda^{2}}{32\pi^{2}}(\mu^{2})^{\epsilon} \left[\frac{1}{\epsilon} + G(\epsilon, \frac{4\pi\mu^{2}}{m^{2}})\right], \qquad (171)$$

for which the Feynman rule deriving from the $\frac{i\Gamma^4(0)}{4!}\phi^4$ interaction is given by

$$-\Gamma^4(0) = -\frac{3i\lambda^2}{32\pi^2}(\mu^2)^\epsilon \left[\frac{1}{\epsilon} + G(\epsilon, \frac{4\pi\mu^2}{m^2})\right]$$
(172)

so that the sum of the 3 fish diagrams plus the vertex counter diagram,

$$\Gamma(s) + \Gamma(t) + \Gamma(u) - \Gamma^{4}(0) = 3\frac{i\lambda^{2}}{32\pi^{2}}(\mu^{2})^{\epsilon} \left[-\gamma + 2 + \ln\frac{4\pi\mu^{2}}{m^{2}} - \frac{1}{3}\sum_{z=s,t,u}\sqrt{1 - \frac{4m^{2}}{z}}\ln\left(\frac{\sqrt{1 - \frac{4m^{2}}{z}} + 1}{\sqrt{1 - \frac{4m^{2}}{z}} - 1}\right) - G(\epsilon, \frac{4\pi\mu^{2}}{m^{2}}) \right], \quad (173)$$

is finite.



Figure 5: Graphical illustration of counterterm approach for vertex.

Subtraction "Schemes"

- **1.** In the so-called minimal subtraction (MS) scheme, we take F = G = 0.
- 2. In the $\overline{\mathrm{MS}}$ subtraction scheme, we choose

$$F = G = -\gamma + \ln 4\pi \tag{174}$$

so as to absorb the obvious constants that appear in both calculations.

As it happens, the $\overline{\rm MS}$ scheme often leads to smaller coefficients for higher order correction expansions, but for the moment we will stick to the $\overline{\rm MS}$ scheme.

In the MS scheme, we have the following identifications with the general form of (after taking $\mu \rightarrow m$ in the basic form to avoid confusion with the renormalization scale μ and going to the dimensionless coupling λ in n dimensions)

$$\Delta \mathcal{L} = \frac{1}{2} (Z_{\phi} - 1) [(\partial_{\mu} \phi)(\partial^{\mu} \phi) - m^{2} \phi^{2}] + \frac{1}{2} Z_{\phi} \delta m^{2} \phi^{2} - \frac{\lambda (\mu^{2})^{\epsilon} (Z_{\lambda} - 1)}{4!} \phi^{4} .$$
(175)

First, at this order only,

$$\frac{1}{2}(Z_{\phi}-1)(\partial_{\mu}\phi)(\partial^{\mu}\phi) = 0$$
(176)

since $\Sigma'(0) = 0$. The two non-trivial identifications are:

$$\frac{1}{2}Z_{\phi}\delta m^{2}\phi^{2} = -\frac{1}{2}\phi^{2}\frac{\lambda m^{2}}{32\pi^{2}}\frac{1}{\epsilon}.$$
(177)

$$-\frac{\lambda(\mu^2)^{\epsilon}}{4!}(Z_{\lambda}-1)\phi^4 = -\frac{\lambda(\mu^2)^{\epsilon}}{4!}\phi^4\frac{3\lambda}{32\pi^2}\frac{1}{\epsilon}.$$
 (178)

Reading off this matching, we find

$$Z_{\phi} = 1$$

$$\delta m^{2} = -\frac{\lambda m^{2}}{32\pi^{2}\epsilon}$$

$$Z_{\lambda} - 1) = \frac{3\lambda}{32\pi^{2}\epsilon}.$$
(179)

What's in the choice of a subtraction scheme?

As we have seen, there is ambiguity in the choice of the finite part of the counterterms. This ambiguity leads to different possible definitions of the coupling constant employed in the perturbative expansions. It turns out to be the case that some coupling constant definitions lead to a systematically better expansion in that, for example, using QED notation,

$$prediction = e^2 + ae^4 \tag{180}$$

where a may be small for one definition of e but big for another definition. It

turns out that this is particularly relevant for QCD.

To see what is at issue more clearly, let us imagine a theory of Quantum Imaginary Dynamics, QID. Assume that in the far distant future someone is able to solve QID exactly so that predictions for important measurements are precisely related to one another.

- At the present time, we imagine that there are two high precision measurements that are made:
 - 1. the QID "Josephson Effect"

$$J_I = 0.20000000 \ (13) \tag{181}$$

2. the magnetic moment of the QID electron

$$a_e^I = 0.333333333(12)$$
. (182)

 The exact solution of QID in terms of a parameter x (to be measured) yields the results

$$J_I = x$$

$$a_e^I = \frac{x}{1-2x}.$$
(183)

Obviously, if x is measured by J_I (so that x = 0.2...) then this exact solution predicts $a_e^I = 0.333...$

• Unfortunately for the inhabitants of QID land, their stupid physicists have not solved QID exactly and must instead resort to QID perturbation theory.

They find it convenient to expand in terms of a parameter y which appears naturally in the calculation of QID perturbation theory diagrams. It happens that

$$x = y(1 + 10y)$$
(184)

but the physicists are unaware of that. They attach no particular significance to x.

• Now, expanding we have

$$J_{I} = y + 10y^{2}$$

$$a_{e}^{I} = y + 12y^{2} + 44y^{3} + \dots$$
(185)

Keeping only the leading "tree-level" terms linear in y, a measurement of $J_I = 0.2 = y$ predicts $a_e^I = 0.2$, in bad disagreement with experiment. This was all the professors could manage.

• After several years of lengthy QID computations, the graduate students obtained the next order, i.e. $\mathcal{O}(y^2)$, results so that now they had

$$y + 10y^2 = 0.2, \quad \Rightarrow \quad y = 0.1$$
 (186)

yielding the prediction $a_e^I = y + 12y^2 = 0.22$, which still gave bad agreement, despite the fact that y = 0.1 seems like a small expansion parameter. The problem is that the coefficients in the expansion of a_e^I in terms of y are very big.

• Finally, some clever physicist says lets choose an expansion parameter such that the coefficients in the expansion are better behaved, and perhaps the expansion parameter has additional physical motivation as well.

An obvious guess, once you compute to order y^2 and realize that $x = y + 10y^2$ has a big y^2 coefficient, is to use x instead of y, so that

 $J_I = x + \mathcal{O}(x^3)$. Then, you find from your a_e^I calculation that

$$a_e^I = x + 2x^2 + \mathcal{O}(x^3)$$
 (187)

which is also clearly better behaved in that the expansion coefficient is much smaller than before when y was used. Using x, one concludes that the measurement $J_I = 0.2 \Rightarrow x = 0.2$ which, in turn, $\Rightarrow a_e^I \sim 0.28$ (up to corrections of order x^3 which they hope are small).

Well this is much better, but still not exactly perfect.

Note that even though x = 0.2 > y = 0.1, the series is much more convergent in x.

• After another 3 years another graduate student manages to compute the 500+ Feynman diagrams (using some advanced algebraic manipulation programs and automated integration routines) needed to obtain the next terms at $\mathcal{O}(x^3)$ for J_I and a_e^I , discovering that $J_I = x + \mathcal{O}(x^4)$ and that $a_e^I = x + 2x^2 + 4x^3 + \mathcal{O}(x^4)$.

The measurement of J_I still gives x = 0.2 and, to the computed order, $a_e^I = 0.312$. At this point, the measured value of $a_e^I = 0.333$ is in

sufficiently good agreement with the prediction based on the $J_I = 0.2$ measurement and the perturbative series that QID is declared a correct theory. (Although some uncertainty must inevitably remain since the $\mathcal{O}(x^4)$ terms would require computing several thousand highly complex Feynman diagrams and the computation is estimated to take more than 5 years — a project that is deemed excessive even for assigning to a graduate student.)

- In QED, we are fortunate to have a physical observation (the charge of the electron) which we use to define our expansion parameter e and we are lucky in that not only is e small (more precisely, $\alpha = \frac{e^2}{4\pi} \ll 1$), but also the coefficients of powers of α that appear in calculating other quantities in terms of α are small.
- In QCD, we have no such direct measurement of g_s and we must be clever in the way in which we define our g_s (by trying to make it as physical as possible) in order to keep our perturbation theory coefficients in the expansion in powers of $\alpha_s = \frac{g_s^2}{4\pi}$ under control.

Experience has shown that the $\overline{\rm MS}$ scheme for defining the renormalized coupling constant is much better in this sense than the $\overline{\rm MS}$ scheme.

Dependence on the Renormalization Scale μ

• Another aspect of scheme-dependence is the choice of the renormalization scale μ .

As one goes to higher and higher orders, there should be less and less dependence on μ if the theory is well behaved and the renormalization program makes sense.

• At leading order (often denoted LO), there is usually substantial dependence on μ of a typical QCD calculation. This dependence upon μ arises because one must use $\alpha_s(\mu)$, i.e. evaluated at some scale μ . (We know very well how α_s depends on μ , as we shall see later in the course.)

As a result, the LO correction has a large theoretical error (sometimes called a "systematic" error from theory).

• At one-loop order (NLO=next-to-leading order), this dependence is typically substantially reduced. A nice little graph is that from the Bern, Dixon and Kosower review (hep-ph/9602280) which might form the basis of a possible final project.

It should be noted that, in general, it is not the case that the LO and NLO curves cross at the μ value which gives experimental agreement. There can be a systematic shift.



Figure 6: The inclusive cross section for single-jet production in $p\overline{p}$ collisions at $\sqrt{s} = 1.8$ TeV and jet transverse energy $E_T = 100$ GeV (using MRS D'_0 structure functions), showing the sensitivity of the LO result to the choice of renormalization scale, μ_R , and the reduced sensitivity at NLO. The CDF data band includes statistical errors only.



Figure 7: Improvement of the scale dependence in higher orders of perturbation theory for $R_{e^+e^-}(\sqrt{s} = 33 \text{ GeV})$ (taken from the book of Ellis, Stirling and Webber).

Another example is provided in the review by Djouadi (hep-ph/0503172). His Fig. 3.7 shows results for the inclusive $pp \rightarrow HW$ cross section at the LHC and Tevatron as a function of Higgs mass, and how these predictions change as one goes from LO, to NLO, to NNLO.



the spread of the cross section when the renormalization and factorization scales (μ_R and μ_F , respectively — we do not have time to get into the definition of the factorization scale) are varied in the range $\frac{1}{3}M_{HV} \leq \mu_R(\mu_F) \leq 3M_{HV}$, the other scale being fixed at $\mu_F(\mu_R) = M_{HV}$.

Here, K is a way of summarizing the net effect on the cross section. It is the multiplicative factor by which the most naive prediction should be multiplied, and is defined in these plots so that K = 1 for LO when $\mu_F = \mu_R = M_{HV}$ (M_{HV} being the invariant mass of the HV system), as indicated by the solid lines in the figures.

As can be seen, except for the accidental cancellation of the scale dependence of the LO cross section at the LHC for $M_H \sim 250$ GeV, the decrease of the scale variation is strong when going from LO to NLO and then to NNLO. For $M_H = 120$ GeV, the uncertainty from the scale choice at the LHC drops from $\sim 10\%$ at LO, to 5% at NLO, and to 2% at NNLO. At the Tevatron, these numbers are $\sim 10\%$ at LO, 7% at NLO, and 3% at NNLO.

Renormalization Group: "Moving" Coupling Constant and Mass in ϕ^4 theory

The understanding of how to use the renormalization process we have just gone through to make predictions for how the coupling λ (as measured by a four-point interaction or scattering) and the mass m (as "measured" by effective strength in probes of mass, e.g. through Yukawa interactions) change as a function of the energy scale of the probe, is of fundamental importance.

To proceed, let us summarize the results so far, and generalize a bit. We recall that

$$\phi = Z_{\phi}^{-1/2} \phi_{0}$$

$$\lambda(\mu^{2})^{\epsilon} = Z_{\lambda}^{-1} Z_{\phi}^{2} \lambda_{0}$$

$$m^{2} = m_{0}^{2} + \delta m^{2}.$$
(188)

Our calculation so far has given

$$\lambda_0 = (\mu^2)^{\epsilon} \lambda Z_\lambda = (\mu^2)^{\epsilon} \lambda \left[1 + \frac{3\lambda}{32\pi^2} \frac{1}{\epsilon} \right]$$
(189)

$$m_0^2 = m^2 - \delta m^2 = m^2 \left[1 + \frac{\lambda}{32\pi^2} \frac{1}{\epsilon} \right] .$$
 (190)

In higher orders this and the analogues for the other cases will take the forms

$$\lambda_{0} = (\mu^{2})^{\epsilon} \left[\lambda + \sum_{k=1}^{\infty} \frac{a_{k}(\lambda)}{\epsilon^{k}} \right]$$

$$m_{0}^{2} = m^{2} \left[1 + \sum_{k=1}^{\infty} \frac{b_{k}(\lambda)}{\epsilon^{k}} \right]$$

$$Z_{\phi} = \left[1 + \sum_{k=1}^{\infty} \frac{c_{k}(\lambda)}{\epsilon^{k}} \right]. \qquad (191)$$

So, let us take $\mu \frac{\partial}{\partial \mu}$ of one of the bare parameters, for example λ_0 . Well, obviously λ_0 , the input Lagrangian parameter, knows nothing about the arbitrary scale μ that we introduced as part of our renormalization process in extra dimensions. However, the renormalized parameters λ and m are implicit functions of the renormalization scale μ : $\lambda(\mu)$, $m(\mu)$. Thus, working to one-loop order we have

$$\mu \frac{\partial \lambda_0}{\partial \mu} = 0$$

= $2\epsilon \lambda (\mu^2)^{\epsilon} \left[1 + \frac{3\lambda}{32\pi^2 \epsilon} \right] + (\mu^2)^{\epsilon} \left[\mu \frac{\partial \lambda}{\partial \mu} + \frac{6\lambda}{32\pi^2 \epsilon} \mu \frac{\partial \lambda}{\partial \mu} \right] .$ (192)

Solving, we find

$$\mu \frac{\partial \lambda}{\partial \mu} = \frac{-2\epsilon \lambda (\mu^2)^{\epsilon} \left[1 + \frac{3\lambda}{32\pi^2 \epsilon}\right]}{(\mu^2)^{\epsilon} \left[1 + \frac{6\lambda}{32\pi^2 \epsilon}\right]}.$$
(193)

Now, we want to take the $\epsilon \to 0$ limit. In so doing, we must stick very consistently to a definite order in perturbation theory, writing

$$\begin{aligned}
\mu \frac{\partial \lambda}{\partial \mu} &= -2\epsilon \lambda \left[1 + \frac{3\lambda}{32\pi^2 \epsilon} - \frac{6\lambda}{32\pi^2 \epsilon} + \mathcal{O}(\lambda^2) \right] & (194) \\
& \epsilon \stackrel{\epsilon}{=} ^0 \frac{3\lambda^2}{16\pi^2} \\
&\equiv \beta(\lambda) \,. & (195)
\end{aligned}$$

(A more careful treatment is possible, and perhaps we shall get to it.)

Similarly, we have

$$\mu \frac{\partial m_0^2}{\partial \mu} = 0$$

= $\mu \frac{\partial m^2}{\partial \mu} \left[1 + \frac{\lambda}{32\pi^2 \epsilon} \right] + m^2 \mu \frac{\partial \lambda}{\partial \mu} \frac{1}{32\pi^2} \frac{1}{\epsilon},$ (196)

which can be solved to give

$$\mu \frac{\partial m^2}{\partial \mu} = -m^2 \mu \frac{\partial \lambda}{\partial \mu} \frac{1}{32\pi^2} \frac{1}{\epsilon} \left[1 - \frac{\lambda}{32\pi^2} \frac{1}{\epsilon} \right] , \qquad (197)$$

into which we substitute the result of Eq. (194) (i.e. $\mu \frac{\partial \lambda}{\partial \mu} = -2\epsilon \lambda \left[1 - \frac{3\lambda}{32\pi^2} \frac{1}{\epsilon}\right]$) to obtain

$$\mu \frac{\partial m^2}{\partial \mu} = -m^2 (-2\epsilon\lambda) \left[1 - \frac{3\lambda}{32\pi^2} \frac{1}{\epsilon} \right] \frac{1}{32\pi^2} \frac{1}{\epsilon} \left[1 - \frac{\lambda}{32\pi^2} \frac{1}{\epsilon} \right]$$
$$\stackrel{\epsilon \Rightarrow 0}{=} m^2 \left[\frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2) \right].$$
(198)

This, we usually write in the form

$$\frac{1}{2}\mu \frac{\partial \ln m^2}{\partial \mu} = \frac{\lambda}{32\pi^2} \equiv \gamma_m(\lambda(\mu)).$$
(199)

So, now the question is how to interpret these $\mu \frac{\partial}{\partial \mu}$ derivative forms. We will give a more precise way of phrasing this interpretation in terms of the renormalization group equations (RGEs), but let us first attempt a naive/intuitive explanation. For this, let us return to the expression for

bare vertex +
$$\sum$$
 fish diagrams + vertex counter term
= $-i\lambda(\mu)(\mu^2)^{\epsilon} + i\frac{\lambda^2}{32\pi^2}(\mu^2)^{\epsilon}3\left[-\gamma + 2 + \ln\frac{4\pi\mu^2}{m^2} - \frac{1}{3}A(s,t,u)\right]$ (2,00)

where

$$A(s,t,u) = \sum_{z=s,t,u} eta(z) \ln rac{eta(z)+1}{eta(z)-1}, \quad ext{with} \quad eta(z) = \sqrt{1 - rac{4m^2}{z}}.$$
 (201)

Let us imagine first performing a measurement of the four-point interaction at an energy such that $s, t, u \sim s_0$. Then, if $\mu^2 \sim s_0$ and $m^2(\mu) \sim s_0$ also, everything in the above expression is simply set by the size of $\lambda(\mu \sim \sqrt{s_0})$. There are no big correction factors coming from the logarithms in the expression.

Next, we imagine increasing the energy at which the experiment operates so that $s, t, u \to fs_0$. Let us examine the behavior of A(s, t, u) in this limit. First, we note that $\beta(s) \sim 1 - \frac{2m^2}{fs_0}$ for large f so that

$$A(s,t,u) \rightarrow 3 \ln rac{2}{\left(rac{-2m^2}{fs_0}
ight)} \sim 3 \ln f + finite$$
 (202)

The idea is to cancel this large logarithmic stuff with some appropriate change of μ that depends on f. We need

$$\ln \mu^2(f) - \frac{1}{3}A(s,t,u) \sim \ln \mu^2(f) - \ln f + finite \sim finite.$$
(203)

The obvious way to accomplish this is if

$$\mu^2(f) \sim f s_0 \,. \tag{204}$$

If we do not make this change in μ as a function of f, then $\lambda(\mu)$ (the leading term) would not be a very good measure of the strength of the 4-point interaction in the high energy experiment. If we do scale up μ^2 by a factor of f, then there are no big logarithmic factors multiplying the $\lambda^2(\mu)$ correction to the leading $\lambda(\mu)$ term in Eq (200). As a result, the leading $\lambda(\mu)$ component of Eq. (200) is a proper measure of the strength of the interaction at this higher energy scale if $\mu^2 \sim fs_0$ since the $\lambda^2(\mu) \times$ modest size stuff is a small correction (assuming, of course, that $\lambda(\mu)$ remains a smallish parameter as μ increases). Thus, we will wish to know how $\lambda(\mu)$ behaves as μ^2 is increased by some factor f. This is clearly determined by the functional form of $\beta(\lambda)$ through the differential equation given in Eq. (195).

In short, to avoid ratios of large scales in the propagator and vertex after renormalization, we should take μ of order the energy scale E of the experimental probe. Then, very roughly,

 $4 - point \ scattering \ amplitude|_{E} \sim -i\lambda(\mu = E), \qquad (205)$

so that the behavior of $\lambda(\mu)$ as a function of μ reveals the physical behavior of 4-point scattering as a function of E.

So, now let us solve the differential equation

$$\mu \frac{\partial \lambda(\mu)}{\partial \mu} = \frac{3\lambda^2(\mu)}{16\pi^2} \tag{206}$$

by writing it in the form

$$\frac{d\lambda}{\lambda^2} = \frac{d\mu}{\mu} \frac{3}{16\pi^2}$$
(207)

which integrates to

$$-\frac{1}{\lambda} = \frac{3}{16\pi^2} \ln \mu + c$$
 (208)

where c can be fixed by specifying λ at some low μ scale, say μ_s :

$$-\frac{1}{\lambda_s} = \frac{3}{16\pi^2} \ln \mu_s + c.$$
 (209)

Subtracting, we obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_s} = \frac{3}{16\pi^2} \ln \frac{\mu_s}{\mu}$$
(210)

which can be rewritten in the form

$$\lambda = \frac{\lambda_s}{1 - \frac{3}{16\pi^2} \lambda_s \ln \frac{\mu}{\mu_s}},\tag{211}$$

where, to repeat, $\lambda_s = \lambda(\mu = \mu_s)$.

Note that λ increases with increasing μ if we start with a small λ_s . At larger μ (shorter distance) this means that the number of orders of λ needed for a reliable result must increase. Eventually, we would leave the domain where λ is sufficiently small that a perturbative result is trustworthy, regardless of how many orders in λ are kept.

Conversely, in $\lambda \phi^4$ theory, perturbation theory becomes more reliable at larger distances (small μ). Further, since λ is small at large distances, the perturbative approach to defining asymptotic states should also be reliable.

However, if the sign of $\mu \frac{\partial \lambda}{\partial \mu}$ were to be negative, then quite the reverse situation arises:

1. short distance behavior would be perturbative;

2. long distance behavior would not be easily computed, and asymptotic states might not be easily defined.

A negative β function is characteristic of non-Abelian groups, most especially QCD based on SU(3).

Anyway, now let's turn to the evolution of m^2 . We had

$$\frac{1}{2}\mu \frac{\partial \ln m^2}{\partial \mu} = \frac{\lambda(\mu)}{32\pi^2}$$
(212)

which implies

$$\ln m^2 = \int \frac{d\mu}{\mu} \frac{\lambda(\mu)}{16\pi^2}.$$
(213)

To integrate this easily, we need to work a bit on our $\lambda(\mu)$ solution:

$$\lambda = \frac{\lambda_s}{1 - \frac{3}{16\pi^2}\lambda_s \ln \frac{\mu}{\mu_s}}$$
$$= \frac{1}{\frac{1}{\frac{1}{\lambda_s} - \frac{3}{16\pi^2} \ln \frac{\mu}{\mu_s}}}$$
$$= \frac{-1}{\frac{3}{16\pi^2} \ln \frac{\mu}{\Lambda}}$$

(214)

provided we define

$$\frac{1}{\lambda_s} \equiv \frac{3}{16\pi^2} \ln \frac{\Lambda}{\mu_s} \,. \tag{215}$$

For ϕ^4 theory, it is convenient to think of using this expression for $\mu < \Lambda$ so that λ as written above will be positive (as needed for stability against large values of ϕ). We then obtain

$$\ln m^{2} = -\frac{1}{3} \int \frac{d \ln \frac{\mu}{\Lambda}}{\ln \frac{\mu}{\Lambda}}$$
$$= -\frac{1}{3} \ln \left(\ln \frac{\mu}{\Lambda} \right) + c, \qquad (216)$$

implying that

$$m^{2} = e^{c} \left(\ln \frac{\mu}{\Lambda} \right)^{-1/3} \equiv \overline{c} \left[-\lambda(\mu) \right]^{1/3} .$$
 (217)

Usually the integration constant is determined by the requirement that

$$m^2(m^2_{physical}) = m^2_{physical} \tag{218}$$

where $m^2_{physical}$ is the physical pole location. (This is not possible in QCD,

but it is OK in the Higgs sector.) Using this condition gives

$$\overline{c}[-\lambda(m_{physical})]^{1/3} = m_{physical}^2, \quad \Rightarrow \quad \overline{c} = m_{physical}^2 [-\lambda(m_{physical})]^{-1/3},$$
(219)

implying that

$$m^{2}(\mu) = m^{2}_{physical} \left[\frac{\lambda(\mu)}{\lambda(m_{physical})} \right]^{1/3} .$$
 (220)

In the present case of ϕ^4 theory, if $\mu \uparrow$ then $\lambda(\mu) \uparrow$ and, hence, $m^2(\mu) \uparrow$. The opposite happens in QCD.

The meaning of this type of $m^2(\mu)$ dependence will only become apparent when we discuss the RGE approach to all this. Basically, while the μ dependence of λ is primarily set by the need to cancel large logarithms, the μ dependence of m^2 is set up to cancel growth of subdominant $\ln \ln$ terms.

We can also examine what happens in the case of Z_{ϕ} . From the power series definition given earlier in Eq. (191) and a careful treatment of the ϵ expansions, we find

$$\mu \frac{\partial \ln Z_{\phi}}{\partial \mu} = -2\lambda \frac{dc_1}{d\lambda}.$$
(221)

Defining

$$\gamma_d = \frac{1}{2} \mu \frac{\partial \ln Z_\phi}{\partial \mu} = \frac{1}{2} \frac{\mu \frac{\partial Z_\phi}{\partial \mu}}{Z_\phi}$$
(222)

yields

$$\gamma_d(\lambda) = -\lambda \frac{dc_1}{d\lambda}.$$
 (223)

In ϕ^4 theory, the first non-zero term for c_1 is at order λ^2 (from the 3-finger or sunset propagator insertion graph) and one finds

$$\gamma_d(\lambda) = \frac{1}{12} \left(\frac{\lambda}{16\pi^2}\right)^2 + \mathcal{O}(\lambda^3) \,. \tag{224}$$

 γ_d is called the "anomalous dimension" for reasons that will become apparent when we turn to the RGEs.

Incidentally, you will have noticed that $\frac{\lambda}{16\pi^2}$ appears as a factor in all of our one-loop type contributions. It is this that is the real expansion parameter. Obviously, the typical one-loop integration factor $\frac{1}{16\pi^2}$ greatly helps the convergence of the perturbation series.

Renormalization Group Equations for ϕ^4 .

The renormalization group is a group of transformations in which $\mu \to e^t \mu$. For our purposes, it is characterized by the $\mu \frac{\partial}{\partial \mu}$ type derivatives.

Let us examine a general Green's function (after amputating the external legs):

$$\Gamma_0^N(p_1 \dots p_N; \lambda_0, m_0, \epsilon) = Z_\phi^{-N/2} \Gamma^N(p_1 \dots p_N; \lambda(\mu), m(\mu), \mu, \epsilon) \quad (225)$$

where Γ^N is the renormalized amputated Green's function and is finite as $\epsilon \to 0$. The Γ_0 is independent of μ so that, using the chain rule,

$$\mu \frac{\partial \Gamma_0^N}{\partial \mu} = 0$$

$$= \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m} - \frac{N}{2} \frac{\mu \frac{\partial Z_{\phi}}{\partial \mu}}{Z_{\phi}} \right] \Gamma^N$$

$$= \left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + m \gamma_m(\lambda) \frac{\partial}{\partial m} - N \gamma_d(\lambda) \right] \Gamma^N. \quad (226)$$

This is the renormalization group equation for Γ^N .

We can derive another equation obeyed by Γ^N by noting that it has an "engineering" dimension equal to

$$4 - N + \epsilon(N - 2) \,. \tag{227}$$

This is most easily seen by noting that the effective term induced in \mathcal{L} by Γ^N is $\propto \Gamma^N \phi^N$ and we know that

$$\int d^{n}x \Gamma^{N} \phi^{N} = dimensionless \tag{228}$$

while ϕ has dimensions $M^{\frac{n}{2}-1}$ as argued earlier. So, if Γ^N has dimension $M^{n-N(\frac{n}{2}-1)}$ then

$$\int d^{n}x \Gamma^{N} \phi^{N} \propto M^{-n} M^{n-N(\frac{n}{2}-1)} M^{N(\frac{n}{2}-1)} \propto M^{0}.$$
 (229)

Substituting $n = 4 - 2\epsilon$ gives

$$M^{n-N(\frac{n}{2}-1)} = M^{4-2\epsilon - N(1-\epsilon)} = M^{4-N+\epsilon(N-2)}.$$
 (230)

This dimension can only be carried by the dimensionful parameters of Γ^N . Hence, we must have

$$\left[\mu\frac{\partial}{\partial\mu} + s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} - \left\{4 - N + \epsilon(N-2)\right\}\right]\Gamma^N(sp_1\dots sp_N; m, \lambda, \mu, \epsilon) = 0, \quad (231)$$

where we have used *s* to set the scale of the momenta. In writing the above, we have used the fact that Γ^N will be a sum of terms, each of which is a product of various powers of our dimension-carrying objects and that these powers are "probed" by the logarithmic $\mu \frac{\partial}{\partial \mu}$, ... derivatives. For example,

$$\mu \frac{\partial}{\partial \mu} \mu^{p} = p \mu^{p}$$

$$m \frac{\partial}{\partial m} m^{q} = q m^{q}$$

$$s \frac{\partial}{\partial s} (sp_{1} \cdot sp_{2})^{r} = s \frac{\partial}{\partial s} s^{2r} (p_{1} \cdot p_{2})^{r} = 2r s^{2r} (p_{1} \cdot p_{2})^{r} = 2r (sp_{1} \cdot sp_{2})^{r}$$
(232)

and similarly for other invariants formed from momenta.

We use this 2nd equation to eliminate $\mu \frac{\partial}{\partial \mu}$ from Eq. (226) to obtain in the

limit of $\epsilon \rightarrow 0$:

$$\left[-s\frac{\partial}{\partial s} - m\frac{\partial}{\partial m} + (4-N) + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_m(\lambda)m\frac{\partial}{\partial m} - N\gamma_d(\lambda)\right]\Gamma^N(sp;m,\lambda,\mu) = 0, \quad (233)$$

or, using the notation $4 - N = d_N$, the "naive" engineering dimension in 4 dimensions,

$$\left[-s\frac{\partial}{\partial s}+\beta(\lambda)\frac{\partial}{\partial \lambda}+(\gamma_m(\lambda)-1)m\frac{\partial}{\partial m}+d_N-N\gamma_d(\lambda)\right]\Gamma^N(sp;m,\lambda,\mu)=0.$$
 (234)

(At all stages above, one should keep in mind that $\lambda = \lambda(\mu)$ and $m = m(\mu)$.) Eq. (234) expresses the fact that a change in s may be compensated by a change in m and λ and an overall engineering factor. So we expect the solution to be of the form:

$$\Gamma^{N}(sp; m, \lambda, \mu) = f(s)\Gamma^{N}(p; m(s), \lambda(s), \mu)$$
(235)

in which case

$$\frac{\partial}{\partial s}\Gamma^{N}(sp;m,\lambda,\mu) = \frac{\partial f(s)}{\partial s}\Gamma^{N}(p;m(s),\lambda(s),\mu) + f(s) \left[\frac{\partial m(s)}{\partial s}\frac{\partial \Gamma^{N}(p;\ldots)}{\partial m} + \frac{\partial \lambda(s)}{\partial s}\frac{\partial \Gamma^{N}(p;\ldots)}{\partial \lambda}\right]$$
(236)

$$srac{\partial}{\partial s}\Gamma^{m N}(sp;m,\lambda,\mu) \quad = \quad \left[srac{\partial f}{\partial s} + f(s)srac{\partial m(s)}{\partial s}rac{\partial}{\partial m} + f(s)srac{\partial\lambda(s)}{\partial s}rac{\partial}{\partial\lambda}
ight]\Gamma^{m N}(p;m(s),\lambda(s),\mu)$$
$$= \left[s\frac{\partial f}{\partial s} + f(s)s\frac{\partial m(s)}{\partial s}\frac{\partial}{\partial m} + f(s)s\frac{\partial\lambda(s)}{\partial s}\frac{\partial}{\partial\lambda}\right]\frac{1}{f(s)}\Gamma^{N}(sp;m,\lambda,\mu)$$
(237)

which we choose to rewrite in the form:

$$\left[-s\frac{\partial}{\partial s} + \frac{s}{f(s)}\frac{\partial f(s)}{\partial s} + s\frac{\partial m(s)}{\partial s}\frac{\partial}{\partial m} + s\frac{\partial \lambda(s)}{\partial s}\frac{\partial}{\partial \lambda}\right]\Gamma^{N}(sp;m,\lambda,\mu) = 0$$
(238)

We now compare Eq. (234) to Eq. (238). Matching coefficients, we conclude

$$s \frac{\partial \lambda(s)}{\partial s} = \beta(\lambda(s))$$
 (239)

$$s\frac{\partial m(s)}{\partial s} = m[\gamma_m(\lambda(s)) - 1]$$
(240)

$$\frac{s}{f(s)}\frac{\partial f(s)}{\partial s} = d_N - N\gamma_d(\lambda(s)). \qquad (241)$$

The last equation can be integrated to give

$$f(s) = s^{d_N} \exp\left[-\int_1^s \frac{N\gamma_d(\lambda(s'))ds'}{s'}\right], \qquad (242)$$

as can be explicitly checked:

$$s\frac{\partial f}{\partial s} = d_N s^{d_N} \exp\left[\ldots\right] + s^{d_N} s\left(\frac{-N\gamma_d(\lambda(s))}{s}\right) \exp\left[\ldots\right]$$
$$= [d_N - N\gamma_d(\lambda(s))] s^{d_N} \exp\left[\ldots\right]$$
$$= [d_N - N\gamma_d(\lambda(s))] f(s) . \tag{243}$$

Substituting the form of f(s) given above into Eq. (235), we obtain

$$\Gamma^{N}(sp;m,\lambda,\mu) = s^{d_{N}=4-N} \exp\left[-\int_{1}^{s} \frac{N\gamma_{d}(\lambda(s'))ds'}{s'}\right] \Gamma^{N}(p;m(s),\lambda(s),\mu)$$
(244)
(244)
where $\lambda(s)$ and $m(s)$ are solutions of the equations (239) and (240) given

where $\lambda(s)$ and m(s) are solutions of the equations (239) and (240) given earlier, with implicit boundary conditions of

$$\lambda(s=1) = \lambda$$
, and $m(s=1) = m$. (245)

These equations are, of course, nothing more than the moving coupling constant and moving mass equations that we have already discussed where we would write $\mu = s\mu_{s=1}$ from which we see that $\mu \frac{\partial}{\partial \mu} = s \frac{\partial}{\partial s}$.

To make the boundary conditions explicit, we can write

$$\lambda(s) = \overline{\lambda}(s, \lambda), \dots \tag{246}$$

Sometimes, it is convenient to use

$$t = \ln s, \quad \Rightarrow \quad s \frac{\partial}{\partial s} = \frac{\partial}{\partial t},$$
 (247)

in which case, we would have

$$\Gamma^{N}(sp;m,\lambda,\mu) = s^{d_{N}=4-N} \exp\left[-\int_{0}^{t} N\gamma_{d}(\overline{\lambda}(t',\lambda))dt'\right] \Gamma^{N}(p;\overline{m}(t,\lambda,m),\overline{\lambda}(t,\lambda),\mu). \quad (248)$$

In any case, our solution for $\Gamma^N(sp; m, \lambda, \mu)$ says that the *N*-point function at sp can be obtained from that at p by:

- 1. supplying naive dimensionful scaling;
- 2. including an extra scale dependence that comes from the wave-function renormalization associated with each external leg;

If γ_d were a constant then $\int_0^t \gamma_d dt' = \gamma_d \ln s$ and the $\exp[\ldots] \rightarrow s^{-N\gamma_d}$.

As a result, γ_d is referred to as the anomalous dimension associated with each external leg.

3. changing m and λ to their moving values.

The form of $\beta(\lambda)$ and the ultimate fate of λ

Some of the possibilities for the running coupling constant can be illustrated by the sample graph for $\beta(\lambda)$ shown in Fig. 9, which might be the result if one could compute exactly in the context of some theory (here we are going beyond ϕ^4 possibly).



Figure 9: Graphical illustration of a possible $\beta(\lambda)$. Arrows indicate the direction of flow of λ as t increases.

The points $\lambda = 0, \lambda_1, \lambda_2$ where $\beta(\lambda)$ vanishes are called fixed points. If the value of λ is such that $\beta(\lambda) = 0$ then λ will not evolve any further once

 λ reaches such a value. More precisely, using our notation of $\overline{\lambda}(t,\lambda)$, where $\overline{\lambda}(t=0,\lambda) = \lambda$ — i.e. the value λ referenced in the functional form is the boundary condition value, then if $\frac{d}{dt}\overline{\lambda}(t,\lambda) = \beta(\overline{\lambda}) = 0$ we conclude that $\overline{\lambda}(t,\lambda)$ does not change after reaching such a fixed point.

Note that since perturbation theory tells us (independent of the type of theory) that $\beta(\lambda) \to 0$ as $\lambda \to 0$, $\lambda = 0$ is always a fixed point.

Furthermore, we can distinguish two types of fixed points.

1. Consider $\overline{\lambda}$ near λ_1 :

For $\overline{\lambda} < \lambda_1$, $\beta(\overline{\lambda}) > 0$ and so $\frac{d}{dt}\overline{\lambda} > 0$ implying that $\overline{\lambda}$ is driven towards λ_1 as $t \uparrow$.

For $\lambda_1 < \overline{\lambda} < \lambda_2$, $\beta(\overline{\lambda}) < 0$ and so $\frac{d}{dt}\overline{\lambda} < 0$, implying that $\overline{\lambda}$ is again driven towards λ_1 as $t \uparrow$.

In short, for $0 < \overline{\lambda} < \lambda_2$, $\overline{\lambda}$ is always driven $\rightarrow \lambda_1$ as t increases (i.e. as momenta become larger). Such a fixed point is called an "ultraviolet stable" fixed point.

2. Similarly, for $\overline{\lambda}$ near $\overline{\lambda} \sim 0$ or $\overline{\lambda}$ near $\overline{\lambda} \sim \lambda_2$, $\overline{\lambda}$ is driven to these values as $t \to 0$, i.e. for small momenta.

Thus, $\lambda = 0$ and $\lambda = \lambda_2$ are "infrared stable' fixed points.

Of course, if $\beta(\lambda)$ starts at 0 at $\lambda = 0$ and then becomes negative for $\lambda > 0$, then for $\overline{\lambda} > 0$ we have $\frac{d}{dt}\overline{\lambda} = \beta(\overline{\lambda}) < 0$, implying that $\overline{\lambda} \to 0$ as $t \uparrow$. In this case, $\lambda = 0$ is an ultraviolet stable fixed point.

Note that since $\beta(\lambda) = \frac{3\lambda^2}{32\pi^2} > 0$ for λ near 0, ϕ^4 is a theory for which $\lambda = 0$ is infrared stable.

It turns out that only gauge theories can yield $\beta(\lambda) < 0$ for λ near 0, and thus, only these can be "asymptotically free" (i.e. $\lambda \to 0$ for large t, i.e. large momenta).

Let us now return to our original $\beta(\lambda)$ graph. As we have seen, for $0 < \overline{\lambda} < \lambda_2$,

$$\lim_{t \to \infty} \lambda(t, \lambda) = \lambda_1$$
$$\lim_{t \to \infty} \Gamma^N(p_i, \overline{\lambda}(t, \lambda), \overline{m}(t, m, \lambda), \mu) = \Gamma^N(p_i, \lambda_1, \overline{m}(t, m, \lambda_1), \mu)$$
(249)

Let us not worry about \overline{m} for now, and instead focus on the effects of the presence of γ_d . Suppose $\beta(\lambda)$ has a simple 0 at λ_1 [i.e. $\beta(\lambda) = a(\lambda_1 - \lambda)$ with a > 0 according to the graph] and $\gamma_d(\lambda_1) \neq 0$. Then, from $\frac{d\overline{\lambda}}{dt} = a(\lambda_1 - \overline{\lambda})$

we obtain

$$\overline{\lambda}(t,\lambda) = \lambda_1 + (\lambda - \lambda_1)e^{-at}.$$
(250)

Let us use this to compute the prefactor in front of $\Gamma^N(sp:...)$ in our earlier equation:

$$\Gamma^{N}(sp;m,\lambda,\mu) = s^{d_{N}=4-N} \exp\left[-\int_{0}^{t} N\gamma_{d}(\overline{\lambda}(t',\lambda))dt'\right] \Gamma^{N}(p;\overline{m}(t,\lambda,m),\overline{\lambda}(t,\lambda),\mu). \quad (251)$$

We have [shifting to the integration variable $y = \overline{\lambda}(t', \lambda)$ and using $\frac{d}{dt'}\overline{\lambda}(t', \lambda) = \beta(\overline{\lambda}(t', \lambda))$]

$$\begin{split} \int_{0}^{t} \gamma_{d}(\overline{\lambda}(t',\lambda)) dt' &= \int_{\lambda}^{\overline{\lambda}} \frac{\gamma_{d}(y) dy}{\beta(y)} \\ &\simeq \frac{-\gamma_{d}(\lambda_{1})}{a} \int_{\lambda}^{\overline{\lambda}} \frac{d\lambda'}{\lambda' - \lambda_{1}} \\ &= -\frac{\gamma_{d}(\lambda_{1})}{a} \ln \left(\frac{\overline{\lambda} - \lambda_{1}}{\lambda - \lambda_{1}} \right) \\ &= \gamma_{d}(\lambda_{1}) t \\ &= \gamma_{d}(\lambda_{1}) \ln s \,. \end{split}$$

(252)

Then, plugging in, we obtain

$$\Gamma^{N}(sp,\lambda,\mu) = s^{d_{N}-N\gamma_{d}(\lambda_{1})}\Gamma^{N}(p_{i},\lambda_{1},\mu), \qquad (253)$$

implying that, in the high-momentum limit, *each* of the N fields scales with an anomalous dimension given by $\gamma_d(\lambda_1)$.

A few final remarks/reminders:

A general computation of $\beta(\lambda)$ is not possible, but near $\lambda = 0$ perturbation theory is OK, and this region, fortunately, is very useful for comparison to experiment.

For instance, for Bjorken scaling, ..., our results indicate that to all intensive purposes the quarks behave as if almost free when considering the asymptotic limit in $p, Q^2, \ldots \rightarrow \infty$ provided $\lambda = 0$ is an ultraviolet stable fixed point. As we have said, only gauge theories such as QCD have this property.

Before proceeding onto QCD and QED, let me return to using the power series of Eqs. (191) for a more complete and extendable derivation of, for example, the form of $\beta(\lambda)$. Our equations were:

$$\lambda_0 \;\; = \;\; (\mu^2)^\epsilon \left[\lambda + \sum_{k=1}^\infty rac{a_k(\lambda)}{\epsilon^k}
ight]$$

$$m_0^2 = m^2 \left[1 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right]$$
$$Z_{\phi} = \left[1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k} \right].$$
(254)

The results obtained so far to one-loop are, from Eqs. (189) and (190):

$$a_1 = \frac{3\lambda^2}{32\pi^2}, \quad b_1 = \frac{\lambda}{32\pi^2}, \quad c_1 = 0.$$
 (255)

Since the coefficients inside the [...]'s do not depend on the mass scales, m, μ , this scheme that we are employing is also called the *mass-independent* renormalization scheme. The m, μ dependence can be avoided to arbitrarily high order.

Intuitively, the counterterms have just the minimal structure needed to cancel ∞ 's coming from big momenta in the loop integrals, and in the *momenta* $\rightarrow \infty$ limit, all masses are irrelevant. Mass dependence can then only appear through the finite parts of the counterterms, and we have chosen a scheme in which they do not.

To see more generally why this scheme is so useful, let us reexamine the coupling constant relation (as an example). Differentiating the top equation above with respect to μ at fixed λ_0 (just as we did before) gives

$$0 = 2\epsilon \left(\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}\right) + \mu \frac{\partial \lambda}{\partial \mu} \left(1 + \sum_{k=1}^{\infty} \frac{a'_k(\lambda)}{\epsilon^k}\right).$$
(256)

Now, we know that λ and $\mu \frac{\partial \lambda}{\partial \mu}$ are analytic at $\epsilon \to 0$. As a result, we can write

$$\mu \frac{\partial \lambda}{\partial \mu} = d_0(\lambda) + d_1(\lambda)\epsilon + d_2(\lambda)\epsilon^2 + \dots \qquad (257)$$

We can then examine the preceding equation in the form:

$$2\epsilon\lambda + (2a_1 + \mu\frac{\partial\lambda}{\partial\mu}) + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \left[\frac{da_k}{d\lambda} \mu \frac{\partial\lambda}{\partial\mu} + 2a_{k+1} \right] = 0$$
(258)

keeping in mind the substitution of Eq. (257). Matching powers of ϵ gives

$$\mathcal{O}(\epsilon^{n>1}) \hspace{0.2cm} \leftrightarrow \hspace{0.2cm} d_n + \sum_{k=1}^{\infty} rac{da_k}{d\lambda} d_{n+k} = 0 \Rightarrow d_{n\geq 2} = 0$$

$$\mathcal{O}(\epsilon^{1}) \quad \leftrightarrow \quad 2\lambda + d_{1} = 0 \Rightarrow d_{1} = -2\lambda$$

$$\mathcal{O}(\epsilon^{0}) \quad \leftrightarrow \quad 2a_{1} + d_{0} + d_{1}\frac{da_{1}}{d\lambda} = 0 \Rightarrow d_{0} = -2a_{1} + 2\lambda\frac{da_{1}}{d\lambda}$$

$$\mathcal{O}(\frac{1}{\epsilon^{r} \geq 1}) \quad \leftrightarrow \quad 2a_{r+1} + d_{1}\frac{da_{r+1}}{d\lambda} + d_{0}\frac{da_{r}}{d\lambda} = 0 \Rightarrow \dots \qquad (259)$$

Regarding the $\epsilon^{n>1}$ result, we obtain a set of linear equations for d_2, d_3, d_4, \ldots , all with 0 on the right-hand side. So long as the matrix involved has an inverse, the only solution is $d_2 = d_3 = \ldots = 0$. There is no reason for the matrix defined by the $\frac{da_k}{d\lambda}$ not to have an inverse.

The most important result from the above is that from reading off the d_0 and d_1 coefficients:

$$\mu \frac{\partial \lambda}{\partial \mu} = d_0 + d_1 \epsilon = (-2a_1 + 2\lambda \frac{da_1}{d\lambda}) - 2\lambda \epsilon$$
$$\stackrel{\epsilon \to 0}{\to} -2(1 - \lambda \frac{d}{d\lambda})a_1(\lambda)$$
$$= \beta(\lambda).$$
(260)

Note that $\beta(\lambda)$ depends only λ and is determined by the residue of the

simplest $\frac{1}{\epsilon}$ single pole term. In leading order, we have computed $a_1(\lambda) = \frac{3\lambda^2}{32\pi^2}$ from which the form of β that we have been using is obtained:

$$egin{array}{rll} eta(\lambda)&=&-2(1-\lambdarac{d}{d\lambda})rac{3\lambda^2}{32\pi^2}+\dots\ &=&-2(\lambda^2-2\lambda^2)rac{3}{32\pi^2}\ &=&rac{3\lambda^2}{16\pi^2}, \end{array}$$

as obtained earlier.

The $r \geq 1$ case can be rewritten as

$$egin{array}{rl} 2a_{r+1}+d_1rac{da_{r+1}}{d\lambda}+d_0rac{da_r}{d\lambda}&=&0\ \Rightarrow 2(1-\lambdarac{d}{d\lambda})a_{r+1}&=&-d_0rac{da_r}{d\lambda}\ &=&(2a_1+d_1rac{da_1}{d\lambda})rac{da_r}{d\lambda} \end{array}$$

(261)

$$= (2a_1 - 2\lambda \frac{da_1}{d\lambda}) \frac{da_r}{d\lambda}$$
$$= 2(1 - \lambda \frac{d}{d\lambda}) a_1 \frac{da_r}{d\lambda}. \qquad (262)$$

This result is remarkable in that it says that all the higher pole terms can be computed once the r = 1 single pole residue is known. This is equivalent to saying that the leading logarithms, the next-to-leading logarithms, etc. can be calculated to all orders by using the renormalization group equation with the computation of just a few low-order terms.

We can repeat this same sort of analysis for m^2 . We have

$$m_0^2 = m^2 \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right]$$
(263)

where $b_0 = 1$ and $b_1 = \frac{\lambda}{32\pi^2}$ (to the order we have computed). In fact, $b_0 = 1$ to all orders since it is δm^2 that has the ϵ singularities. Taking the $\mu \frac{\partial}{\partial \mu}$

derivative gives

$$0 = \mu \frac{\partial m^2}{\partial \mu} \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right] + m^2 \left[\mu \frac{\partial b_0}{\partial \mu} + \sum_{k=1}^{\infty} \frac{\mu \frac{\partial b_k}{\partial \mu}}{\epsilon^k} \right] , \qquad (264)$$

where $\mu \frac{\partial b_0}{\partial \mu} = 0$. Now since b_k depends on μ only through λ , we can rewrite this as

$$\mu \frac{\partial m^2}{\partial \mu} \left[b_0 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k} \right] = -m^2 \sum_{k=1}^{\infty} \frac{\mu \frac{\partial \lambda}{\partial \mu} \frac{\partial o_k}{d\lambda}}{\epsilon^k}.$$
 (265)

At this point, we recall that

$$\mu \frac{\partial \lambda}{\partial \mu} = d_0 + d_1 \epsilon = -2(1 - \lambda \frac{d}{d\lambda})a_1 - 2\lambda\epsilon, \qquad (266)$$

and proceed with power series matching to obtain:

$$\mathcal{O}(\epsilon^{0}) \quad \leftrightarrow \quad \mu \frac{\partial m^{2}}{\partial \mu} = 2\lambda m^{2} \frac{db_{1}}{d\lambda}$$
(267)
$$\mathcal{O}(\frac{1}{\epsilon^{r} \geq 1}) \quad \leftrightarrow \quad \mu \frac{\partial m^{2}}{\partial \mu} b_{r} = -m^{2} \left[-2(1-\lambda \frac{d}{d\lambda})a_{1} \frac{db_{r}}{d\lambda} - 2\lambda \frac{db_{r+1}}{d\lambda} \right]$$
(268)

where this latter equation can be rearranged by canceling a common factor of $2m^2$ to read

$$\lambda \frac{db_{r+1}}{d\lambda} = \lambda \frac{db_1}{d\lambda} b_r - \frac{db_r}{d\lambda} (1 - \lambda \frac{d}{d\lambda}) a_1.$$
 (269)

Obviously, Eq. (267) defines a running mass equation of the form:

$$\gamma_{m} \equiv \frac{1}{2} \mu \frac{\partial \ln m^{2}}{\partial \mu} = \frac{1}{2} \frac{\mu \frac{\partial m^{2}}{\partial \mu}}{m^{2}}$$
$$= \lambda \frac{db_{1}}{d\lambda}$$
$$= \frac{\lambda}{32\pi^{2}}$$
(270)

to the one-loop order computed, in agreement with our earlier result. This is supplemented with the recursion relation for the higher $\frac{1}{\epsilon}$ power singularity residues given in Eq. (269).

Finally, we have the Z_{ϕ} expansion:

$$Z_{\phi} = 1 + \sum_{k=1}^{\infty} \frac{c_k(\lambda)}{\epsilon^k}, \qquad (271)$$

from which we compute

$$\frac{1}{2}\mu \frac{\partial Z_{\phi}}{\partial \mu} = \frac{1}{2}\mu \frac{\partial \lambda}{\partial \mu} \left[\frac{dc_1}{d\lambda} \frac{1}{\epsilon} + \frac{dc_2}{d\lambda} \frac{1}{\epsilon^2} + \dots \right]$$
$$= \gamma_d \left[1 + \frac{c_1}{\epsilon} + \frac{c_2}{\epsilon^2} \right]$$
(272)

using $\gamma_d \equiv \frac{1}{2} \mu \frac{\partial Z_{\phi}}{\partial \mu} Z_{\phi}^{-1}$. Once again, we insert

$$u\frac{\partial\lambda}{\partial\mu} = d_0 + d_1\epsilon \tag{273}$$

with $d_0 = \beta(\lambda)$ and $d_1 = -2\lambda$, and match powers of ϵ . The main result is the ϵ^0 coefficient matching equation:

$$\frac{1}{2}d_1\frac{dc_1}{d\lambda} = \gamma_d \tag{274}$$

which converts to

$$\gamma_d = -\lambda \frac{dc_1}{d\lambda}, \qquad (275)$$

as quoted earlier. The more singular $\frac{1}{\epsilon^{r\geq 1}}$ matching equations give recursion relations for $c_{k\geq 2}$. For example, the $\mathcal{O}(\epsilon^{-1})$ matching gives

$$d_1 \frac{dc_2}{d\lambda} + \frac{1}{2} d_0 \frac{dc_1}{d\lambda} = \gamma_d c_1 \tag{276}$$

which allows a computation of c_2 once the residue of the single pole, c_1 , is known.

(

The prototype here is, of course, QED. Our Feynman rules are:

$$\gamma e^{-}e^{+} \ vertex \quad \rightarrow \quad -ie\gamma_{\mu}$$

$$photon \ propagator \quad \rightarrow \quad \frac{-ig_{\mu\nu}}{k^{2}}$$

$$fermion \ propagator \quad \rightarrow \quad \frac{i}{k-m}.$$
(277)

Degree of Divergence

Denoting by L the number of loops, IB the number of internal vector boson propagators and IF the number of internal fermion lines,

$$D = 4L - 2IB - IF.$$
⁽²⁷⁸⁾

We want to eliminate IB and IF in terms of external particle counts: EB, the number of external photons; and EF, the number of external fermion or

antifermion lines. Note that EF must be even since every incoming fermion line must exit somewhere. Let us use the notation V for the number of internal vertices, each of which comes with a $\delta^4(\ldots)$. And, there is one overall $\delta^4(\sum p_{external})$ that must pop out.

Thus, the number of independent loops is given by

$$L = IB + IF - V + 1.$$
 (279)

The total number of fermion lines available to attach to vertices is 2IF + EF (each internal line can run from one vertex to another, but each external line has only one end to attach to a vertex). Since each vertex eats up 2 fermion lines, we have

$$2IF + EF = 2V$$
, or $IF = V - \frac{1}{2}EF$. (280)

Similarly, the number of photon lines available to attach to vertices is 2IB + EB, but, each vertex accepts only one photon line. Hence,

$$2IB + EB = V$$
, or $IB = \frac{1}{2}(V - EB)$. (281)

Substituting L into the D expression, and then the results for IF and IB, we obtain

$$D = 2IB + 3IF - 4V + 4 = (V - EB) + (3V - \frac{3}{2}EF) - 4V + 4 = 4 - \frac{3}{2}EF - EB.$$
(282)

Note: there is no dependence on V, the number of vertices, *i.e.* the order of the perturbative graph.

Generalization to QCD

If we go to QCD, we have to add the 3 and 4 gluon vertices. There are also ghost-ghost gluon vertices, but these act completely parallel to the 3-gluon vertex and need not be treated separately — for every diagram containing two internal gluons connected to two 3-gluon vertices, there is a corresponding diagram in which the two internal gluons are replaced by ghosts. (Of course, there are no external ghosts.) The important new thing to keep track of is that the 3G (and also ghost-ghost gluon) vertex is proportional to a single power of momenta.

Thus, our D expression is modified to

$$D = 4L - IF - 2IB + V_{3G}$$
(283)

while L is as before

$$L = IB + IF - V + 1, \quad V = V_{GFF} + V_{3G} + V_{4G}.$$
(284)

Substituting into D we obtain

$$D = 2IB + 3IF - 4V + 4 + V_{3G}.$$
 (285)

Now, we also have, just like QED,

$$2IF + EF = 2V_{GFF}, \quad \rightarrow \quad IF = V_{GFF} - \frac{1}{2}EF, \qquad (286)$$

and the generalized result

$$2IB + EB = 3V_{3G} + V_{GFF} + 4V_{4G}, \quad \rightarrow \quad 2IB = 3V_{3G} + V_{GFF} + 4V_{4G} - EB,$$
(287)

where the coefficients are simply the number of gluons entering a given kind of vertex.

We now substitute these results into the last expression for D to obtain:

$$D \;\;=\;\; (3V_{3G}+V_{GFF}+4V_{4G}-EB)+(3V_{GFF}-rac{3}{2}EF)-4V+4+V_{3G}$$

$$egin{array}{rll} &=& 4(V_{3G}+V_{GFF}+V_{4G})-4V+4-EB-rac{3}{2}EF \ &=& 4-EB-rac{3}{2}EF \,, \end{array}$$

i.e. exactly the same result as in QED!

As expected, in both cases we are dealing with what we hope are renormalizable theories since they have dimensionless coupling constants (in 4 dimensions) and the degree of divergence is purely determined by the external line counts. This means that there are only a certain subset of diagrams that can be primitively divergent. Hopefully, this subset of diagrams can be counter-termed by $\Delta \mathcal{L}$ structures that are like those already present in \mathcal{L} .

The basic divergent diagrams in QED and QCD are summarized in the table below. The superficial degrees of divergence are those given by Eq. (288).

(288)

The actual degrees of divergence will be discussed below.

Tabulation of Divergent Diagrams (Tadpole & ghost diagrams not shown)



For QED, just eliminate all diagrams with 3g or 4g vertices.

The above table does not show the diagrams in which a single photon or gluon attaches to a closed fermion loop, nor the QCD diagram in which a single gluon attaches to a closed gluon loop.

- The superficial D for these diagrams is D = 3.
- But, in fact the diagrams are zero.

This is because they correspond to the vacuum expectation value of an electromagnetic (color) current coupling to the polarization vector for a photon (gluon).

But, the vacuum expectation value of any such current must be zero if the vacuum is not to have a preferred 4-vector direction, contrary to invariance of the vacuum under Lorentz transformations.

• In the case of QED, the vanishing of this type of diagram can also be derived assuming that the vacuum is charge conjugation invariant.

A non-zero value of the vacuum expectation value of the electromagnetic current would change sign under C.

- Another way of realizing that these diagrams must be zero is to note that it would be inconsistent for the *vacuum* to carry a net electromagnetic or color current.
- In fact, as discussed momentarily, it is easy at one-loop to explicitly show that these diagrams are zero.

In general, there are many subtleties that can cause the actual degree of divergence to differ from the superficial value of D. First is the fact that sometimes the naive degree of divergence is reduced by virtue of gauge invariance requirements on the structure of the diagram or by symmetry. The important cases are:

1. the bubble insertions into a gluon or photon propagator.

Gauge invariance demands that the sum of the diagrams be proportional to $k_{\mu}k_{\nu} - k^2g_{\mu\nu}$, i.e. there are two powers of *external* momentum that must pop out, thereby decreasing the value of D by 2 units. Actually, this works separately for fermion and gluon loops.

2. the gauge loop correction to a fermion propagator.

We can understand easily at one loop what the issue is. Consider very large values of the loop integration momentum l. Since the internal fermion propagator goes like $1/l = l/l^2$ while the internal gauge propagator goes like $1/l^2$, the divergence would be $\int d^4 l \frac{l}{(l^2)^2}$ which is odd under $l \to -l$. Thus, the leading term cannot survive. Instead of being linearly divergent, the diagram is only log divergent.

3. the loop corrections to the four-gluon vertex.

Again, there is a gauge requirement that means this correction must be proportional to appropriate gauge structures for the external gluons. Roughly, one must have a momentum structure that is the Fourier transform of $(F_{\mu\nu})^4$, a dimension 8 form. So, even though D = 0 superficially, in fact D < 0. This works separately for fermion and gluon loops.

4. the fermion loop possibility for a three-photon or a three-gluon vertex.

In the three-photon case, there is the theorem that the fermion-loop correction must be 0 by virtue of charge conjugation symmetry (Furry's theorem). This is also needed for renormalizability in the sense that if we had a divergence for this graph, then the counter term (a 3 photon vertex) would correspond to an interaction that is not present in the QED

Lagrangian. This goes "counter" to the counter-term Lagrangian approach. In fact, it would mean a break down in gauge invariance; there should be no self-interactions of photons regardless of the order of perturbation theory considered.

In QCD, the fermion loop correction to the three-gluon vertex is non-zero. The actual D is 0 by virtue of the leading contribution being odd under $l \rightarrow -l$ for the loop integration, $\int d^4l \left(\frac{1}{l}\right)^3$.

5. the gluon loop corrections to the three-gluon vertex.

Here, gauge invariance structure requirements for the vertex (which means it must look like our bare vertex which was linear in *external* momenta arranged in the appropriate antisymmetrized form — constructed so that $p^{\alpha} \epsilon^{\beta} \epsilon^{\gamma} V_{\alpha\beta\gamma}(p,q,r) = \ldots = 0$ for example), reduce the degree of divergence from 1 to 0.

The second subtlety is that the divergence of a diagram can actually be larger than D when it contains divergent subdiagrams corresponding to the basic primitive divergences isolated above and discussed below. We will return to this point later.

For now, let us consider just our basic primitively divergent types of graphs. Before considering ghosts, we have:

- 1. gluon (or photon) propagator corrections;
- 2. fermion propagator corrections;
- 3. gluon (or photon) fermion-antifermion vertex corrections;
- 4. three-gluon vertex corrections (but not three-photon vertex corrections, since these are 0 by virtue of Furry's theorem).

Well, I hope it is obvious that all these divergences can be compensated by counter terms of the same form as terms already present in the bare Lagrangian.

After including ghosts, the additional divergent items are:

- 1. ghost propagator corrections;
- 2. gluon-ghost-ghost vertex corrections.

Again, the necessary counterterms have the same form as terms already present in the ghost addition to the bare Lagrangian.

Another interesting point will emerge once we consider dimensional regularization. One finds that the propagator tadpole diagrams are 0 for massless virtual propagators when regularized via dimensional regularization. This is another great advantage of dimensional regularization. This feature is related to the fact that dimensional regularization preserves gauge invariance. This is something that we shall return to once we do explicit calculations.

Let us next briefly return to discuss why, at one loop, the diagrams with one external photon or gluon are simply zero. In the case of QED, we have

$$\epsilon^{\mu} \int d^4 l \frac{\text{Tr}[\gamma_{\mu}(\not l + m)]}{l^2 - m^2} = \epsilon^{\mu} \int d^4 l \frac{4l_{\mu}}{l^2 - m^2} = 0$$
(289)

by $l \rightarrow -l$ symmetry. The same applies to the QCD fermion loop diagram. The QCD gluon loop diagram is zero partly because of similar symmetry arguments, but also requires using the same result for a *massless* gluon propagator in the internal loop as stated in the previous paragraph.

Next, let us return to the fact that a diagram can have D < 0, for example, and actually be divergent. An example of this kind is photon exchange between two scattering electrons in which a fermion loop is inserted

in the exchanged photon. Naively, D = -2 for this diagram (EF = 4, EB = 0), but the fermion loop insertion actually gives a D = 0 logarithmic divergence. This, however, is rendered finite once the photon propagator counter term component of $\Delta \mathcal{L}$ is introduced. So, the value of a diagram's superficial divergence D only applies once all divergent subdiagrams (at lower order in perturbation theory) have been rendered finite by the counterterm $\Delta \mathcal{L}$.

Before turning to the explicit calculations, it's useful to discuss a couple of additional things. First, let us understand a little bit more about nonrenormalizable cases. Suppose we lived in 6 dimensions instead of 4. Then, sticking to QED for simplicity (but a similar result appears in QCD), we would have

$$D = 6L - IF - 2IB = 4IB + 5IF - 6V + 6$$

= $(2V - 2EB) + (5V - \frac{5}{2}EF) - 6V + 6$
= $V + 6 - 2EB - \frac{5}{2}EF$. (290)

We see that D increases with the vertex number V which we know is a problem. To be more specific, suppose we had a diagram with 2 incoming

fermions, 2 outgoing fermions, one incoming photon and one outgoing photon. Suppose we constructed a V = 8 diagram for this case, being careful not to choose a diagram with subdiagrams having a higher degree of superficial divergence. One such diagram is constructed as follows. Take the two external photons and attach them to a closed fermion loop which then emits two virtual photons (#1 and #2). The first of these emits an external fermion (#1) and a virtual fermion, the latter turning into a 2nd external fermion (#2) and another (#3) virtual photon. Virtual photons #2 and #3 then both attach to a fermion line that begins with external fermion #3 and ends with external fermion #4, with a virtual fermion propagator between the attachments of the #2 and #3 virtual photons. This diagram would have D = 0. A counter term to correct for this would have the structure $(\overline{\psi}\psi)^2 A^2$ and/or $(\overline{\psi}A\psi)^2$ (I have not worked it out) with infinite coefficient constructed so that the $\frac{1}{\epsilon}$ structure of the counterterm cancels the $\frac{1}{\epsilon}$ of the explicit diagram. Well, obviously such a counterterm is not a structure that appears in the bare Lagrangian.

The moral is that one would have to introduce counter terms for diagrams with any arbitrarily large number of external fermions and photons. This is characteristic of a non-renormalizable theory.

Finally, we want to consider the connection between renormalizability and dimensionless coupling constant: gauge theory version.

Consider a QED-like theory with just one type of interaction or vertex, and

define:

- B = # of boson lines entering or exiting a vertex;
- F = # of fermion lines entering or exiting a vertex;
- d = # of derivatives associated with the vertex interaction.

Then we can generalize our earlier results as follows.

2IF + EF = FV2IB + EB = BV

(291)

and as before the number of loops is

$$L = IB + IF - V + 1, \qquad (292)$$

from which we derive (noting that each derivative at a vertex gives a momentum power upstairs — just like the 3-gluon vertex example)

D = 4L - 2IB - IF + dV

$$= 4(IB+IF-V+1) - 2IB - IF + dV$$

$$= 2IB + 3IF - 4V + dV + 4$$

$$= (BV - EB) + \frac{3}{2}(VF - EF) + dV - 4V + 4$$
$$= 4 - \frac{3}{2}EF - EB + V(B + \frac{3}{2}F + d - 4)$$

$$\equiv 4 - \frac{3}{2}EF - EB + V\delta V. \qquad (293)$$

As we have come to understand, standard renormalizability corresponds to $\delta V = 0$. We will now show that the dimensionality of the coupling constant associated with the vertex is related to δV . Let us recall:

- $\dim[\psi] = \dim[\overline{\psi}] = M^{3/2}$;
- dim $[A] = M^1$;
- dim of a derivative $= M^1$.

Then, since

$$d^4xg_{vertex}(\overline{\psi} \text{ or } \psi)^F(A)^B(\partial_x)^d = M^0$$
 (294)

we must have

$$M^{-4}(M)^{\dim[g]}M^{3F/2}M^{B}M^{d} = M^{0}$$
(295)

or, equivalently,

$$\dim[g] = 4 - B - \frac{3}{2}F - d = -\delta V.$$
(296)

From this result, we see that in a vector fermion theory such as QED (also QCD) only a theory with dimensionless coupling constant(s) is renormalizable.

If $\delta V > 0$, then D increases with the number of vertices V so that higher order diagrams are increasingly divergent and a process involving arbitrary numbers of external particles will be divergent if a sufficiently high order of perturbation is considered. In this case, the coupling constant has negative mass dimension $M^{-\delta V}$. This is the case of non-renormalizability.

If $\delta V < 0$, then D decreases with the number of vertices, implying that only some lower order diagrams will be divergent. The coupling constant has positive mass dimension $M^{-\delta V} = M^{|\delta V|}$ in this case. This is the case of super-renormalizability.

Explicit 1-loop renormalization and computations for QED

At the moment I plan to follow Ryder to a large extent, so that will be a useful reference.

We will carry out the procedure in the Feynman version of the Lorentz gauge, for which the appropriate \mathcal{L} is

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi - eA^{\mu}\overline{\psi}\gamma_{\mu}\psi$$
$$-\frac{1}{4}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) - \frac{1}{2}(\partial_{\mu}A^{\mu})(\partial_{\nu}A^{\nu}).$$
(297)

First, we must do some dimensional analysis to see what the dimensions of things are in n dimensions in order that the mass dimension $\dim[\int d^n x \mathcal{L}] = 0$.

• From $i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi$ we determine that $\dim[\psi] = rac{n-1}{2}$ so that

$$\dim[i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi] = 2\frac{n-1}{2} + 1 = n$$
(298)

cancels $\dim[d^n x] = -n$.
- If $\dim[m] = 1$ then the $m\overline{\psi}\psi$ form will also have net dimension of n as required.
- From the $-\frac{1}{4}(\partial_{\mu}A_{\nu} \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} \partial^{\nu}A^{\mu})$ part of \mathcal{L} we find that $\dim[A] = \frac{n}{2} 1$.
- This is consistent also for the gauge fixing term of \mathcal{L} .
- Thus,

$$\dim\left[-eA^{\mu}\overline{\psi}\gamma_{\mu}\psi\right] = \dim\left[e\right] + 2\frac{n-1}{2} + \frac{n}{2} - 1 = n$$
(299)

requires

$$\dim[e] = 2 - \frac{n}{2}. \tag{300}$$

Thus, in close analogy to what we did in ϕ^4 theory, we will replace

$$e \to e(\mu)^{2-\frac{n}{2}} = e\mu^{\epsilon}. \tag{301}$$

As regards external spinors, e.g. $u(p), \ldots$, these may be kept as in n = 4 dimensions. We only need to go to $n \neq 4$ dimensions for stuff in the loop computations.

This brings up the question of how we deal with Dirac algebra and matrices in n dimensions. There are actually two approaches.

- 1. Dimensional regularization, which is the approach we follow and in which there will be some changes from 4 dimensions.
- 2. Dimensional reduction, in which the numerator algebra is done as if in 4 dimensions and only the non-Dirac part of the propagators are treated in *n* dimensions.

At one-loop the two approaches give the same results. There is an ongoing debate as to the number of loops to which this equivalence extends.

In dimensional regularization, we modify the Dirac algebra as follows. We begin with the anticommutator

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu} \tag{302}$$

where we consider $g_{\mu\nu}$ to be defined in n dimensions so that $\delta^{\mu}_{\mu} = n$. For consistency, this then implies that

$$\gamma^{\mu}\gamma_{\mu} = n, \quad \gamma_{\mu}\gamma^{\nu}\gamma^{\mu} = (2-n)\gamma^{\nu}, \quad \gamma_{\rho}\gamma_{\mu}\gamma_{\nu}\gamma^{\rho} = 4g_{\mu\nu} + (n-4)\gamma_{\mu}\gamma_{\nu}.$$
(303)

We will also define

$$\operatorname{Tr}[I] = 2^{n/2} \tag{304}$$

(any regular function approaching 4 as $n \to 4$ would have been ok) for which consistency with $\gamma^{\mu}\gamma_{\mu} = n$ then requires

$$\operatorname{Tr}[\gamma_{\mu}\gamma_{\nu}] = 2^{n/2}g_{\mu\nu}, \quad \operatorname{Tr}[\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}] = 2^{n/2}[g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}].$$
(305)

However, γ_5 becomes a problem in this approach as we need all 4 gamma matrices to define it. In our QED renormalization we don't have to face this problem, but in parity-violating theories we do. This is a long story and is related to "anomalies" that can destroy the renormalizability of a theory. We don't have time to go into this during these lectures.

So now let us do a few QED calculations.

• The photon loop correction to the fermion propagator.

$$\begin{split} -i\Sigma(p) &= (-ie\mu^{\epsilon})^{2} \int \frac{d^{n}k}{(2\pi)^{n}} \gamma^{\mu} \frac{i}{\not p - \not k - m} \frac{-ig_{\mu\nu}}{k^{2}} \gamma^{\nu} \\ &= -e^{2}\mu^{2\epsilon} \int \frac{d^{n}k}{(2\pi)^{n}} \frac{\gamma_{\mu}(\not p - \not k + m)\gamma^{\mu}}{[(p-k)^{2} - m^{2}]k^{2}} \\ &= -e^{2}\mu^{2\epsilon} \int d\alpha \int \frac{d^{n}k}{(2\pi)^{n}} \frac{\gamma_{\mu}(\not p - \not k + m)\gamma^{\mu}}{[\alpha(p-k)^{2} - \alpha m^{2} + (1-\alpha)k^{2}]^{2}} \end{split}$$

$$= -e^{2}\mu^{2\epsilon} \int d\alpha \int \frac{d^{n}k'}{(2\pi)^{n}} \frac{\gamma_{\mu}[(1-\alpha)\not\!p - \not\!k' + m]\gamma^{\mu}}{[k'^{2} - \alpha m^{2} + \alpha(1-\alpha)p^{2}]^{2}}$$

$$= -e^{2}\mu^{2\epsilon} \int d\alpha \gamma_{\mu}[(1-\alpha)\not\!p + m]\gamma^{\mu} \int \frac{d^{n}k'}{(2\pi)^{n}} \frac{d^{n}k'}{[k'^{2} - \alpha m^{2} + \alpha(1-\alpha)p^{2}]^{2}} (306)$$

In the above, we introduced $k' = k - \alpha p$ in order to write the denominator in a diagonal form, i.e. no terms of the form $k' \cdot p$. At this point, the k'disappears because it is odd under $k' \rightarrow -k'$, and the integral then reduces to a logarithmically divergent integral.

We now do the same thing we did before of performing a Wick rotation on the k'_0 integration and then reducing the integration to something that corresponds to a certain Γ function. The result is (where we use $n = 4 - 2\epsilon$ at a certain point — don't forget that Ryder uses $n = 4 - \epsilon$ if you are looking at his stuff):

$$\begin{split} -i\Sigma(p) &= -ie^{2}\mu^{2\epsilon}\frac{\Gamma(2-n/2)}{(4\pi)^{n/2}}\int d\alpha\gamma\mu[(1-\alpha)p'+m]\gamma^{\mu}[\alpha m^{2}-\alpha(1-\alpha)p^{2}]^{n/2-2} \\ &= -ie^{2}\mu^{2\epsilon}\frac{\Gamma(2-n/2)}{(4\pi)^{n/2}}\int d\alpha[(1-\alpha)p'(2-n)+mn][\alpha m^{2}-\alpha(1-\alpha)p^{2}]^{n/2-2} \\ &= -ie^{2}\frac{\Gamma(\epsilon)}{16\pi^{2}}\int d\alpha[(1-\alpha)p'(-2+2\epsilon)+m(4-2\epsilon)]\left[\frac{\alpha m^{2}-\alpha(1-\alpha)p^{2}}{4\pi\mu^{2}}\right]^{-\epsilon} \\ &= -ie^{2}\frac{(\frac{1}{\epsilon}-\gamma)}{16\pi^{2}}\int d\alpha[(1-\alpha)p'(-2+2\epsilon)+m(4-2\epsilon)]\left[\frac{\alpha m^{2}-\alpha(1-\alpha)p^{2}}{4\pi\mu^{2}}\right]^{-\epsilon} \end{split}$$

$$= -ie^{2}\frac{(\frac{1}{\epsilon} - \gamma)}{16\pi^{2}}\int d\alpha[(1 - \alpha)p(-2 + 2\epsilon) + m(4 - 2\epsilon)] \left[1 - \epsilon \ln\left(\frac{\alpha m^{2} - \alpha(1 - \alpha)p^{2}}{4\pi\mu^{2}}\right)\right]$$

$$= -i\frac{e^{2}}{16\pi^{2}\epsilon}(-p' + 4m) - i\frac{e^{2}}{16\pi^{2}}\left\{p'(1 + \gamma) - 2m(1 + 2\gamma) + 2\int d\alpha[p(1 - \alpha) - 2m]\ln\left(\frac{\alpha m^{2} - \alpha(1 - \alpha)p^{2}}{4\pi\mu^{2}}\right)\right\}$$
(307)

where the singularity of interest is contained entirely in the first term.

Derivation of the integral in a general form

The result for the $d^n k'$ integral employed in going from the last line of Eq. (306) to the first line of Eq. (307) can be read off of our old ϕ^4 case, but it is useful to derive a more general result for later use. We consider an integral over l and then Wick rotate the l_0 component to $i\bar{l}_0$, so that $l^2 \rightarrow -(\bar{l}_0^2 + l_1^2 + ...) = -\bar{l}_E^2$. Using these manipulations below we have the following sequence of steps:

$$\frac{d^{n}l}{(2\pi)^{n}} \frac{1}{[l^{2} - a^{2}]^{A}} = i \int \frac{d^{n}l_{E}}{(2\pi)^{n}} \frac{1}{[-l_{E}^{2} - a^{2}]^{A}} \\
= i \frac{2\pi^{n/2}}{(2\pi)^{n}\Gamma\left(\frac{n}{2}\right)} \int_{0}^{\infty} \frac{l^{n-1}dl}{[-l^{2} - a^{2}]^{A}} \quad \text{with } l = |\vec{l}_{E}| \\
= i \frac{2\pi^{n/2}}{(2\pi)^{n}\Gamma\left(\frac{n}{2}\right)} (-1)^{A} \int_{0}^{\infty} \frac{l^{n-1}dl}{[l^{2} + a^{2}]^{A}} \\
= i \frac{2\pi^{n/2}}{(2\pi)^{n}\Gamma\left(\frac{n}{2}\right)} (-1)^{A} \int_{0}^{\infty} \frac{dt}{2\sqrt{t}} \frac{t^{\frac{n-1}{2}}}{[t+a^{2}]^{A}} \\
= i \frac{\pi^{n/2}}{(2\pi)^{n}\Gamma\left(\frac{n}{2}\right)} (-1)^{A} \int_{0}^{\infty} dt \frac{t^{\frac{n}{2}-1}}{[t+a^{2}]^{A}} \\
= i \frac{\pi^{n/2}}{(2\pi)^{n}\Gamma\left(\frac{n}{2}\right)} (-1)^{A} \frac{1}{[a^{2}]^{A-\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \\
= i (-1)^{n/2} \frac{1}{(4\pi)^{n/2}} \frac{1}{[-a^{2}]^{A-\frac{n}{2}}} \frac{\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \\
= i (-1)^{A} \frac{1}{(4\pi)^{n/2}} \frac{1}{[a^{2}]^{A-\frac{n}{2}}} \frac{\Gamma\left(A-\frac{n}{2}\right)}{\Gamma(A)} \tag{308}$$

• The photon propagator correction

Next, we have the correction to the photon propagator coming from the fermion loop diagram. For reasons that will become apparent after we do the usual geometric sum of iterated 1PI diagrams, we wish to denote it by $i\Pi_{\mu\nu}$, where the μ and ν indices refer to the Lorentz indices of the photons entering and exiting the diagram. The form is (remembering the minus sign for a closed fermion loop and shifting to $p' = p - \alpha k$ at a certain stage):

$$\begin{split} \tilde{E}\Pi_{\mu\nu} &= -(-ie\mu^{\epsilon})^{2} \int \frac{d^{n}p}{(2\pi)^{n}} \operatorname{Tr}\left(\gamma_{\mu}\frac{i}{\not{p}-m}\gamma_{\nu}\frac{i}{\not{p}-\not{k}-m}\right) \\ &= -e^{2}\mu^{2\epsilon} \int \frac{d^{n}p}{(2\pi)^{n}} \frac{\operatorname{Tr}\left[\gamma_{\mu}(\not{p}+m)\gamma_{\nu}(\not{p}-\not{k}+m)\right]}{(p^{2}-m^{2})[(p-k)^{2}-m^{2}]} \\ &= -e^{2}\mu^{2\epsilon} \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \frac{\operatorname{Tr}\left[\gamma_{\mu}(\not{p}'+\alpha\not{k}+m)\gamma_{\nu}(\not{p}'-(1-\alpha)\not{k}+m)\right]}{[p'^{2}-m^{2}+k^{2}\alpha(1-\alpha)]^{2}} \end{split}$$
(309)

At this point, let us work on simplifying the numerator. We must remember that terms that are odd in p' will integrate to 0. Further, we must recall that the trace of an odd number of γ matrices is 0. Using these two ingredients, we obtain:

$$N = [p'^{\rho}p'^{\sigma} - k^{\rho}k^{\sigma}\alpha(1-\alpha)]\mathrm{Tr}[\gamma_{\mu}\gamma_{\rho}\gamma_{\nu}\gamma_{\sigma}] + m^{2}\mathrm{Tr}[\gamma_{\mu}\gamma_{\nu}]$$

$$= [p'^{\rho}p'^{\sigma} - k^{\rho}k^{\sigma}\alpha(1-\alpha)]2^{n/2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho}) + m^{2}2^{n/2}g_{\mu\nu}]$$

$$= 2^{n/2} \left\{ 2p'_{\mu}p'_{\nu} - 2\alpha(1-\alpha)(k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}) - g_{\mu\nu}[p'^{2} + k^{2}\alpha(1-\alpha) - m^{2}] \right\} (310)$$

where we organized the $k^2 g_{\mu\nu}$ term in to two pieces so that the coefficient of the final $g_{\mu\nu}$ piece above has precisely the same form as the denominator.

An aside on some useful integrals

Now we need some theorems for integrals containing integrated momenta in the numerator. What we, in particular, wish to show is that the $2p'_{\mu}p'_{\nu}$ terms cancels the $-g_{\mu\nu}[p'^2 + k^2\alpha(1-\alpha) - m^2]$ term, which will leave the gauge invariant piece proportional to $(k_{\mu}k_{\nu} - k^2g_{\mu\nu})$.

A good approach is to begin with what we have already shown, which I write using a general notation.

$$\int \frac{d^{n}l}{(2\pi)^{n}} \frac{1}{[l^{2} + M^{2} + 2l \cdot q]^{A}} = \int \frac{d^{n}l}{(2\pi)^{n}} \frac{1}{[(l+q)^{2} + M^{2} - q^{2}]^{A}}$$
$$= i(-1)^{n/2} \frac{\Gamma(A - \frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)} \frac{1}{(M^{2} - q^{2})^{A - \frac{n}{2}}},$$
(311)

where the above expression follows from Eq. (308) using the identification of $-a^2 = M^2 - q^2$.

Starting from this expression, we may generate other useful expressions by differentiation.

$$\int \frac{d^{n}l}{(2\pi)^{n}} \frac{l\mu}{[l^{2} + M^{2} + 2l \cdot q]^{A}} = -\frac{1}{2} \frac{1}{A - 1} \frac{\partial}{\partial q^{\mu}} \int \frac{d^{n}l}{(2\pi)^{n}} \frac{1}{[l^{2} + M^{2} + 2l \cdot q]^{A - 1}}$$

$$= -\frac{1}{2} \frac{1}{A - 1} \frac{\partial}{\partial q^{\mu}} \left[i(-1)^{n/2} \frac{\Gamma(A - 1 - \frac{n}{2})}{(4\pi)^{n/2} \Gamma(A - 1)} \frac{1}{(M^{2} - q^{2})^{A - 1 - \frac{n}{2}}} \right]$$

$$= -i(-1)^{n/2} \frac{1}{2} \frac{\Gamma(A - 1 - \frac{n}{2})}{(4\pi)^{n/2} \Gamma(A)} [-(A - 1 - \frac{n}{2})](-2q\mu) \frac{1}{(M^{2} - q^{2})^{A - \frac{n}{2}}}$$

$$= -i(-1)^{n/2} \frac{\Gamma(A - \frac{n}{2})}{(4\pi)^{n/2} \Gamma(A)} \frac{q\mu}{(M^{2} - q^{2})^{A - \frac{n}{2}}}.$$
(312)

From this result, we may proceed to

$$\int \frac{d^{n}l}{(2\pi)^{n}} \frac{l\mu l\nu}{[l^{2} + M^{2} + 2l \cdot q]^{A}}$$

$$= -\frac{1}{2} \frac{1}{A - 1} \frac{\partial}{\partial q^{\mu}} \int \frac{d^{n}l}{(2\pi)^{n}} \frac{l\nu}{[l^{2} + M^{2} + 2l \cdot q]^{A - 1}}$$

$$= -\frac{1}{2} \frac{1}{A - 1} \frac{\partial}{\partial q^{\mu}} \left[i(-1)^{n/2} \frac{-\Gamma(A - 1 - \frac{n}{2})}{(4\pi)^{n/2}\Gamma(A - 1)} \frac{q\nu}{(M^{2} - q^{2})^{A - 1 - \frac{n}{2}}} \right]$$

$$= i(-1)^{n/2} \frac{1}{2} \frac{\Gamma(A - 1 - \frac{n}{2})}{(4\pi)^{n/2}\Gamma(A)} \left\{ [-(A - 1 - \frac{n}{2})](-2q\mu)q\nu \frac{1}{(M^{2} - q^{2})^{A - \frac{n}{2}}} + \frac{g\mu\nu}{(M^{2} - q^{2})^{A - 1 - \frac{n}{2}}} \right\}$$

$$= i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}\Gamma(A)} \left[\frac{1}{2} g\mu\nu \frac{\Gamma(A - 1 - \frac{n}{2})}{(M^{2} - q^{2})^{A - 1 - \frac{n}{2}}} + \frac{\Gamma(A - \frac{n}{2})q\mu q\nu}{(M^{2} - q^{2})^{A - 1 - \frac{n}{2}}} \right].$$
(313)

Back to the main $\Pi_{\mu\nu}$ calculation

The case of current interest, referring back to our expression for the numerator N of Eq. (310), is l = p', q = 0, $M^2 = k^2 \alpha (1 - \alpha) - m^2$ and A = 2, for which we have

$$\int \frac{d^{n}l}{(2\pi)^{n}} \frac{2l_{\mu}l_{\nu}}{[l^{2}+M^{2}]^{2}} = i(-1)^{n/2} 2 \frac{1}{(4\pi)^{n/2}\Gamma(2)} \left[\frac{1}{2} g_{\mu\nu} \frac{\Gamma(1-\frac{n}{2})}{(M^{2})^{1-\frac{n}{2}}} \right]$$
$$= i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}} \left[g_{\mu\nu} \frac{\Gamma(1-\frac{n}{2})}{(M^{2})^{1-\frac{n}{2}}} \right], \qquad (314)$$

which we wish to compare to the explicit $g_{\mu\nu}$ term of N which is reduced as follows:

$$-g_{\mu\nu} \int \frac{d^{n}l}{(2\pi)^{n}} \frac{l^{2} + M^{2}}{[l^{2} + M^{2}]^{2}} = -g_{\mu\nu} \int \frac{d^{n}l}{(2\pi)^{n}} \frac{1}{[l^{2} + M^{2}]^{1}}$$

$$= -g_{\mu\nu}i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}\Gamma(1)} \left[\frac{\Gamma(1 - \frac{n}{2})}{(M^{2})^{1 - \frac{n}{2}}} \right]$$

$$= -i(-1)^{n/2} \frac{1}{(4\pi)^{n/2}} \left[\frac{g_{\mu\nu}\Gamma(1 - \frac{n}{2})}{(M^{2})^{1 - \frac{n}{2}}} \right], \quad (315)$$

which does indeed cancel the $2p'_{\mu}p'_{
u}=2l_{\mu}l_{
u}$ term. Of course, there were

simpler ways to get this one result, but the above general procedure will be useful in other computations.

So, now the remainder of the $i\Pi_{\mu\nu}$ expression is

$$\begin{split} i\Pi_{\mu\nu}(k) &= -e^{2}\mu^{2\epsilon} 2^{n/2} [k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}] \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \left\{ -\frac{2\alpha(1-\alpha)}{[p'^{2} + k^{2}\alpha(1-\alpha) - m^{2}]^{2}} \right\} \\ &= -e^{2}\mu^{2\epsilon} 2^{2-\epsilon} [k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}] \int d\alpha \left\{ -2\alpha(1-\alpha)i(-1)^{2-\epsilon} \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2)} \frac{1}{[k^{2}\alpha(1-\alpha) - m^{2}]^{\epsilon}} \right\} \\ &= ie^{2}\mu^{2\epsilon} 2^{2-\epsilon} [k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}] \int d\alpha \left\{ 2\alpha(1-\alpha) \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2)} \frac{1}{[m^{2} - k^{2}\alpha(1-\alpha)]^{\epsilon}} \right\} \\ &= i\frac{e^{2}}{2\pi^{2}} [k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}] \int d\alpha \left\{ \alpha(1-\alpha)\Gamma(\epsilon) \left[\frac{m^{2} - k^{2}\alpha(1-\alpha)}{2\pi\mu^{2}} \right]^{-\epsilon} \right\} \\ &= i\frac{e^{2}}{2\pi^{2}} [k_{\mu}k_{\nu} - k^{2}g_{\mu\nu}] \left\{ \frac{1}{6} \left(\frac{1}{\epsilon} - \gamma \right) - \int d\alpha\alpha(1-\alpha)\ln\left[\frac{m^{2} - k^{2}\alpha(1-\alpha)}{2\pi\mu^{2}} \right] + \mathcal{O}(\epsilon) \right\}, \quad (316)$$

where the $\ln [\ldots]$ came from $\frac{1}{\epsilon}(-\epsilon \ln [\ldots])$. You will notice some differences between this and Ryder's expression. Partly it is $2\epsilon_{me} = \epsilon_{Ryder}$, partly because I used what Ryder calls $f(d) = 2^{n/2}$, whereas he set f(d) = 1, and partly because I believe he has the wrong sign for the $m^2 - k^2 \alpha (1-\alpha)$ after performing the Wick rotation and properly accounting

for the $(-1)^{n/2} = (-1)^{2-\epsilon} = (-1)^{-\epsilon}$ which can be absorbed in the form

$$(-1)^{-\epsilon} [k^2 \alpha (1-\alpha) - m^2]^{-\epsilon} = [m^2 - k^2 \alpha (1-\alpha)]^{-\epsilon}.$$
 (317)

Please let me know if you disagree with my sign after checking this through. There was a similar discrepancy in the sign of the ln argument in the $-i\Sigma$ expression for exactly the same reason. Incidentally, my signs agree with those in Ramond's QFT textbook, Eqs. (8.2.20) and (8.2.32), once you realize that he is using a Euclidean notation so that to compare to my expressions you must take $p_E \rightarrow -p$ and $p_E^2 \rightarrow -p^2$ where p_E is the Euclidean momentum employed by Ramond (called p in his book).

The most important property of the result of Eq. (309) is the $[k_{\mu}k_{\nu}-k^2g_{\mu\nu}]$ structure. This is the form required by gauge invariance in the form $k^{\mu}\Pi_{\mu\nu} = 0$. In fact, this structure is the only form that $\Pi_{\mu\nu}$ could have and obey the gauge invariance requirement. It is because we had to get proportionality to this structure that the naive D = 2 divergence level was reduced to D = 0. Two of the momenta powers were eaten up in creating this form. In the explicit calculation above, the quadratically divergent terms cancelled one another.

We will see that the above GI structure is also critical in order to

retain zero mass for the photon after including one-loop corrections and performing dimensional regularization. You should also know that not every regularization procedure produces this kind of explicitly gauge invariant form.

We shall return to these issues shortly. But first, we need to compute

• The vertex correction.

The result of the photon exchange contribution to the vertex will be denoted by $-ie\Lambda_{\mu}$ in 4 dimensions or, in $n = 4 - 2\epsilon$ dimensions we will pull out the necessary mass scale and write $-ie\mu^{\epsilon}\Lambda_{\mu}$. We have from the diagram (using q for the momentum on the photon, p' for the outgoing electron momentum, and p for the incoming electron momentum):

$$-ie\mu^{\epsilon}\Lambda_{\mu} = (-ie\mu^{\epsilon})^{3}\int \frac{d^{n}k}{(2\pi)^{n}} \frac{-ig^{\nu\rho}}{k^{2}}\gamma_{\nu}\frac{i}{p^{\prime}-k^{\prime}-m}\gamma_{\mu}\frac{i}{p^{\prime}-k^{\prime}-m}\gamma_{\rho}$$

$$= -(e\mu^{\epsilon})^{3}\int \frac{d^{n}k}{(2\pi)^{n}}\frac{\gamma_{\nu}(p^{\prime}-k^{\prime}+m)\gamma_{\mu}(p^{\prime}-k^{\prime}+m)\gamma^{\nu}}{k^{2}[(p^{\prime}-k)^{2}-m^{2}][(p-k)^{2}-m^{2}]}$$

$$= -2(e\mu^{\epsilon})^{3}\int_{0}^{1}d\alpha\int_{0}^{1-\alpha}d\beta\int \frac{d^{n}k}{(2\pi)^{n}}\frac{\gamma_{\nu}(p^{\prime}-k^{\prime}+m)\gamma_{\mu}(p^{\prime}-k^{\prime}+m)\gamma^{\nu}}{[k^{2}-m^{2}(\alpha+\beta)-2k\cdot(\alpha p+\beta p^{\prime})+\alpha p^{2}+\beta p^{\prime}2]^{3}},$$
(318)

where we used the Feynman parameter trick for three denominators in the

form:

$$\frac{1}{abc} = 2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{[a(1-\alpha-\beta)+\alpha b+\beta c]^3}.$$
 (319)

We will now shift to $k' = k - \alpha p - \beta p'$ in order to diagonalize the denominator above so that we get (after redefining $k' \rightarrow k$)

$$\Lambda_{\mu} = -2i(e\mu^{\epsilon})^{2} \int_{0}^{1} d\alpha \int_{0}^{1-\alpha} d\beta \int \frac{d^{n}k}{(2\pi)^{n}} \frac{\gamma_{\nu}(p'(1-\beta) - \alpha p' - k' + m)\gamma_{\mu}(p'(1-\alpha) - \beta p' - k' + m)\gamma^{\nu}}{[k^{2} - m^{2}(\alpha + \beta) + \alpha(1-\alpha)p^{2} + \beta(1-\beta)p'^{2} - \alpha\beta 2p' \cdot p]^{3}} \\
\equiv \Lambda_{\mu}^{(1)} + \Lambda_{\mu}^{(2)},$$
(320)

where $\Lambda_{\mu}^{(1)}$ is the divergent piece where we keep the two k terms in the numerator (the linear terms vanish by $k \to -k$ symmetry) and $\Lambda_{\mu}^{(2)}$ is the remaining convergent piece containing no k's in the numerator. Let us write

$$\gamma_{\nu} k \gamma_{\mu} k \gamma^{\nu} = k_{\rho} k_{\sigma} \gamma_{\nu} \gamma^{\rho} \gamma_{\mu} \gamma^{\sigma} \gamma^{\nu}$$
(321)

and then use Eq. (313) to write

$$\int \frac{d^{n}k}{(2\pi)^{n}} \frac{k\rho k\sigma}{[k^{2} - m^{2}(\alpha + \beta) + \alpha(1 - \alpha)p^{2} + \beta(1 - \beta)p'^{2} - \alpha\beta 2p' \cdot p]^{3}}$$

$$= i(-)^{n/2} \frac{1}{(4\pi)^{n/2}\Gamma(3)} \frac{1}{2}g\rho\sigma \frac{\Gamma(2 - \frac{n}{2})}{[-m^{2}(\alpha + \beta) + \alpha(1 - \alpha)p^{2} + \beta(1 - \beta)p'^{2} - \alpha\beta 2p' \cdot p]^{2 - \frac{n}{2}}}$$

$$= i\frac{1}{4(4\pi)^{n/2}}g\rho\sigma \frac{\Gamma(\epsilon)}{[m^{2}(\alpha + \beta) - \alpha(1 - \alpha)p^{2} - \beta(1 - \beta)p'^{2} + \alpha\beta 2p' \cdot p]^{\epsilon}}.$$
(322)

Next, we use Eq. (303) to write

$$g_{\rho\sigma}\gamma_{\nu}\gamma^{\rho}\gamma_{\mu}\gamma^{\sigma}\gamma^{\nu} = \gamma_{\nu}\gamma^{\rho}\gamma_{\mu}\gamma_{\rho}\gamma^{\nu}$$
$$= \gamma_{\nu}(2-n)\gamma_{\mu}\gamma^{\nu}$$
$$= (2-n)^{2}\gamma_{\mu}.$$
(323)

Putting all this into our expression for Λ_{μ} we get for just the coefficient of the singular $\frac{1}{\epsilon}$ part of $\Gamma(\epsilon)$ and setting n = 4 ($\epsilon = 0$) everywhere else:

$$\Lambda_{\mu}^{(1)} = -2ie^2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left[i \frac{1}{4(4\pi)^2 \epsilon} \right] 4\gamma_{\mu} + finite$$
$$= \frac{e^2}{16\pi^2} \frac{1}{\epsilon} \gamma_{\mu} + finite. \qquad (324)$$

For completeness, we could write out the finite part of $\Lambda_{\mu}^{(1)}$ as well and, of course, we have $\Lambda_{\mu}^{(2)}$ which is finite from the beginning. For the latter, we can simply set n = 4 and consider the non-k parts of the numerator. Using the integral of Eq. (311), we obtain

$$\Lambda_{\mu}^{(2)} = \frac{e^2}{16\pi^2} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{\gamma_{\nu}(p'(1-\beta) - \alpha p' + m)\gamma_{\mu}(p'(1-\alpha) - \beta p' + m)\gamma^{\nu}}{[-m^2(\alpha+\beta) + \alpha(1-\alpha)p^2 + \beta(1-\beta)p'^2 - \alpha\beta 2p' \cdot p]}, \quad (325)$$

which we could compute explicitly if we really wanted to, as we should have to for precision comparisons with data, but will for the moment leave as it is. (Note: in this case I have an opposite overall sign wrp to Ryder. I think I am right, but please check.) Meanwhile, it will be useful to keep in mind that the finite part of $\Lambda_{\mu}^{(1)}$ is also proportional to γ_{μ} .

The 1-loop renormalization of QED

To summarize, we have found the singular structures:

$$\Sigma(p) = \frac{e^2}{16\pi^2\epsilon}(-\not p + 4m) + finite \qquad (326)$$
$$\Pi_{\mu\nu}(k) = \frac{e^2}{12\pi^2\epsilon}(k_{\mu}k_{\nu} - k^2g_{\mu\nu}) + finite \qquad (327)$$

$$\Lambda_{\mu}^{(1)}(p,q,p') = \frac{e^2}{16\pi^2\epsilon} \gamma_{\mu} + finite.$$
 (328)

Note that for the $1/\epsilon$ terms above,

$$\Lambda_{\mu} = -\frac{\partial \Sigma(p)}{\partial p^{\mu}}.$$
(329)

This is one consequence of a very general result called the QED Ward identity. An aside on the Ward identity

- This Ward identity holds to all orders in perturbation and it can be proved in a variety of ways. A path integral proof is given in Ryder, Sec. 7.4. Here, I will give a diagrammatic type of proof.
- The general statement of the Ward identity is

$$\frac{\partial S_F^{-1}(p)}{\partial p^{\mu}} = \Gamma_{\mu}(p, 0, p) , \qquad (330)$$

where S_F is the complete electron propagator and $e\Gamma_{\mu}$ is the complete photon-fermion-antifermion vertex function:

$$S_F(p)^{-1} = -i[\not p - m - \Sigma(p)], \quad e\Gamma_\mu = -ie\gamma_\mu - ie\Lambda_\mu.$$
(331)

• The Ward identity is actually the $q^{\mu} \rightarrow 0$ limit of a more general identity called the Ward-Takahashi identity, which reads:

$$q^{\mu}\Gamma_{\mu}(p,q,p+q) = S_F^{-1}(p+q) - S_F^{-1}(p).$$
(332)

To see how this comes about, let us imagine a diagram for $-i\Sigma(p)$. It will have an e line coming in and and e line exiting, and this e line will undergo various interactions at vertices within the diagram. There will also in general be internal closed e loops — we shall return to those in a moment. We establish a labeling for the vertices to which the photons internal to the diagram are attached, beginning with γ^{λ_0} at the entry point of the e line and ending with γ^{λ_1} at the exit point of the e line. In between, we have vertices ranging from γ^{λ_1} to $\gamma^{\lambda_{n-1}}$. The fermion line running from the γ^{λ_0} vertex to the γ^{λ_1} vertex will have momentum p_1 , and so forth with the final fermion line from $\gamma^{\lambda_{n-1}}$ to γ^{λ_n} having p_n .

We can generate a set of diagrams for the (amputated) γee vertex from the above diagram contributing to the propagator by inserting the $\gamma(q)$ (qis taken as ingoing) at every possible location along the e line and shifting all the momenta of the internal e propagators subsequent to the insertion point up by the additional momentum q. Each such insertion will have the standard $-ie\gamma^{\mu}$ interaction. When we multiply by q_{μ} , as we wish to do when we want to get the WT identity, this becomes an insertion of $-ieq_{\mu}\gamma^{\mu}$.

One then notes the following simple identity (factors of the charge e are removed in defining Λ_{μ} and Γ_{μ}):

$$-iq_{\mu}\gamma^{\mu} = -i[(\not p_{i} + \not q - m) - (\not p_{i} - m)], \qquad (333)$$

where p_i is the momentum on the *e* line to which we attached the $\gamma(q)$. Now, on either side of this insertion, we have an *e* propagator. Including these, we have the structure

$$\frac{i}{\not p_i + \not q - m} (-i \not q) \frac{i}{\not p_i - m} = \left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not q - m}\right).$$
(334)

Including still more neighboring vertices and propagators, we end up with

an expression of the form

$$\dots \left(\frac{i}{p_{i+1}'+q'-m}\right)\gamma^{\lambda_i} \left(\frac{i}{p_i'-m}-\frac{i}{p_i'+q'-m}\right)\gamma^{\lambda_i-1} \left(\frac{i}{p_{i-1}'-m}\right)\gamma^{\lambda_i-2}\dots$$
(335)

Now consider what happens if we insert $-i \not q$ into line i - 1. We generate a structure of the form

$$\dots \left(\frac{i}{p_{i+1}'+q'-m}\right)\gamma^{\lambda_i} \left(\frac{i}{p_i'+q'-m}\right)\gamma^{\lambda_i-1} \left(\frac{i}{p_{i-1}'-m}-\frac{i}{p_{i-1}'+q'-m}\right)\gamma^{\lambda_i-2}\dots$$
(336)

Clearly the 1st term in the 2nd expression cancels the 2nd term in the 1st expression. This will continue to work in this way until we get to the very ends of the fermion line, leaving us with two uncanceled forms so that we end up with (still need to return to closed fermion loops that we can show don't contribute)

$$[q^{\mu}\Lambda_{\mu}(p,q,p+q)] = -\gamma^{\lambda n} \left(\frac{i}{p'_{n}+q'-m}\right) \gamma^{\lambda n-1} \dots \left(\frac{i}{p'_{i+1}+q'-m}\right) \gamma^{\lambda i} \left(\frac{i}{p'_{i}+q'-m}\right) \gamma^{\lambda i-1} \left(\frac{i}{p'_{i-1}+q'-m}\right) \gamma^{\lambda i-2} \dots \times \gamma^{\lambda 1} \left(\frac{i}{p'_{1}+q'-m}\right) \gamma^{\lambda 0} + \gamma^{\lambda n} \left(\frac{i}{p'_{n}-m}\right) \gamma^{\lambda n-1} \dots \left(\frac{i}{p'_{i+1}-m}\right) \gamma^{\lambda i} \left(\frac{i}{p'_{i-1}-m}\right) \gamma^{\lambda i-1} \left(\frac{i}{p'_{i-1}-m}\right) \gamma^{\lambda i-2} \dots \gamma^{\lambda 1} \left(\frac{i}{p'_{1}-m}\right) \gamma^{\lambda 0} = -[-i\Sigma(p+q)] + [-i\Sigma(p)]$$

$$(337)$$

where the $-i\gamma^{\mu}$ part of $-i\not q$ is included in the Γ^{μ} as being the vertex to which the external photon attaches (remember that in our definitions we have removed the explicit charge part, e, of the coupling — the full amputated vertex is $e\Gamma_{\mu}$).

• If we include the bare vertex part of Γ_{μ} (i.e. $\Gamma_{\mu}=-i\gamma_{\mu}+$ higher order) and recall that

$$S_F^{-1}(p) = -i[\not p - m - \Sigma(p)]$$
(338)

so that

$$S_F^{-1}(p+q) - S_F^{-1}(p) = -i \not q + i \Sigma(p+q) - i \Sigma(p), \qquad (339)$$

we obtain the net result,

$$q^{\mu}\Gamma_{\mu}(p,q,p+q) = S_F^{-1}(p+q) - S_F^{-1}(p).$$
(340)

In the limit as $q \rightarrow 0$, we can write

$$S_F^{-1}(p+q) - S_F^{-1}(p) = q^{\mu} \frac{\partial S_F^{-1}}{\partial p^{\mu}}.$$
 (341)

Inserting above, and matching coefficients of q^{μ} as $q^{\mu} \rightarrow 0$ yields the Ward identity:

$$\Gamma_{\mu}(p,0,p) = \frac{\partial S_F^{-1}}{\partial p^{\mu}}.$$
(342)

It is always useful to remember this by simply referring to the zeroth order version of the above which says that

$$-i\gamma_{\mu} = \frac{\partial \left[\frac{(\not p - m)}{i}\right]}{\partial p^{\mu}}.$$
 (343)

• Returning to the issue of closed loops, I hope the following is obvious.

After inserting $-i \not q$ at every possible location on the closed loop, since the loop is closed one ends up with the closed loop written in terms of the unshifted p_i 's that appeared in the loop minus exactly the same structure with every $p_i \rightarrow p_i + q$. In other words, we will have a result proportional to

$$\operatorname{Tr}\left\{-\gamma^{\lambda_{n}}\left(\frac{i}{p_{n}+q-m}\right)\cdots\left(\frac{i}{p_{i+1}+q-m}\right)\gamma^{\lambda_{i}}\left(\frac{i}{p_{i}+q-m}\right)\gamma^{\lambda_{i-1}}\left(\frac{i}{p_{i-1}+q-m}\right)\gamma^{\lambda_{i-2}}\cdots\left(\frac{i}{p_{1}+q-m}\right)\gamma^{\lambda_{i-2}}\cdots\left(\frac{i}{p_{i+1}+q-m}\right)\gamma^{\lambda_{i-2}}\cdots\left($$

But, every one of these p_i 's is given in terms of the loop integration variable, lets use p_1 as the free loop momentum, plus momenta that are brought in at the vertices that are present on the closed loop. These "external" momenta can be regarded as fixed as we perform the p_1 loop integration. But, then the term with all the p_i replaced by $p_i + q$ can be converted to the same form as the term without any q's added simply by shifting the integration variable, $p_1 + q \rightarrow p_1$, which will also shift all the other $p_i + q \rightarrow p_i$. Because there was a relative minus sign between the $p_i + q$ and the p_i structures, they cancel after this shift.

It should be noted that this shift is not always valid. Theories such as QED for which this shift *is* valid are called "non-anomalous". If the theory has an anomaly, this shift is not valid. Anomalies can emerge when considering axial vector currents such as those encountered in the electroweak interactions. We will not have time to pursue this topic, but you will want to read about it in one of the standard texts (e.g. Peskin section 19.2). In the context of dimensional regularization, the problem is associated with defining γ_5 in

 $n \neq 4$ dimensions. One finds (see Peskin section 19.4) that the electroweak theory will be anomalous (certain unphysical states would fail to cancel and the S matrix would not be unitary) unless quarks and leptons come in just the right representations. The condition that the electroweak theory be anomaly free is Tr[Q] = 0, where Q is the charge operator and Tr[Q] is the sum over all the charges of the fermions of the theory. For one family of quarks and leptons we have

Tr[Q] =
$$3\left(\frac{2}{3} - \frac{1}{3}\right) + (0 - 1) = 0$$
, (345)

where the first term comes from the quarks with three colors and the second term comes from the leptons. Thus, if quarks have fractional charge, they must have three colors for this to work. The structure of each family and the need for complete families is no accident!

Back to the renormalization program

The Ward identity is very critical to proving renormalizability to all orders. As we have seen, it is already incorporated in our explicit one-loop results.

So let us now construct the counter-term Lagrangian, $\Delta \mathcal{L}$.

• We begin with an appropriate term to counter the singularity in $-i\Sigma(p)$. There are actually two independent singularities:

$$-i\Sigma(p) = -irac{e^2}{16\pi^2\epsilon}(-\not p + 4m) + finite$$
 (346)

which must be countered by terms of the form

$$\Delta \mathcal{L} = i B \overline{\psi} \, \partial \psi - A \overline{\psi} \psi \,. \tag{347}$$

The Feynman rules that go with these $\Delta \mathcal{L}$ terms are simply $iB \not p$ and -iA, respectively. (We always have to multiply by i from the $\exp[i\Delta \mathcal{L}]$ expansion to first order, and $\not p \rightarrow -i \not p$ in momentum space.) What we need is for

$$-i\Sigma(p) - iA + iB \not p = -i\frac{e^2}{16\pi^2\epsilon}(-\not p + 4m) + finite - iA + iB \not p = finite.$$
(348)

This is the case provided

$$A = -\frac{me^2}{4\pi^2\epsilon}, \quad B = -\frac{e^2}{16\pi^2\epsilon}.$$
 (349)

So, if we match $\mathcal{L} + \Delta \mathcal{L}$ to \mathcal{L}_0 for the purely fermionic part, we must have

$$i(1+B)\overline{\psi}\,\partial\!\!\!/\psi - (m+A)\overline{\psi}\psi = i\overline{\psi}_0\,\partial\!\!\!/\psi_0 - m_0\overline{\psi}_0\psi_0\,. \tag{350}$$

As we know, this is accomplished by wave function and mass renormalization in the form

$$\psi_0 = \sqrt{Z_2}\psi, \quad m_0 = m + \delta m, \qquad (351)$$

where the convention is to call the QED wave function renormalization factor Z_2 . From this we see that

$$Z_{2} = 1 + B = 1 - \frac{e^{2}}{16\pi^{2}\epsilon},$$

$$m_{0} = Z_{2}^{-1}(m+A) = m\left(1 + \frac{e^{2}}{16\pi^{2}\epsilon}\right)\left(1 - \frac{e^{2}}{4\pi^{2}\epsilon}\right) = m\left(1 - \frac{3e^{2}}{16\pi^{2}\epsilon}\right) = m + \delta m(352)$$

To repeat what we have already considered in the ϕ^4 theory case, we have the relation

$$\langle 0|T\{\overline{\psi}_0(x)\psi_0(y)\}|0\rangle = Z_2\langle 0|T\{\overline{\psi}(x)\psi(y)\}|0\rangle$$
(353)

which, in words, says that $Z_2 < 1$ is the probability of finding a full renormalized electron (represented by ψ) propagating in the propagation of

a bare electron (represented by ψ_0). Alternatively, we say that in order to get full one-particle normalization for ψ propagation, we must boost up ψ_0 propagation by $Z_2^{-1} > 1$.

There is one more important feature of the result for δm . Note that $\delta m \to 0$ as $m \to 0$. This is a consequence of chiral symmetry. You should recall that the Dirac \mathcal{L} has an extra symmetry under $e^{-i\alpha\gamma_5}$ chiral transformations in the m = 0 limit. This symmetry is retained by the full QED Lagrangian (keeping m = 0). Thus, this symmetry should be respected to all orders of perturbation theory. In order for this to be the case, a non-zero value for the electron mass should not be generated by loop corrections. We see that our explicit 1-loop result verifies this requirement.

This behavior means that the chiral symmetry has "protected" the electron mass from hierarchy/fine-tuning problems. To understand what we mean by this, let us compare to the ϕ^4 theory. There, the one-loop correction to the scalar propagator gives rise to a quadratically divergent correction to the scalar particle mass-squared. The net result is that

$$m^2 = m_0^2 + \lambda \Lambda^2 \tag{354}$$

where Λ is the cutoff of the loop momentum integration. If the observed

m is of modest observable size (e.g. of order a TeV), then if Λ is very large (for example of order $M_{
m P} \sim 10^{19} {
m ~GeV}$) this modest size becomes unnatural in the sense that a very precise cancellation (to a few parts in 10^{38}) between m_0^2 and $\lambda \Lambda^2$ would be required. This is termed the "fine-tuning" problem that derives from the "hierarchy" problem of understanding how two such diverse scales as $M_{
m P}$ and 1 ${
m TeV}$ can be accommodated in the theory. Of course, this problem for the scalar field is why we have come to favor a new symmetry, such as supersymmetry, in which there are, for example, spin-1/2supersymmetric partners of mass $\sim m_{
m SUSY}$ to the scalar particles that also give quadratically divergent contributions to m^2 . However, these quadratic divergences come with the opposite sign of the fermion loop and the supersymmetry also implies that the couplings involved are of exactly the correct size such that once the loop momentum l exceeds $m_{
m SUSY} \sim 1~{
m TeV}$ there is exact cancellation of the scalar loop correction against the fermion loop correction. In this case, Λ is replaced by $m_{
m SUSY}$ and if $m_{
m SUSY} \sim 1~{
m TeV}$ there is no fine-tuning problem even if the ultimate ultraviolet completion of the theory (but now including SUSY) is at $M_{\rm P}$.

In the electron/QED case, the above kind of problem and the necessity of some such dramatic solution does not arise. Whatever the mass, m, of the electron is, the 1-loop-corrected mass has the form $m(1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda}{\mu})$, where

I have replaced the $\frac{1}{\epsilon}$ by the momentum cutoff equivalent. Now, even if $\Lambda \sim M_{\rm P}$ the correction *factor* is not particularly big ($\sim 1 + \frac{1}{516\pi} \times 19$) and so even if it is gravity that provides the ultraviolet completion of the theory there is no particular fine-tuning problem.

 Now let us turn to the vacuum polarization correction to the photon propagator and see what role it plays.

First, we must figure out what the counter term Lagrangian $\Delta \mathcal{L}$ must be for this part of things. We are dealing with the

$$\mathcal{L} \ni -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_{\mu} A^{\mu})^2 = \frac{1}{2} A^{\mu} g_{\mu\nu} \Box A^{\nu}$$
(355)

part of \mathcal{L} . The allowed counter terms are then of the form

$$\Delta \mathcal{L} = -\frac{C}{4} F_{\mu\nu} F^{\mu\nu} - \frac{E}{2} (\partial_{\mu} A^{\mu})^2 \,. \tag{356}$$

Note that C and E are not necessarily equal in the counter term, even though they were for \mathcal{L} . It is simply a matter of what is required and it

turns out that $C \neq E$ is required. Now, the Feynman rules coming from the above $\Delta \mathcal{L}$ are:

$$-iC(k^{2}g_{\mu\nu} - k_{\mu}k_{\nu}), -iEk^{2}g_{\mu\nu}.$$
(357)

Meanwhile our singularity structure is purely of the form

$$i\Pi_{\mu\nu}(k) = i\frac{e^2}{12\pi^2\epsilon}(k_{\mu}k_{\nu} - k^2g_{\mu\nu}) = -i\frac{e^2}{12\pi^2\epsilon}(k^2g_{\mu\nu} - k_{\mu}k_{\nu}).$$
 (358)

From this, we see that, as far as singularities are concerned we may take E = 0. This is important since it means we can remain in Feynman gauge for which the precise $\mathcal{L} \ni -\frac{1}{2}(\partial_{\mu}A^{\mu})^2$ form and normalization is required. Thus, subsequent calculations in still higher order can again be done in Feynman gauge. This is not to say that the resummed photon propagator will have a Feynman gauge form, but that is a different issue as we shall see.

Anyway, the above requires that

$$C = -\frac{e^2}{12\pi^2\epsilon} \tag{359}$$

at least as far as the singular part is concerned.

Before continuing on, let me return to a schematic derivation of the claim that the Feynman rule for the C term of $\Delta \mathcal{L}$ is indeed as stated above in Eq. (357). We begin by using the antisymmetry to write

$$\mathcal{L}_{FF} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) (\partial^{\mu} A^{\nu}) \,. \tag{360}$$

In order to determine the corresponding Feynman rule, we must compute the S matrix for a photon to turn back into a photon after interacting with \mathcal{L}_{FF} . The Feynman rule for $i\mathcal{M}$ will be extracted from the S form by "pulling away" the usual factors (which you will see shortly). We thus have to compute, leaving aside the C coefficient for the moment,

$$\langle 0|a_{\vec{k}}^{s}\left[i\int d^{4}x\mathcal{L}_{FF}\right]a_{\vec{k}}^{t\dagger}|0\rangle = \langle 0|a_{\vec{k}}^{s}\int d^{4}x\left[-\frac{i}{2}(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu})\right]a_{\vec{k}}^{t\dagger}|0\rangle \quad (361)$$

where, as always, the decomposition of the A_{μ} fields looks like

$$A_{\mu}(x) = \sum_{\vec{q},r} \frac{1}{\sqrt{2VE_{\vec{q}}}} \left(a_{\vec{q}}^r \epsilon_{\mu}^r(\vec{q}) e^{-iq \cdot x} + a_{\vec{q}}^{r \dagger} \epsilon_{\mu}^r(\vec{q}) e^{iq \cdot x} \right)$$
(362)

assuming a real ϵ^r polarization basis. Obviously we must introduce two such expansions. We write the 2nd expansion as a $\sum_{\vec{p},r'}$ expansion. Let us focus first on just one of the two equivalent contraction matchings, or "killing" operations. We contract the $a_{\vec{p}}^{r'}$ with the $a_{\vec{k}}^{t\dagger}$ and the $a_{\vec{k}}^s$ with the $a_{\vec{q}}^{t\dagger}$, yielding $\delta_{r't}\delta_{\vec{p}\vec{k}}\delta_{rs}\delta_{\vec{q}\vec{k}}$. We write the derivatives in terms of the momenta q and p appearing in the expansions. However, after employing the above δ 's, we have p = q = k. The result is:

$$\langle 0|a_{\vec{k}}^{s} \int d^{4}x \left[-\frac{i}{2} (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu}) \right] a_{\vec{k}}^{t\dagger} |0\rangle_{\text{contraction 1}}$$

$$= -\frac{i}{2} \frac{1}{\sqrt{2VE_{\vec{k}}} \sqrt{2VE_{\vec{k}}}} \int d^{4}x e^{-ik \cdot x + ik \cdot x} \left\{ \epsilon_{\nu}^{s} (-ik_{\mu})(+ik^{\mu})\epsilon^{t\nu} - \epsilon_{\mu}^{s} (-ik_{\nu})(+ik^{\mu})\epsilon^{t\nu} \right\}$$

$$= \epsilon^{s\mu} \epsilon^{t\nu} \frac{1}{\sqrt{2VE_{\vec{k}}} \sqrt{2VE_{\vec{k}}}} (2\pi)^{4} \delta^{4} (k-k) \left(\frac{-i}{2}\right) \left[k^{2}g_{\mu\nu} - k_{\mu}k_{\nu}\right]. \tag{363}$$

The other possible contraction arrangement doubles this answer. Removing

the usual factors in extracting $i\mathcal{M}$, we get the Feynman rule for $i\mathcal{M}$ of

$$-i\left[k^2g_{\mu\nu}-k_{\mu}k_{\nu}\right] \tag{364}$$

as claimed.

So, the net form of the pure gauge part of the Lagrangian is:

$$\mathcal{L} + \Delta \mathcal{L} \ni -\left(\frac{1+C}{4}\right) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_{\mu} A^{\mu})^2 \equiv -\frac{Z_3}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_{\mu} A^{\mu})^2 ,$$
(365)

where at 1-loop we have

$$Z_3 = 1 + C = 1 - \frac{e^2}{12\pi^2\epsilon}.$$
(366)

The above pure gauge part of $\mathcal{L}+\Delta\mathcal{L}$ is to be identified with the bare Lagrangian form

$$\mathcal{L}_0 = -\frac{1}{4} F_{0\,\mu\nu} F_0^{\mu\nu} + \text{gauge terms}, \qquad (367)$$

where $F_0^{\mu\nu} = \partial^{\mu}A_0^{\nu} - \partial^{\nu}A_0^{\mu}$. This means that we should relate the unrenormalized ("bare") gauge field to the renormalized gauge field by:

$$A_0^{\mu} = \sqrt{Z_3} A^{\mu} \,. \tag{368}$$

We would have a relation very much like that for the fermion field:

$$\langle 0|T\{A_0^{\mu}(x)A_0^{\nu}(y)\}|0
angle \sim Z_3\langle 0|T\{A^{\mu}(x)A^{\nu}(y)\}|0
angle$$
 (369)

giving rise to the interpretation that Z_3 is the probability that the bare field A_0^{μ} will propagate freely like a non-interacting particle.

In any case, we have seen that we can choose C so that the sum of the actual vacuum polarization diagram $i\Pi_{\mu\nu}$ of Eq. (316) plus the counter term leaves us with a form

$$i\widetilde{\Pi}_{\mu\nu} = i\Pi(k^2) \left[k^2 g_{\mu\nu} - k_{\mu} k_{\nu} \right] \equiv i\Pi(k^2) P_{\mu\nu}(k) , \qquad (370)$$

where $\Pi(k^2)$ is finite, and depends upon the scheme (i.e. on the finite terms in addition to the $\frac{1}{\epsilon}$ that we put into C).

Now a crucial identity:

$$P_{\alpha\beta}(k)\frac{g^{\beta\lambda}}{k^{2}}P_{\lambda\mu}(k) = P_{\alpha\beta}P_{\mu}^{\beta}\frac{1}{k^{2}}$$

$$= (k^{2}g_{\alpha\beta} - k_{\alpha}k_{\beta})(k^{2}g_{\mu}^{\beta} - k^{\beta}k_{\mu})\frac{1}{k^{2}}$$

$$= (k^{4}g_{\alpha\mu} - 2k_{\alpha}k_{\mu}k^{2} + k_{\alpha}k_{\mu}k^{2})\frac{1}{k^{2}}$$

$$= P_{\alpha\mu}(k). \qquad (371)$$

With this in hand, we can iterate this 1PI irreducible object to develop a result for the full propagator:

$$egin{aligned} D_{
ho\sigma}(k) &= \; rac{-ig_{
ho\sigma}}{k^2} + rac{-ig_{
holpha}}{k^2} i \widetilde{\Pi}_{lphaeta}(k^2) rac{-ig_{eta\sigma}}{k^2} \ &+ rac{-ig_{
holpha}}{k^2} i \widetilde{\Pi}_{lphaeta}(k^2) rac{-ig_{eta\lambda}}{k^2} i \widetilde{\Pi}_{\lambda
u}(k) rac{-ig_{
u\sigma}}{k^2} + \dots \ &= \; rac{-ig_{
ho\sigma}}{k^2} + rac{-i(k^2g_{
ho\sigma}-k_{
ho}k_{\sigma})}{k^4} \left[\Pi(k^2) + \Pi^2(k^2) + \dots
ight] \end{aligned}$$

$$= \frac{-ig_{\rho\sigma}}{k^2} + \frac{-i(k^2g_{\rho\sigma} - k_{\rho}k_{\sigma})}{k^4} \frac{\Pi(k^2)}{1 - \Pi(k^2)}$$
$$= \frac{-ig_{\rho\sigma}}{k^2} \frac{1}{1 - \Pi(k^2)} + \frac{ik_{\rho}k_{\sigma}}{k^4} \frac{\Pi(k^2)}{1 - \Pi(k^2)}.$$
(372)

We can basically ignore the 2nd "gauge" term of this expression; it will not contribute to any physical amplitude because of the $k^{\mu}\mathcal{M}_{\mu} = 0$ type of identity (which is actually a consequence of the Ward-Takahashi identity). From this, we see that so long as $\Pi(k^2)$ does not have a singular behavior as $k^2 \to 0$ (e.g. $\Pi(k^2) = \frac{a}{k^2}$), which it does not by explicit calculation, the full photon propagator will still have a pole at $k^2 = 0$ implying that renormalization has not ended up giving a mass to the photon. We see that this is intimately related to the gauge invariant structure, namely $\widetilde{\Pi}_{\mu\nu}(k) \propto P_{\mu\nu}(k)$. Note that had there been such a singular behavior for $\Pi(k^2)$, the full photon propagator would behave as $\frac{1}{k^2-a}$, corresponding (roughly, i.e. modulo whatever other finite terms are present for $k^2 \to a$) to a photon mass-squared = a.

• Finally, we have the vertex correction to deal with.
To cancel the infinity in $\Lambda^{(1)}_{\mu}$ we must introduce a counter term of the form:

$$\Delta \mathcal{L} \ni -De\mu^{\epsilon} \overline{\psi} \, \mathcal{A}\psi \tag{373}$$

which in the standard fashion will give rise to a Feynman rule of form:

$$-iDe\mu^{\epsilon}\gamma_{\mu}.$$
 (374)

D must be chosen, therefore, so that

$$-ie\mu^{\epsilon}(\Lambda^{(1)}_{\mu}+D\gamma_{\mu})=finite$$
, with $\Lambda^{(1)}_{\mu}=rac{e^2}{16\pi^2\epsilon}\gamma_{\mu}+finite$. (375)

We make the simple choice of

$$D = -\frac{e^2}{16\pi^2\epsilon}.$$
(376)

The full structure is then:

$$\mathcal{L} + \Delta \mathcal{L} \quad \ni \quad -(1+D)e\mu^{\epsilon}A^{\mu}\overline{\psi}\gamma_{\mu}\psi \equiv \quad -Z_{1}e\mu^{\epsilon}A^{\mu}\overline{\psi}\gamma_{\mu}\psi ,$$
 (377)

with

$$Z_1 = 1 + D = 1 - \frac{e^2}{16\pi^2\epsilon}.$$
 (378)

Thus, in the end we have the three multiplicative renormalization factors:

$$Z_1 = Z_2 = 1 - \frac{e^2}{16\pi^2\epsilon}, \quad Z_3 = 1 - \frac{e^2}{12\pi^2\epsilon}.$$
 (379)

And, you should of course realize that $Z_1 = Z_2$ is no pure coincidence; it is required by the Ward identity as we have already stressed.

Finally, we can work out the relation between the bare charge, e_0 , and the renormalized charge, e. We must have

$$\mathcal{L} + \Delta \mathcal{L} = \mathcal{L}_0, \qquad (380)$$

which for the photon fermion fermion interaction term leads to the requirement

$$-Z_1 e \mu^{\epsilon} A^{\mu} \overline{\psi} \gamma_{\mu} \psi = -e_0 A_0^{\mu} \overline{\psi}_0 \gamma_{\mu} \psi_0 = -e_0 \sqrt{Z_3} (\sqrt{Z_2})^2 A^{\mu} \overline{\psi} \gamma_{\mu} \psi , \qquad (381)$$

where we inserted the relations of the bare fields to the renormalized fields

established earlier. This implies the relation

$$e_0 = e\mu^{\epsilon} \frac{Z_1}{Z_2 Z_3^{1/2}} = e\mu^{\epsilon} Z_3^{-1/2}, \qquad (382)$$

where the latter equality follows from the Ward identity requirement (and also our explicit 1-loop result) of $Z_1 = Z_2$.

• Summary

We have absorbed all the infinite quantities by establishing appropriate relations between the renormalized fields and renormalized mass and the bare fields and bare mass. The fact that we were able to do so, keeping a \mathcal{L} of the same form as the original means that, to this 1-loop order, QED is indeed renormalizable. The proof that the renormalization procedure can be carried out to all orders appears in Ryder and other books.

Here, I wish to focus on some more implications of the 1-loop results.

• The asymptotic behavior of $e(\mu)$.

We begin with the result of Eq. (382):

$$e_{0} = e\mu^{\epsilon} Z_{3}^{-1/2}$$

$$= e\mu^{\epsilon} \left(1 - \frac{e^{2}}{12\pi^{2}\epsilon}\right)^{-1/2}$$

$$= e\mu^{\epsilon} \left(1 + \frac{e^{2}}{24\pi^{2}\epsilon}\right) + \mathcal{O}(e^{4}). \quad (383)$$

We now differentiate in the usual fashion:

$$0 = \mu \frac{\partial e_0}{\partial \mu} = \epsilon e \mu^{\epsilon} \left(1 + \frac{e^2}{24\pi^2 \epsilon} \right) + \mu \frac{\partial e}{\partial \mu} \left[\mu^{\epsilon} \left(1 + \frac{e^2}{24\pi^2 \epsilon} \right) + e \mu^{\epsilon} \frac{e}{12\pi^2 \epsilon} \right]$$
$$= \epsilon e \mu^{\epsilon} \left(1 + \frac{e^2}{24\pi^2 \epsilon} \right) + \mu \frac{\partial e}{\partial \mu} \mu^{\epsilon} \left[1 + \frac{e^2}{8\pi^2 \epsilon} \right].$$
(384)

Solving for $\mu \frac{\partial e}{\partial \mu}$ gives

$$\mu rac{\partial e}{\partial \mu} \hspace{0.1 in} = \hspace{0.1 in} -\epsilon e rac{1+rac{e^2}{24\pi^2\epsilon}}{1+rac{e^2}{8\pi^2\epsilon}}$$

$$= -\epsilon e \left(1 - \frac{e^2}{12\pi^2 \epsilon} + \mathcal{O}(e^4) \right)$$

$$\stackrel{\epsilon \to 0}{\to} \frac{e^3}{12\pi^2}$$

$$\equiv \beta(e). \qquad (385)$$

Thus, as in ϕ^4 theory, $\beta(e) > 0$. The solution of the above equation, which is obvious after rewriting it in the form:

$$\mu \frac{\partial e^2}{\partial \mu} = \frac{(e^2)^2}{6\pi^2} \tag{386}$$

is

$$e^{2}(\mu) = \frac{e^{2}(\mu_{0})}{1 - \frac{e^{2}(\mu_{0})}{6\pi^{2}} \ln \frac{\mu}{\mu_{0}}}$$
(387)

and we see that $e^2(\mu)$ increases with increasing μ or decreasing distance scale. The singular point at

$$\mu = \mu_0 \exp\left(\frac{6\pi^2}{e^2(\mu_0)}\right) \tag{388}$$

is sometimes referred to as the Landau singularity.

This increase of $e(\mu)$ at short distances has the following interpretation. We begin at large distances and imagine the photon probing an electron, e^- . At large distances, the photon has lots of sequential fermion loop (bubble) insertions of e^+e^- pairs in its propagator before it interacts with the electron. These bubbles polarize (like a dielectric medium) so that the e^+ of each bubble insertion spends more time near the source e^- than does the e^- of each bubble. Thus, the fundamental charge of the source e^- is substantially shielded by the "polarization of the vacuum" (that is where the name comes from). As we probe on shorter and shorter distance scales (higher momentum scales), we penetrate further and further inside this cloud of bubbles and we see more and more of the bare e^- charge.

• What is the best evidence for a running α ? An analysis by OPAL of their LEP data on small angle Bhabha scattering claims to be the best.



Figure 12: |t| spectrum normalized to the BHLUMI theoretical prediction for a fixed coupling $(\Delta \alpha = 0)$. The points show the combined OPAL data with statistical error bars. The solid line is our fit. The horizontal line (Ratio=1) is the prediction if α were fixed. The dot-dashed curve is the prediction of running α determined by vacuum polarization with only virtual e⁺e⁻ pairs, the dashed curve includes all charged lepton pairs and the dotted curve the full Standard Model prediction, with both lepton and quark pairs.

Figure 10: The comparison between Bhabha scattering data and QED (including some quark-related diagrams) predictions as a function of the momentum transfer t. LEP data from OPAL. CERN-PH-EP/2005-14.

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• The anomalous magnetic moment of the electron.

For this we will need to return to $\Lambda_{\mu}^{(2)}$. To set the stage, we first derive the standard tree-level result that $g_e = 2$, the so-called Dirac magnetic moment for the electron.

Before proceeding, we need to prove the Gordon identity. One begins with the vertex γ_{μ} . This we sandwich between spinors in the form $\overline{u}(p')\gamma_{\mu}u(p)$ to establish the interaction of an electron with the photon at tree-level. Recalling that

$$\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu}, \quad \gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu} = -2i\sigma_{\mu\nu} \tag{389}$$

we can write, using the Dirac equation for the spinors,

$$\begin{split} \overline{u}(p')\gamma_{\mu}u(p) &= \frac{1}{2m}\overline{u}(p')(p'\gamma_{\mu}+\gamma_{\mu}p)u(p) \\ &= \frac{1}{2m}\overline{u}(p')(p'^{\nu}\gamma_{\nu}\gamma_{\mu}+\gamma_{\mu}\gamma_{\nu}p^{\nu})u(p) \\ &= \frac{1}{2m}\overline{u}(p')(p'^{\nu}[g_{\mu\nu}+i\sigma_{\mu\nu}]+[g_{\mu\nu}-i\sigma_{\mu\nu}]p^{\nu})u(p) \end{split}$$

$$= \frac{1}{2m} \overline{u}(p') [(p'_{\mu} + p_{\mu}) + i\sigma_{\mu\nu}q^{\nu}] u(p) . \qquad (390)$$

This is the so-called Gordon identity.

To use this identity, we first consider the full vertex Γ_{μ} . Given the 4-vectors available, the most general possible expression is

$$\Gamma_{\mu}(p,q,p'=p+q) = \gamma_{\mu}A + (p'+p)_{\mu}B + (p'-p)_{\mu}C.$$
(391)

We have already seen that the other possible form, $\sigma_{\mu\nu}q^{\nu}$, is equivalent to a linear combination of the first two terms above when the structure is sandwiched between spinors. Further, the coefficients A, B, and C could all contain Dirac matrices contracted with vectors, *i.e.* p, or p'. But, if we plan to sandwich Γ_{μ} between spinors, $\overline{u}(p')\Gamma_{\mu}u(p)$, then $p \to m$ and $p' \to m$ and so there is for our purposes no need to consider A, B, Cas anything other than simple numbers. Given that we will also require $p^2 = p'^2 = m^2$, the only non-trivial number is $q^2 = -2p \cdot p' + 2m^2$. A, B, C can be functions of q^2 . However, we can simplify even further. Gauge invariance for the on-shell external in and out electron situation being considered requires that $q^{\mu}\overline{u}(p')\Gamma_{\mu}u(p) = 0$. This follows from the Dirac equation for the γ_{μ} term, and from $(p'-p) \cdot (p'+p) = p'^2 - p^2 = m^2 - m^2$ for the 2nd term, but is not true for the 3rd term. Thus, C = 0 is required.

At this point, it is conventional to replace the $(p' + p)_{\mu}$ term using the Gordon identity, and so we can say that the most general structure for the vertex is

$$\overline{u}(p')\Gamma_{\mu}u(p) = \overline{u}(p')\left[\gamma_{\mu}F_{1}(q^{2}) + \frac{i\sigma_{\mu\nu}q^{\nu}}{2m}F_{2}(q^{2})\right]u(p).$$
(392)

 F_1 and F_2 are called form factors. The hall-mark of an elementary particle is that at tree-level $F_1 = 1$ and $F_2 = 0$.

To see how this structure is related to the magnetic moment of the electron, we must allow the electron to interact with a classical vector potential, $A_{cl}^{\mu}(x) = (0, \vec{A}_{cl}(\vec{x}))$. It is the Fourier transform of this potential that appears in the $i\mathcal{M}$ matrix element for the electron scattering from this classical potential. To see exactly how, you consider a

$$\Delta H_{int} = \int d^3 \vec{x} e A^{\mu}_{cl} j_{\mu}, \quad \text{with} \quad j_{\mu}(x) = \overline{\psi}(x) \Gamma_{\mu} \psi(x). \tag{393}$$

In the leading order of perturbation theory, the S-matrix element for

scattering from this classical field is³

$$i\mathcal{M}(2\pi)\delta(p^{0}'-p^{0}) = -ie\overline{u}(p')\Gamma_{\mu}u(p)\widehat{A}^{\mu}_{cl}(p'-p)$$
(394)

where $\widehat{A}_{cl}^{\mu}(p'-p)$ is the 4-dimensional Fourier transform of $A_{cl}^{\mu}(x)$ which for a time-independent potential always takes the form

$$\widehat{A}^{\mu}_{cl}(p'-p) = (2\pi)\delta(p^{0}'-p^{0})\widetilde{A}^{\mu}_{cl}(\vec{q})$$
(395)

where $\widetilde{A}^{\mu}_{cl}(\vec{q})$ is the 3-dimensional Fourier transform. The result is that

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma_{\mu}u(p)\overline{A}^{\mu}_{cl}(\vec{q}).$$
(396)

In the case of $A^{\mu}_{cl}(x)=(0,ec{A_{cl}}(ec{x}))$, as appropriate for a magnetic field,

³The $(2\pi)^3 \delta^3 (\vec{p}' - \vec{p})$ is missing since 3-momentum conservation does not apply when dealing with scattering from a classical potential which is "infinitely massive" and can absorb any amount of momentum. To get the expression given, one simply considers the 1st order term in the expansion of $\exp[-i \int dt \Delta H_{int}]$ sandwiched between initial and final states, $\langle p'| - i \int dt \Delta H_{int} |p\rangle$, does the 1-particle contractions, and removes the usual $1/\sqrt{2VE}$ factors for the initial and final particles.

the above expression for $i\mathcal{M}$ takes the form⁴

$$i\mathcal{M} = ie\widetilde{A}^{i}_{cl}(\vec{q})\overline{u}(p')\left[\gamma^{i}F_{1}(q^{2}) + \frac{i\sigma^{i\nu}q_{\nu}}{2m}F_{2}(q^{2})\right]u(p).$$
(397)

The coefficient of A_{cl} vanishes as $q \to 0$ and so we must be careful to extract the linear term in q. We need the nonrelativistic expansion of the spinors u(p) and u(p'). The relevant expansion is

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \overline{\sigma}} \xi \end{pmatrix} \simeq \sqrt{m} \begin{pmatrix} (1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m})\xi \\ (1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m})\xi \end{pmatrix}.$$
 (398)

Then, the F_1 term can be simplified in the form:

$$egin{aligned} &\overline{u}(p')\gamma^{i}u(p)\ &=&\sqrt{m}\left({\xi'}^{\dagger}\left[1-rac{ec{p}'\cdotec{\sigma}}{2m}
ight]\,,{\xi'}^{\dagger}\left[1+rac{ec{p}'\cdotec{\sigma}}{2m}
ight]
ight)\left(egin{array}{c}0&1\1&0\end{array}
ight) \end{aligned}$$

⁴The usual — sign has been compensated by my having 2 up i indices rather than one up and one down. This is just the first of many such little sign changes associated with raised or lowered spatial indices. You have to be careful to get them all in correctly.

$$\times \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix} \sqrt{m} \begin{pmatrix} \left[1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \\ \left[1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \end{pmatrix}$$

$$= m \left(\xi'^{\dagger} \left[1 - \frac{\vec{p}' \cdot \vec{\sigma}}{2m}\right], \xi'^{\dagger} \left[1 + \frac{\vec{p}' \cdot \vec{\sigma}}{2m}\right] \right) \begin{pmatrix} -\sigma^{i} & 0 \\ 0 & \sigma^{i} \end{pmatrix} \begin{pmatrix} \left[1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \\ \left[1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \end{pmatrix}$$

$$= m \left(\xi'^{\dagger} \left[-\sigma^{i} + \frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^{i}\right], \xi'^{\dagger} \left[\sigma^{i} + \frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^{i}\right] \right) \begin{pmatrix} \left[1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \\ \left[1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \xi \end{pmatrix}$$

$$= m \xi'^{\dagger} \left(\left[-\sigma^{i} + \frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^{i}\right] \left[1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] + \left[\sigma^{i} + \frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^{i}\right] \left[1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}\right] \right) \xi$$

$$= 2m \xi'^{\dagger} \left(\frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^{i} + \sigma^{i} \frac{\vec{p} \cdot \vec{\sigma}}{2m} \right) \xi = \xi'^{\dagger} \left(\vec{p}'^{j} \sigma^{j} \sigma^{i} + \sigma^{i} \vec{p}^{j} \sigma^{j} \right) \xi$$

$$= \xi'^{\dagger} \left[\left(\vec{p}'^{i} + i \epsilon^{jik} \sigma^{k} \right) + \vec{p}^{j} (\delta^{ij} + i \epsilon^{ijk} \sigma^{k}) \xi \right]$$

$$= \xi'^{\dagger} \left[\left(\vec{p}'^{i} + \vec{p}^{i} \right) - i \left(\vec{p}'^{j} - \vec{p}^{j} \right) \epsilon^{ijk} \sigma^{k} \right) \xi$$

$$= \xi'^{\dagger} \left[\left(\vec{p}'^{i} + \vec{p}^{i} \right) - i q^{j} \epsilon^{ijk} \sigma^{k} \right) \xi , \qquad (399)$$

where we used $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$. The first of these terms substituted into Eq. (397) gives the contribution of the operator $[\vec{p}_{op} \cdot \vec{A} + \vec{A} \cdot \vec{p}_{op}]$ coming from minimal substitution in the standard kinetic energy term of nonrelativistic quantum mechanics. The 2nd term is the magnetic moment interaction we are after, which we rewrite in the form

$$\overline{u}(p')\gamma^{i}u(p) \ni 2m\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^{j}\sigma^{k}\right)\xi.$$
(400)

The F_2 term already contains an explicit factor of q, so we can evaluate it using the leading term in the spinor expansion and the fact that

$$\sigma^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0\\ 0 & \sigma^k \end{pmatrix}, \qquad (401)$$

as derived long ago, to obtain (sign change since we raise the index on q_j)

$$\overline{u}(p')\left(\frac{i}{2m}\sigma^{i\nu}q_{\nu}\right)u(p) = 2m\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^{j}\sigma^{k}\right)\xi.$$
 (402)

Thus, the full spin-dependent structure takes the form

$$\overline{u}(p')\left(\gamma^{i}F_{1}+\frac{i\sigma^{i\nu}q_{\nu}}{2m}F_{2}\right)u(p)\overset{q\to 0}{\to}2m\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^{j}\sigma^{k}\left[F_{1}(0)+F_{2}(0)\right]\right)\xi,$$
(403)

which gives, see Eq. (397),

$$i\mathcal{M} = ie\widetilde{A}_{cl}^{i}(\vec{q})\overline{u}(p') [\ldots] u(p)$$

$$\stackrel{q \to 0}{\to} ie\widetilde{A}_{cl}^{i}(\vec{q})2m\xi'^{\dagger} \left(\frac{-i}{2m}\epsilon^{ijk}q^{j}\sigma^{k} [F_{1}(0) + F_{2}(0)]\right)\xi$$

$$= -i(2m)e\xi'^{\dagger} \left(\frac{-1}{2m}\sigma^{k} [F_{1}(0) + F_{2}(0)]\right)\xi\widetilde{B}^{k}(\vec{q}), \quad (404)$$

where

$$\widetilde{B}^{k}(\vec{q}) = -i\epsilon^{ijk}q^{i}\widetilde{A}^{j}_{cl}(\vec{q})$$
(405)

is the Fourier transform of the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ produced by $\vec{A}_{cl}(\vec{x})$. We interpret this expression as the Born approximation to the scattering of the electron from a potential. The potential takes the form⁵ of a magnetic moment interaction

$$V(\vec{x}) = -\langle \vec{\mu} \rangle \cdot \vec{B}(\vec{x})$$
(407)

⁵The 2m factor in (404) is part of our relativistic normalization and should be dropped in comparing to the usual Born approximation — see Peskin Eq. (4.123) and surrounding discussion. The general rule in relativistic normalization is that

$$i\mathcal{M} = -i2mV_{cl}(\vec{q}) , \qquad (406)$$

where $\widetilde{V}_{cl}(\vec{q})$ is the 3-dimensional Fourier transform of the effective spatial potential $V_{cl}(\vec{x})$.

with the identification of

$$\langle \vec{\mu} \rangle = rac{e}{m} \left[F_1(0) + F_2(0) \right] {\xi'}^{\dagger} rac{\vec{\sigma}}{2} \xi \,.$$
 (408)

This should be compared to the standard form of

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S} \tag{409}$$

where \vec{S} is the electron spin operator $\frac{\vec{\sigma}}{2}$. Doing so, we find

$$g = 2 \left[F_1(0) + F_2(0) \right] = 2 + 2F_2(0) = 2 + \mathcal{O}(\alpha) ,$$
 (410)

where g is called the Lande g-factor.

You might ask how I can write so confidently that $F_1(0) = 1$ without doing the loop calculation, *i.e.* without evaluating the finite part of $\Lambda_{\mu}^{(1)}$ (remember $\Lambda_{\mu}^{(1)} \propto \gamma_{\mu}$ and thus would contribute to F_1 .). Well, it is because $F_1(0)$ defines the charge! To see this, we must again turn to a non-relativistic reduction but now in the case of a classical vector potential of the form $A_{cl}^{\mu}(\vec{x}) = (\phi(\vec{x}), \vec{0})$. The Fourier transform of this A_{cl}^{μ} is again what appears in $i\mathcal{M}$ and so we have, referring to Eq. (396):

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma^0 u(p)\widetilde{\phi}(\vec{q}).$$
(411)

For definition of the charge, we want to consider a potential that is essentially constant as a function of \vec{x} over some large region in which the charge is being probed. In this case, $\tilde{\phi}(\vec{q})$ will be concentrated at $\vec{q} = 0$. Thus, it is appropriate to take $\vec{q} \to 0$ in defining the charge, and the F_2 term then drops out and we evaluate F_1 at $\vec{q} = 0$. Then, we only need to compute

$$\overline{u}(p')\gamma^0 u(p) \stackrel{q \to 0}{\to} u^{\dagger}(p)u(p) = 2m\xi^{\dagger}\xi = 2m$$
(412)

so that we end up with

$$i\mathcal{M} = -ieF_1(0)\widetilde{\phi}(\vec{q})2m$$
 (413)

which is the Born approximation for scattering from a potential of the form

$$V(\vec{x}) = eF_1(0)\phi(\vec{x}), \qquad (414)$$

where, see previous footnote, the -i2m factor appears in $i\mathcal{M}$ multiplying the fourier transform of the classical potential. This means that $F_1(0)$ is the electric charge of the electron in units of e, which by definition means that $F_1(0) = 1$ is required after carrying out the renormalization program. Since $F_1(0) = 1$ at the 0-loop level (tree-level), radiative corrections to $F_1(q^2)$ should vanish as $q^2 \to 0$.

To compute $F_2(0)$, we first identify that part of $\Lambda_{\mu}^{(2)}$ that gives rise to the $i\sigma_{\mu\nu}q^{\nu}$ structure that defines F_2 . Referring to Eq. (325), we find

$$\overline{u}(p')\Lambda_{\mu}^{(2)}u(p) = \overline{u}(p')\left\{\frac{e^2}{16\pi^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{\gamma_{\nu}(p'(1-\beta)-\alpha p'+m)\gamma_{\mu}(p'(1-\alpha)-\beta p'+m)\gamma^{\nu}}{[-m^2(\alpha+\beta)+\alpha(1-\alpha)p^2+\beta(1-\beta)p'^2-\alpha\beta 2p'\cdot p]}\right\}u(p)$$
(415)

To proceed, we need to do some work on the numerator. Remembering that $\gamma_{\nu} \not a \not b \gamma^{\nu} = 4a \cdot b$, $\gamma_{\nu} \not a \not b \not c \gamma^{\nu} = -2 \not c \not b \not a$, $\gamma_{\nu} \not a \gamma^{\nu} = -2 \not a$, we have

$$\gamma_{\nu}(p'(1-\beta) - \alpha p' + m)\gamma_{\mu}(p'(1-\alpha) - \beta p' + m)\gamma^{\nu}$$

$$= -2(p'(1-\alpha) - \beta p')\gamma_{\mu}(p'(1-\beta) - \alpha p) + 4m(p_{\mu}(1-\alpha) - \beta p'_{\mu}) + 4m(p'_{\mu}(1-\beta) - \alpha p_{\mu}) - 2\gamma_{\mu}m^{2}$$

$$\rightarrow -2(p'(1-\alpha) - \beta m)\gamma_{\mu}(p'(1-\beta) - \alpha m) + 4m(p_{\mu}(1-\alpha) - \beta p'_{\mu}) + 4m(p'_{\mu}(1-\beta) - \alpha p_{\mu}) - 2\gamma_{\mu}m^{2}$$
(416)

where the \rightarrow indicates that we used the Dirac equation for the spinors.

We now need a couple of intermediate results.

$$\not p \gamma_{\mu} \not p' = 2p_{\mu} \not p' - \gamma_{\mu} \not p \not p'$$

$$= 2p_{\mu} \not p' - \gamma_{\mu} 2p \cdot p' + \gamma_{\mu} \not p' \not p$$

$$= 2p_{\mu} \not p' - \gamma_{\mu} 2p \cdot p' + 2p'_{\mu} \not p - \not p' \gamma_{\mu} \not p$$

$$\rightarrow 2p_{\mu} m - \gamma_{\mu} 2p \cdot p' + 2p'_{\mu} m - \gamma_{\mu} m^{2}$$

$$= 2(p_{\mu} + p'_{\mu})m - 3m^{2} \gamma_{\mu}$$

$$(417)$$

where in the last step we used the kinematics $p^2 = m^2$, $p'^2 = m^2$ and $(p - p')^2 = 2m^2 - 2p \cdot p' = q^2 = 0$ which implies $2p \cdot p' = 2m^2$. We also have

$$\not p \gamma_{\mu} = 2p_{\mu} - \gamma_{\mu} \not p \to 2p_{\mu} - \gamma_{\mu} m , \qquad (418)$$

$$\gamma_{\mu} \not p' = 2p'_{\mu} - \not p' \gamma_{\mu} \to 2p'_{\mu} - \gamma_{\mu} m.$$
(419)

Altogether, we get for the numerator

$$N = -2 \left[(1 - \alpha)(1 - \beta)(2(p + p')\mu m - 3m^{2}\gamma\mu) - \beta(1 - \beta)m(2p'_{\mu} - m\gamma\mu) - \alpha(1 - \alpha)m(2p\mu - m\gamma\mu) + \alpha\beta m^{2} \right] + 4m \left[p'_{\mu}(1 - 2\beta) + p_{\mu}(1 - 2\alpha) \right] - 2\gamma\mu m^{2}$$
$$= -2\gamma\mu m^{2}[-3(1 - \alpha)(1 - \beta) + \beta(1 - \beta) + \alpha(1 - \alpha) + \alpha\beta + 1] + 4p_{\mu}m[\beta - \alpha\beta - \alpha^{2}] + 4p'_{\mu}m[\alpha - \beta\alpha - \beta^{2}]$$
(420)

Meanwhile, the denominator takes the form (using the same kinematics as already discussed)

$$D = -m^2(\alpha + \beta)^2$$
. (421)

Because of the symmetry of the denominator, we can rewrite \boldsymbol{N} in the form

$$N = -2\gamma_{\mu}m^{2}[4(\beta + \alpha) - (\alpha + \beta)^{2} - 2] + 2(p_{\mu} + p'_{\mu})m[(\alpha + \beta) - (\alpha + \beta)^{2}]$$
(422)

so that we now have

$$\begin{split} \overline{u}(p')\Lambda_{\mu}^{(2)}u(p) \\ &= \overline{u}(p')\frac{-e^2}{16\pi^2m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2} \\ &\quad \left(-2\gamma\mu m^2[4(\beta+\alpha)-(\alpha+\beta)^2-2]+2(p\mu+p'_{\mu})m[(\alpha+\beta)-(\alpha+\beta)^2]\right)u(p) \\ &= \overline{u}(p')\frac{-e^2}{16\pi^2m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2} \\ &\quad \left(-2\gamma\mu m^2[4(\beta+\alpha)-(\alpha+\beta)^2-2]+2[2m\gamma\mu-i\sigma\mu\nu q^{\nu}]m[(\alpha+\beta)-(\alpha+\beta)^2]\right)u(p) \\ &= \overline{u}(p')\frac{-e^2}{16\pi^2m^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)^2} \\ &\quad \left(-2\gamma\mu m^2[2(\beta+\alpha)+(\alpha+\beta)^2-2]-2i\sigma\mu\nu q^{\nu}m[(\alpha+\beta)-(\alpha+\beta)^2]\right)u(p). \end{split}$$

We actually don't care about the γ_{μ} part. For the anomalous magnetic moment we only want the coefficient of $+\frac{i\sigma_{\mu\nu}}{2m}$ which contains the two integrals

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{(\alpha+\beta)} = \int_0^1 d\alpha \ln(\alpha+\beta) \frac{\beta=1-\alpha}{\beta=0} = -\int_0^1 d\alpha \ln\alpha = -[\alpha \ln\alpha - \alpha]_0^1 = 1 \quad (424)$$

and

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta = \frac{1}{2}$$
(425)

yielding

$$\frac{i\sigma_{\mu\nu}q^{\nu}}{2m}\left[\frac{e^2}{8\pi^2}\right] = \frac{i\sigma_{\mu\nu}q^{\nu}}{2m}\left[\frac{\alpha}{2\pi}\right] \,. \tag{426}$$

The above is the well-known result from Schwinger in 1948 for the anomalous magnetic moment addition to the bare contribution yielding a total of

$$\frac{i\sigma_{\mu\nu}q^{\nu}}{2m}\left[1+\frac{\alpha}{2\pi}\right].$$
(427)

where the relation to the usual g_e is

$$\frac{g_e}{2} = 1 + \frac{\alpha}{2\pi}$$
 (428)

Please note that I got the correct sign for my sign of the denominator and that Ryder slipped in an extra sign.

The electron anomalous magnetic moment

At present, $\frac{g_e}{2}$ has been calculated to order α^5 (see arXiv:1205.5368) yielding impressive agreement between theory and experiment. Until very recently, the comparison was

$$a_{e}^{th} = \frac{1}{2}(g_{e} - 2)$$

= (1159652.4 ± 0.4) × 10⁻⁹
$$a_{e}^{exp} = (1159652.4 \pm 0.2) × 10^{-9}.$$
 (429)

Actually, experiment (denoted by HV for Harvard) has gotten a lot better recently:

$$a_e^{exp} = a_e(HV) = (1159652180.73 \pm 0.28) \times 10^{-12}$$
. (430)

The latest theoretical prediction is

$$a_e^{th} = 1159652181.78(6)(4)(3)(77) \times 10^{-12}$$
(431)

where α is being determined using h/m_{Rb} , the Rydberg constant and m_{Rb}/m_e . In the above, the first, second, third, and fourth uncertainties come from the eighth-order term, the tenth-order term, the hadronic corrections, and the above means of determining the fine- structure constant, respectively.

Going to a still higher accuracy and higher order in α for the theoretical prediction of the QED contribution is difficult to justify at the moment. The above prediction already includes the very small hadronic and electroweak loop diagrams:

 $a_e(\text{had. vac pol}) = 1.875(18) \times 10^{-12}$

 $a_e({\sf NLO} \; {\sf had.} \; {\sf vac} \; {\sf pol}) \; = \; -0.225(5) imes 10^{-12}$

 a_e (had. light by light scattering) = $0.035(10) \times 10^{-12}$

 $a_e(\text{weak}) = 0.0297(5) \times 10^{-12}$. (432)

At this point, agreement is impressive:

$$a_e(HV) - a_e(theory) = -1.06(0.82) \times 10^{-12}$$
. (433)

More rigorous comparision between experiment and theory is hindered by the uncertainty of $\alpha^{-1}(\text{Rb10})$,

$$\alpha^{-1}(\mathsf{Rb10}) = 137.035999049(90). \tag{434}$$

as determined from the precise measurements of h/m_{Rb} , the Rydberg constant and m_{Rb}/m_e .

Note that the sum $1.685(21) \times 10^{-12}$ of the hadronic contributions is now larger than the exp. / theory difference. It is thus desirable to reexamine and update the values of the hadronic contributions.

The result of Eq. (431) shows clearly that the largest source of uncertainty is the fine-structure constant. To put it differently, it means that a non-QED α , even the best one available at present, is too crude to test QED to the extent achieved by the theory and measurement of a_e . Thus it makes more sense to test QED by an alternative approach, namely, compare α^{-1} (Rb10)

with α^{-1} obtained from theory and measurement of a_e . This leads to

$$\alpha^{-1}(a_e) = 137.0359991736(68)(46)(26)(331), \quad (435)$$

where the first, second, third, and fourth uncertainties come from the eighth-order and the tenth-order QED terms, the hadronic and electroweak terms, and the measurement of $a_e(HV)$, respectively.

From the broader perspective, the SM calculation agrees very well with the best available laboratory measurements, and it is this kind of agreement that makes us think that this whole business of QED and renormalization really does make sense.

In the past, one employed a_e as the means of determining α , then you can use this value of α to predict a_{μ} . This was the approach used in the PDG prior to the 2011 update. However, the determination of α from Rb10 is (at the moment) regarded as a better approach.

The muon anomalous magnetic moment

The above sets the stage for computing a_{μ} .

Representative diagrams contributing to the muon anomalous magnetic moment appear below.



Figure 11: Lowest order Electromagnetic (EM), Electroweak (EW) and Hadronic (HD) contributions to a_{μ} .

In fact, the QED part has been computed to order $(\alpha/\pi)^5$ with α determined as described above from the Rb10 data, leading to

$$a_{\mu}^{QED} = 116584718.95(0.08) \times 10^{-11}$$
. (436)

The story regarding the other contributions is too lengthy for us to cover

here. Suffice it to say that in the end it is the HD contributions that lead to the biggest error. Indeed, there are 2 ways of computing these that give slightly discrepant results from one another:

- using e^+e^- collision data directly to measure the hadronic bubble insertion — the integral over the bubble is most sensitive to low energy data, such as around the $\rho(770)$ resonance, such data having been obtained some time ago and not necessarily that accurate;
- using $\tau \rightarrow \nu_{\tau} + hadrons$ decays (which are precisely measured) and then using isospin symmetry to relate to $e^+e^- \rightarrow hadrons$ at low energy, the latter involving some uncertainties related to isospin breaking.

Also, different groups have worked on the theoretical prediction as summarized in the figure below.



Using either the direct e^+e^- data approach or the τ -decay data approach one finds a discrepancy between the SM prediction for a_{μ} and the experimental measurement of a_{μ} .

For the direct e^+e^- measurement approach, the PDG summary is:

$$a_{\mu}^{th} = \frac{1}{2}(g_e - 2) = a_{\mu}^{QED} + a_{\mu}^{EW} + a_{\mu}^{HAD}$$
$$= (116591803 \pm 1 \pm 42 \pm 26) \times 10^{-11}, \quad (437)$$

to be compared to

$$a_{\mu}^{exp} = (11659208.9 \pm 5.4 \pm 3.3) \times 10^{-10}$$
. (438)

In the theory prediction, the errors are due to the electroweak, lowest-order hadronic, and higher-order hadronic contributions, respectively. In the experimental result, the errors are from statistical and systematic errors, respectively.

The result is that

$$\Delta a_{\mu} = a_{\mu}^{exp} - a_{\mu}^{th} = 288(63)(49) \times 10^{-11}.$$
(439)

corresponding to a 3.6 σ discrepancy. Using τ data instead of e^+e^- data directly in a critical low energy regime for the HD contribution reduces the discrepancy to 2.4 σ .

Please look at the PDG chapter on this for more details.

One interpretation of this discrepancy should it turn out to be real is the existence of supersymmetric particles, which would also contribute loop

corrections to a_{μ} . Generically, supersymmetric models predict an additional contribution

$$a_{\mu}^{SUSY} \sim \operatorname{sign}(\mu) \times 130 \times 10^{-11} \left(\frac{100 \text{ GeV}}{m_{SUSY}}\right)^2 \tan\beta$$
 (440)

where m_{SUSY} is a representative supersymmetric mass scale, $\tan \beta = 3-40$ a potential enhancement factor, and $\operatorname{sign}(\mu) = \pm 1$. Supersymmetric particles in the mass range 100 - 500 GeV could be the source of the deviation, Δa_{μ} . If so, those particles should be directly observed at the Large Hadron Collider at CERN.

How to use the Ward identity in the proof of renormalizability

I will give just a simple illustration of why the WI is so useful: Overlapping Divergences.

Consider the diagram for $-i\Sigma$ that corresponds to two overlapping virtual photon loops. Labeling the photon vertices as ρ, μ, ρ, μ (after already using the g_{\dots} of the photon propagators) and using k_1 and k_2 for the $\mu \to \mu$ photon and the $\rho \to \rho$ photon, respectively, the expression for the diagram takes the form

$$\int d^4k_1 d^4k_2 \frac{1}{k_1^2} \frac{1}{k_2^2} \gamma^{\rho} \frac{1}{\not p - \not k_2 - m} \gamma^{\mu} \frac{1}{\not p - \not k_1 - \not k_2 - m} \gamma_{\rho} \frac{1}{\not p - \not k_1 - m} \gamma_{\mu},$$
(441)

which is superficially D = 1 divergent. If k_1 is held fixed, then the d^4k_2 integral has D = 0. In other words the overlapping divergences from the two integrations cannot be separated out. The WI allows one to circumvent this difficulty. The idea is to remember that the WI says that $\frac{\partial \Sigma}{\partial p^{\alpha}} = -\Lambda_{\alpha}$ and to begin by showing that in Λ_{α} the divergences can be separated. In fact, this latter separation is always possible. There are 3 associated Λ_{α} graphs corresponding to the 3 different locations in the Σ graph at which a photon can be attached on an internal electron propagator. Let us consider,

for example, the diagram in which the photon (index α) is attached in the left-most location. For our purposes, the photon can have q = 0. The expression for this diagram takes the form

$$\int d^4k_1 d^4k_2 rac{1}{k_1^2} \gamma^
ho rac{1}{\not p - \not k_2 - m} \gamma^\mu rac{1}{\not p - \not k_1 - \not k_2 - m} \gamma_
ho rac{1}{\not p - \not k_1} \gamma_lpha rac{1}{\not p - \not k_1} \gamma_\mu rac{1}{k_2^2} .
onumber \ (442)$$

Now, for fixed k_2 , we have $\int \frac{d^4k_1}{k_1^5}$ which is convergent and only $\int d^4k_2$ must be dealt with, but this can be done using the usual procedures (dimensional regularization). The story is the same for all the diagrams. But, now we can figure out how to write Σ by using the WI differential equation.

At this point, I could discuss the 2-loop calculation for the photon propagator. A version of this calculation appears in Itzykson and Zuber (who employ Euclidean conventions). The discussion in the case of the QCD correction to the photon propagator appears towards the end of these pdf notes. You could go and look at this later presentation and remove the color factors and change g^2 to e^2 and so forth. It is more convenient to delay this two-loop calculation until we can see its relevance to the QCD calculation of the correction to the famous R_{had} ratio of e^+e^- annihilation.

Renormalization of QCD

Because the gluon-quark-quark and 3-gluon vertex are both given by g times appropriate structure, we could compute the 1-loop corrections to either in order to figure out how g behaves as a function of the scale μ . Hopefully, they would continue to be tied together at 1-loop. This is, indeed, the case. For us, it is simpler to use the gluon-quark-quark vertex to define g since then we can relate the required calculations in part to those already done for QED. We will do our calculations in Feynman gauge, which means that at some points we will have to consider ghosts. Relevant Feynman rules can be found in the Path Integral Notes pdf file.



The calculation is much like the QED one. For the main "non-tadpole" diagram,⁶ we have the $-i\Sigma^{AB}$ contribution written below, where A, B are

⁶The tadpole diagrams are the gluon attachment with (a) fermion loop at the top, (b) gluon loop at the top and (c) ghost loop at the top — all are zero as we will discuss in the case of the corresponding gluon propagator corrections.

the initial and final quark color indices. We have:

$$-i\Sigma^{AB} = \int \frac{d^{n}k}{(2\pi)^{n}} (ig\mu^{\epsilon}\gamma_{\mu}) \frac{i}{\not p - \not k - m} (ig\mu^{\epsilon}\gamma_{\nu}) \frac{-ig^{\mu\nu}}{k^{2}} (L^{c})_{AD} (L^{c})_{DB}.$$
(443)

(Here, L_{AB}^{a} is our notation for the specific fundamental triplet representation matrix.) Thus, we very quickly obtain

$$-i\Sigma^{AB} = C_F \delta^{AB} \times -i\Sigma(QED, e \to g) = -i\frac{g^2}{16\pi^2\epsilon}(-\not p + 4m)C_F \delta^{AB}.$$
(444)
(444)

Using $C_F = 4/3$ for SU(3), and using the p coefficient to determine Z_2 , we obtain

$$Z_2 = 1 - \frac{g^2}{16\pi^2\epsilon} C_F = 1 - \frac{g^2}{12\pi^2\epsilon}.$$
 (445)

$i\Pi$

The one loop diagrams are: gluon-loop insertion; ghost-loop insertion; fermion-loop insertion; gluon loop quartic vertex "tadpole"; gluon-loop true tadpole; ghost-loop true tadpole; the fermion-loop true tadpole. The loop integration momentum will typically be called p and the external gluon momentum will be called k.

The fermion-loop true tadpole is 0 for exactly the same reason as in QED $-\int \frac{d^n p}{(2\pi)^n} \frac{\text{Tr}[\gamma_\alpha(\not p+m)]}{p^2-m^2} = 0$ due to $\text{Tr}[\gamma_\alpha] = 0$ and oddness of $\text{Tr}[\gamma_\alpha \not p] = 4p_\alpha$ under $p \to -p$, whereas the denominator is even. Here α was the Lorentz index of the virtual q = 0 gluon running from the propagating gluon and attaching to the fermion loop.

The gluon-loop true tadpole and ghost-loop true tadpole are both 0 for the same basic reason: they both are proportional to $\int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2} \times p^{\alpha} = 0$.

The quartic vertex "tadpole" is 0 for a more subtle reason. The gluon loop integral does not contain any numerator momenta. Thus, the integral in question is

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2}.$$
(446)

To evaluate this, we temporarily introduce an m^2 and then later take $m^2 \rightarrow 0$. We have, using our master formula,

$$\int \frac{d^{n}p}{(2\pi)^{n}} \frac{1}{p^{2} - m^{2}} = (-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \frac{\Gamma(1 - \frac{n}{2})}{\Gamma(1)} [-m^{2}]^{n/2 - 1}$$

$$\stackrel{m^{2} \to 0}{\to} 0, \qquad (447)$$

provided that n/2 - 1 > 0, which is of course the case for $n \to 4$. Note that

for $n \rightarrow 2$, we cannot use this argument and there is in fact a pole at $n \rightarrow 2$ reflecting the basic quadratic divergence of this integral.

To make this more explicit and to illustrate some important lessons we evaluate the basic integral using a different set of tricks. We write

$$\int \frac{d^{n}p}{(2\pi)^{n}} \frac{1}{p^{2}} = \int \frac{d^{n}p}{(2\pi)^{n}} \frac{1}{p^{2}} \frac{(p+k)^{2}}{(p+k)^{2}}$$

$$= \int d\alpha \int \frac{d^{n}p}{(2\pi)^{n}} \frac{(p+k)^{2}}{[\alpha(p+k)^{2} + (1-\alpha)p^{2}]^{2}}$$

$$= \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \frac{(p'+(1-\alpha)k)^{2}}{[p'^{2} + \alpha(1-\alpha)k^{2}]^{2}} \text{ using shift to } p' = p + \alpha k$$

$$= \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \left[\frac{1}{[p'^{2} + \alpha(1-\alpha)k^{2}]} + \frac{(1-\alpha)(1-2\alpha)k^{2}}{[p'^{2} + \alpha(1-\alpha)k^{2}]^{2}} \right] \text{ dropping terms odd in } p'$$

$$= \int d\alpha (-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \left[\frac{\Gamma(1-\frac{n}{2})}{\Gamma(1)} \frac{1}{[\alpha(1-\alpha)k^{2}]^{1-\frac{n}{2}}} + \frac{\Gamma(2-\frac{n}{2})(1-\alpha)(1-2\alpha)k^{2}}{[\alpha(1-\alpha)k^{2}]^{2-\frac{n}{2}}} \right]$$

$$= \int d\alpha (-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \frac{1}{[\alpha(1-\alpha)k^{2}]^{2-\frac{n}{2}}} \left[\Gamma(1-\frac{n}{2})k^{2}\frac{\alpha}{2}\alpha(1-\alpha) + \Gamma(2-\frac{n}{2})k^{2}(1-\alpha)^{2} \right] ,$$

$$(448)$$

where in the last step we used

$$\alpha(1-\alpha)\Gamma(1-\frac{n}{2}) + (1-\alpha)(1-2\alpha)\Gamma(2-\frac{n}{2})$$

= $\alpha(1-\alpha)\left[\Gamma(1-\frac{n}{2}) - \Gamma(2-\frac{n}{2})\right] + (1-\alpha)^2\Gamma(2-\frac{n}{2}) = \alpha(1-\alpha)\Gamma(1-\frac{n}{2})\frac{n}{2} + (1-\alpha)^2\Gamma(2-\frac{n}{2}).$ (449)
Now, as $n \to 4$, we can still check that we get 0 by virtue of cancellation of the two terms. We must use (as $n \to 4$)

$$\Gamma(1-\frac{n}{2}) = \frac{\Gamma(2-\frac{n}{2})}{1-\frac{n}{2}} \sim -\Gamma(2-\frac{n}{2}), \quad \int d\alpha \alpha (1-\alpha) = \frac{1}{6}, \quad \int d\alpha (1-\alpha)^2 = \frac{1}{3}.$$
(450)

But, we want to keep the form given so that we can see how this term combines with the other 1-loop Π contributions when $n \neq 4$. In particular, we want to see that we get a gauge-invariant structure even when $n \neq 4$, thereby demonstrating that dimensional regularization preserves gauge invariance.

Of course, we have not written down this "tadpole" contribution with all the correct factors and couplings and color structure. To see what we get when we do that, we start with our basic Feynman expression, including color factors, in the form (using gluon color labels ordered clockwise as a, d, c, b going with μ, σ, ρ, ν — also note contraction symmetry factor of 1/2 and use $g_{\rho}^{\rho} = n$):⁷

$$\begin{split} i\Pi^{ab}_{\mu\nu} & \ni \quad \frac{1}{2} \int \frac{d^{n}p}{(2\pi)^{n}} \frac{-ig\rho\sigma}{p^{2}} \delta^{cd}(ig^{2}) \\ & \left[ic^{dce} ic^{eba} (g^{\sigma\nu}g^{\mu\rho} - g^{\mu\sigma}g^{\nu\rho}) + ic^{dbe} ic^{eca} (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + ic^{dae} ic^{ebc} (g^{\sigma\nu}g^{\rho\mu} - g^{\sigma\rho}g^{\mu\nu}) \right] \end{split}$$

⁷The version of the Feynman rule employed below is taken from Fig. 12 of QFT-III notes in which the canonical ic^{dce}, \ldots color factors are separated out. Thus the $+ig^2$ overall factor is correct.

$$= \frac{1}{2}g^{2}\int \frac{d^{n}p}{(2\pi)^{n}} \frac{1}{p^{2}} \left[0 + (-C_{A}\delta^{ab})[(n-1)g_{\mu\nu}] + (C_{A}\delta^{ab})[-(n-1)g_{\mu\nu}] \right]$$

$$= -g^{2}C_{A}\delta^{ab}(-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\int d\alpha \frac{(n-1)g_{\mu\nu}}{\left[\alpha(1-\alpha)k^{2}\right]^{2-\frac{n}{2}}} \left[\Gamma(1-\frac{n}{2})k^{2}\frac{n}{2}\alpha(1-\alpha) + \Gamma(2-\frac{n}{2})k^{2}(1-\alpha)^{2} \right]$$
(451)

where the color diagram created by $\delta^{cd}ic^{dae}ic^{ebc} \Rightarrow +C_A\delta^{ab}$ (no crossed gluons) whereas that created by $\delta^{cd}ic^{dbe}ic^{eca}$ has a crossed gluon configuration that must be "unwrapped" (leading to a minus sign because of c... antisymmetry) to obtain a diagram that leads to $-C_A\delta^{ab}$.

So, now we must work on the other diagrams.

The fermion-loop diagram works just like QED except for the extra color factor of $\frac{1}{2}$ from our representation matrix product trace and a factor of n_f which is the number of different "flavors" of quarks that should be put into the fermion loop. Referring back to Eq. (309), we have (for the singular piece):

$$i\Pi_{\mu\nu}^{ab} \ni \frac{ig^2}{12\pi^2} \delta^{ab} (k_{\mu}k_{\nu} - k^2 g_{\mu\nu}) \Gamma(2 - \frac{n}{2}) (\frac{1}{2}n_f) , \qquad (452)$$

where as $\epsilon \to 0$, $\Gamma(2 - \frac{n}{2}) = \Gamma(\epsilon) \to \frac{1}{\epsilon}$.

Next comes the gluon-loop diagram. Here, the color calculation just gives

us $C_A \delta^{ab}$. Below, we discuss the Feynman-momentum space calculation. Starting with a gluon vertex μ, ρ, σ (moving clockwise) on the left and ρ', ν, σ' on the right, and using p flowing into the left-hand vertex along the (bottom) σ, σ' gluon line and p + k flowing out of the left-hand vertex along the (top) ρ, ρ' line, we have

$$i\Pi_{\mu\nu}^{ab} = C_A \delta^{ab} \frac{(-ig\mu^{\epsilon})^2}{2} \int \frac{d^n p}{(2\pi)^n} \frac{N_{\mu\nu}}{(p+k)^2 p^2} = C_A \delta^{ab} \frac{(-ig\mu^{\epsilon})^2}{2} \int d\alpha \int \frac{d^n p}{(2\pi)^n} \frac{N_{\mu\nu}}{[(p+\alpha k)^2 + \alpha(1-\alpha)k^2]^2}$$
(453)

where

$$N_{\mu\nu} = [g_{\mu\sigma}(-k+p)\rho + g_{\mu}\rho(p+2k)\sigma + g_{\rho\sigma}(-2p-k)\mu] \times [g_{\sigma'\nu}(p-k)_{\rho'} + g_{\rho'\sigma'}(-2p-k)\nu + g_{\rho'\nu}(p+2k)_{\sigma'}](-ig^{\rho\rho'})(-ig^{\sigma\sigma'})$$

$$= -g_{\mu\nu}[5k^{2} + 2p^{2} + 2k \cdot p] - (n-6)k\mu k\nu - (4n-6)p\mu p\nu - (2n-3)(p\mu k\nu + p\nu k\mu))$$

$$p = p' - \alpha k - g_{\mu\nu}[k^{2}(5 - 2\alpha(1-\alpha)) + 6(p')^{2}\frac{n-1}{n}] - k\mu k\nu[(6-4n)\alpha(1-\alpha) + n-6]$$

$$\int \frac{d^{n}p'}{d^{n}p'} (-1)^{n/2}\frac{i}{(4\pi)^{n/2}}\frac{1}{[\alpha(1-\alpha)k^{2}]^{\epsilon}} \left\{ -g_{\mu\nu} \left[k^{2}[5 - 2\alpha(1-\alpha)]\Gamma(2-\frac{n}{2})\right] \right\}$$

$$+3(n-1)k^2lpha(1-lpha)\Gamma(1-rac{n}{2})igg]-k\mu k
u\left([6-4n]lpha(1-lpha)+n-6
ight)\Gamma(2-rac{n}{2})$$

$$\stackrel{n \to 4}{\to} \frac{i}{(4\pi)^2} \Gamma(\epsilon) \left\{ -g_{\mu\nu} \left(k^2 [5 - 11\alpha(1 - \alpha)] \right) - k_{\mu} k_{\nu} \left(-10\alpha(1 - \alpha) - 2 \right) \right\}$$

$$\stackrel{\int d\alpha}{\to} \frac{i}{(4\pi)^2} \Gamma(\epsilon) \left\{ -\frac{19}{6} g_{\mu\nu} k^2 + \frac{11}{3} k_{\mu} k_{\nu} \right\}, \qquad (454)$$

where we isolated only the singular part. This leads to

$$i\Pi^{ab}_{\mu\nu} = C_A \delta^{ab} \left(\frac{-g^2}{2}\right) \frac{i}{(4\pi)^2} \Gamma(\epsilon) \left\{-\frac{19}{6}g_{\mu\nu}k^2 + \frac{11}{3}k_{\mu}k_{\nu}\right\},$$
 (455)

which can be compared to Ryder's result. However, we really want to back track to the expression obtained after $\int d^n p'$. At that stage our result is

$$\begin{split} i\Pi^{ab}_{\mu\nu} &= C_A \delta^{ab} \left(\frac{-g^2}{2} \right) \int d\alpha (-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \frac{1}{[\alpha(1-\alpha)k^2]^\epsilon} \Biggl\{ -g_{\mu\nu} \Biggl[k^2 [5 - 2\alpha(1-\alpha)] \Gamma(2-\frac{n}{2}) \\ &+ 3(n-1)k^2 \alpha(1-\alpha) \Gamma(1-\frac{n}{2}) \Biggr] - k\mu k_\nu \left([6 - 4n] \alpha(1-\alpha) + n - 6 \right) \Gamma(2-\frac{n}{2}) \Biggr\} \end{split}$$

(456)

So, finally, we have to compute the ghost-loop diagram. You will notice that if we sum the "tadpole" and the gluon-loop diagram, we do not have the required gauge invariant structure yet. Also, both diagrams have a $\Gamma(1 - \frac{n}{2})$ piece (reflecting a quadratic divergence) and these do not cancel, suggesting that we could have a hierarchy problem. The ghost-loop diagram will take care of both problems.

The expression we must evaluate is (remember - sign for ghost loop)

$$i\Pi_{\mu\nu}^{ab} = (-1)C_A \delta^{ab} \int \frac{d^n p}{(2\pi)^n} \frac{i}{p^2} \frac{i}{(p+k)^2} (-ig)^2 (p+k) \mu p_{\nu}$$

$$= -g^2 C_A \delta^{ab} \int d\alpha \int \frac{d^n p}{(2\pi)^n} \frac{(p+k) \mu p_{\nu}}{[(p+\alpha k)^2 + \alpha(1-\alpha)k^2]^2}$$

$$= -g^2 C_A \delta^{ab} \int d\alpha \int \frac{d^n p'}{(2\pi)^n} \frac{(p') \nu (p') \mu - \alpha(1-\alpha) k \mu k_{\nu}}{[(p')^2 + \alpha(1-\alpha)k^2]^2} \quad \text{dropping terms odd in } p'$$

$$= -g^2 C_A \delta^{ab} \int d\alpha (-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \frac{1}{[\alpha(1-\alpha)k^2]^{\epsilon}} \left[\frac{1}{2} g \mu \nu k^2 \alpha (1-\alpha) \Gamma(1-\frac{n}{2}) - \alpha(1-\alpha) k \mu k_{\nu} \Gamma(2-\frac{n}{2}) \right]$$
(457)

We are now in a position to collect our results together. Let us first examine the $\Gamma(1-\frac{n}{2})$ terms. These are (from the quartic "tadpole" loop,

gluon loop and ghost loop, respectively — the fermion loop did not have a quartic divergence since it was proportional to the GI structure on its own)

$$\begin{split} i\Pi_{\mu\nu}^{ab} & \ni \quad g_{\mu\nu}(-1)^{n/2} \frac{ig^2}{(4\pi)^{n/2}} C_A \delta^{ab} \Gamma(1-\frac{n}{2}) \int d\alpha \frac{k^2 \alpha (1-\alpha)}{[k^2 \alpha (1-\alpha)]^{\epsilon}} \left[-(n-1)\frac{n}{2} + \frac{3(n-1)}{2} - \frac{1}{2} \right] \\ & = \quad g_{\mu\nu}(-1)^{n/2} \frac{ig^2}{(4\pi)^{n/2}} C_A \delta^{ab} \Gamma(1-\frac{n}{2}) \int d\alpha \frac{k^2 \alpha (1-\alpha)}{[k^2 \alpha (1-\alpha)]^{\epsilon}} \left[(1-\frac{n}{2})(n-2) \right] \\ & = \quad g_{\mu\nu}(-1)^{n/2} \frac{ig^2}{(4\pi)^{n/2}} C_A \delta^{ab} \Gamma(2-\frac{n}{2}) \int d\alpha \frac{k^2 \alpha (1-\alpha)}{[k^2 \alpha (1-\alpha)]^{\epsilon}} (n-2) \,. \end{split}$$
(458)

Note that all the singularities at $1 - \frac{n}{2} = 0$ from $\Gamma(1 - \frac{n}{2})$ have been canceled by the explicit factor of $1 - \frac{n}{2}$ that emerged after summing all the gauge-related diagrams. We now combine the residual above with all the other $g_{\mu\nu}$ terms and also accumulate the $k_{\mu}k_{\nu}$ terms to obtain:

$$i\Pi_{\mu\nu}^{ab} = (-1)^{n/2} \frac{ig^2}{(4\pi)^{n/2}} C_A \delta^{ab} \Gamma(2 - \frac{n}{2}) \int \frac{d\alpha}{[k^2 \alpha (1 - \alpha)]^\epsilon} \left\{ g_{\mu\nu} k^2 \left[-(n - 1)(1 - \alpha)^2 + \frac{1}{2} [5 - 2\alpha (1 - \alpha)] + (n - 2)\alpha (1 - \alpha) \right] + k_{\mu} k_{\nu} \left[0 + \frac{1}{2} [(6 - 4n)\alpha (1 - \alpha) + n - 6] + \alpha (1 - \alpha) \right] \right\}$$

$$(459)$$

In the $g_{\mu\nu}$ coefficient expression, the 1st term comes from the $\Gamma(2-\frac{n}{2})$ quartic "tadpole" terms, the 2nd term from the gluon loop stuff proportional to $\Gamma(2-\frac{n}{2})$ and the last term is the residual after summing all the $\Gamma(1-\frac{n}{2})$ stuff so that it reduces to $\Gamma(2-\frac{n}{2})$ as described just above. In the $k_{\mu}k_{\nu}$ coefficient expression, the 1st 0 entry is for the quartic "tadpole" graph, the 2nd entry is for the gluon-loop graph, and the 3rd term is the $\Gamma(2-\frac{n}{2})$ part of the ghost-loop graph.

The above expression can be simplified by making the $g_{\mu\nu}$ coefficient explicitly symmetric in $\alpha \to 1 - \alpha$ using the symmetry of the denominator. This means we want to write

$$(1-\alpha)^{2} = \frac{1}{2} \left[(1-\alpha)^{2} + \alpha^{2} \right] = \frac{1}{2} - \alpha (1-\alpha) .$$
(460)

The $g_{\mu
u}k^2$ coefficient then becomes

$$-(n-1)\left[\frac{1}{2} - \alpha(1-\alpha)\right] + \frac{1}{2}\left[5 - 2\alpha(1-\alpha)\right] + (n-2)\alpha(1-\alpha)$$

$$= -n\left[\frac{1}{2} - 2\alpha(1-\alpha)\right] + \left[\frac{1}{2} - \alpha(1-\alpha) + \frac{5}{2} - \alpha(1-\alpha) - 2\alpha(1-\alpha)\right]$$

$$= -\frac{n}{2}\left[1 - 4\alpha(1-\alpha)\right] + \left[1 - 4\alpha(1-\alpha) + 2\right]$$

$$= (1 - \frac{n}{2})(1 - 2\alpha)^{2} + 2$$
(461)

Meanwhile, the $k_{\mu}k_{\nu}$ coefficient is:

$$rac{1}{2}\left[(6-4n)lpha(1-lpha)+n-6
ight]+lpha(1-lpha)$$

$$= n \left[-2\alpha(1-\alpha) + \frac{1}{2} \right] + \left[4\alpha(1-\alpha) - 3 \right]$$

= $\frac{n}{2} \left[1 - 4\alpha(1-\alpha) \right] - \left[1 - 4\alpha(1-\alpha) + 2 \right]$
= $\left(\frac{n}{2} - 1 \right) (1 - 2\alpha)^2 - 2.$ (462)

Inserting these results, we obtain:

$$i\Pi_{\mu\nu}^{ab} = (-1)^{n/2} \frac{ig^2}{(4\pi)^{n/2}} C_A \delta^{ab} \Gamma(2-\frac{n}{2}) \int \frac{d\alpha}{[k^2\alpha(1-\alpha)]^\epsilon} \left(g_{\mu\nu}k^2 - k_{\mu}k_{\nu}\right) \left[(1-\frac{n}{2})(1-2\alpha)^2 + 2\right] (463)$$

Note that after summing all the contributions to $i\Pi_{\mu\nu}^{ab}$, we obtain an expression that has manifest gauge invariance built in, *i.e.* $k^{\mu}i\Pi_{\mu\nu}^{ab} = 0$. We knew this had to happen, but as you see it is very essential to combine all contributions in order to arrive at a GI invariant form.

It is now easy to isolate the $\frac{1}{\epsilon}$ singularity of the above expression. We find:

$$i\Pi^{ab}_{\mu
u}
ightarrow rac{1}{\epsilon} rac{ig^2}{(4\pi)^2} C_A \delta^{ab} \left(g_{\mu
u} k^2 - k_\mu k_
u
ight) \left(rac{5}{3}
ight) \,.$$
 (464)

We shall see how this fits into things later on, but you should note how this

compares to our QED result (cf. Eq. (358))

$$i\Pi_{\mu\nu} \rightarrow \frac{1}{\epsilon} \frac{ie^2}{(4\pi)^2} \left(g_{\mu\nu}k^2 - k_{\mu}k_{\nu}\right) \left(-\frac{4}{3}\right) \,. \tag{465}$$

Note the opposite sign. This is the signal that QCD will be (or at least can be, assuming not too many flavors of fermions) asymptotically free whereas QED was the opposite.

We now bring in the fermion-loop contribution to the $\frac{1}{\epsilon}$ singularity [see Eq. (452)] and find

$$i\Pi^{ab}_{\mu\nu} \to \frac{1}{\epsilon} \frac{ig^2}{(4\pi)^2} \delta^{ab} \left(g_{\mu\nu}k^2 - k_{\mu}k_{\nu}\right) \left(\frac{5}{3}C_A - \frac{4}{3}(\frac{1}{2}n_f)\right) ,$$
 (466)

where sometimes $\frac{1}{2}n_f$ is written as $C(r)n_f$ where C(r) expresses the normalization of the fermion representation matrices, which we know have traces normalized to $\frac{1}{2}$ in the fundamental representation that we have been assuming.

Finally, we can immediately read off from the above what the the gluon field renormalization factor is. Recall the procedure. We introduce a counter-term $\boldsymbol{\mathcal{L}}$ component of the form

$$\Delta \mathcal{L} \ni -\frac{C}{4} (\partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu}) (\partial^{\mu} A^{a \nu} - \partial^{\nu} A^{a \mu})$$
(467)

which leads to Feynman rule

$$-iC(k^2g_{\mu\nu} - k_{\mu}k_{\nu})\delta^{ab} \tag{468}$$

and we must choose C so that the sum of this and the $\frac{1}{\epsilon}$ component of $i\Pi^{ab}_{\mu\nu}$ is finite. This clearly leads to

$$C = \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \left(\frac{5}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right) \,. \tag{469}$$

The net (quadratic in A) part of the Lagrangian form is then

$$\mathcal{L} + \Delta \mathcal{L} \ni -\frac{1+C}{4} (\partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu}) (\partial^{\mu}A^{a\nu} - \partial^{\nu}A^{a\mu}) \equiv -\frac{Z_{3}}{4} (\partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu}) (\partial^{\mu}A^{a\nu} - \partial^{\nu}A^{a\mu})$$

$$(470)$$

leading to

$$Z_3 = 1 + \frac{g^2}{(4\pi)^2 \epsilon} \left(\frac{5}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right) \,. \tag{471}$$

iΛ

We must now work on the gluon-quark-quark vertex. There are two diagrams. In one, the gluon attaches to the quark line of the gluon loop insertion on a quark line. In the second, the gluon attaches to the gluon line. The result for the first can easily be read off of our earlier QED result. Everything is just as in QED except that we have the group calculation that we did some time ago, which found that the appropriate group factor was $-\frac{1}{2n}$ which can also be written as $C_F - \frac{1}{2}C_A$. We must also remember that the basic vertex in our notation is

$$+ig\mu^{\epsilon}\gamma_{\mu}\times(L^{a}_{AB}) \tag{472}$$

where L^a is the representation matrix and we have supplied the appropriate dimensional factor for n dimensions. Thus, the appropriate notation for the one-loop correction is to write

$$+ig\mu^{\epsilon}\Lambda_{\mu}\times(L^{a}_{AB}). \tag{473}$$

After introducing the above color factor, the divergent component coming

from the fermion attachment graph is

$$\Lambda_{\mu} = \frac{g^2}{16\pi^2\epsilon} \gamma_{\mu} (C_F - \frac{1}{2}C_A) \,. \tag{474}$$

Now, let me show you a quick a dirty approach to rederiving this result. For the momentum space component of the calculation we have (neglecting the electron mass for simplicity):

$$\begin{aligned} +ig\mu^{\epsilon}\Lambda\mu &= (ig\mu^{\epsilon})(ig\mu^{\epsilon})^{2}\int \frac{d^{n}k}{(2\pi)^{n}}\gamma^{\nu}\frac{i}{p^{\prime}+k^{\prime}}\gamma\mu\frac{i}{p^{\prime}+k^{\prime}}\gamma^{\lambda}\left(\frac{-ig_{\nu\lambda}}{k^{2}}\right) \\ &= (ig\mu^{\epsilon})(ig\mu^{\epsilon})^{2}i\int \frac{d^{n}k}{(2\pi)^{n}}\frac{\gamma^{\nu}(p^{\prime}+k^{\prime})\gamma\mu(p^{\prime}+k^{\prime})\gamma\nu}{(p^{\prime}+k)^{2}(p+k)^{2}k^{2}} \\ &\rightarrow (ig\mu^{\epsilon})i(ig)^{2}\int \frac{d^{n}k}{(2\pi)^{n}}\frac{\gamma^{\nu}k^{\prime}\gamma\mu^{k}\gamma\nu}{(k^{2})^{3}} \quad \text{keeping only the most divergent stuff} \\ &= (ig\mu^{\epsilon})(-ig^{2})\int \frac{d^{n}k}{(2\pi)^{n}}(2-n)\frac{k^{\prime}\gamma\mu^{k}}{(k^{2})^{3}} \\ &= (ig\mu^{\epsilon})(-ig^{2})\int \frac{d^{n}k}{(2\pi)^{n}}(2-n)\frac{\gamma^{\rho}\gamma\mu\gamma^{\sigma}k\rhok\sigma}{(k^{2})^{3}} \\ &= (ig\mu^{\epsilon})(-ig^{2})\int \frac{d^{n}k}{(2\pi)^{n}}(2-n)\frac{\gamma^{\rho}\gamma\mu\gamma^{\sigma}\frac{g\rho\sigma}{n}k^{2}}{(k^{2})^{3}} \\ &= (ig\mu^{\epsilon})(-ig^{2})\int \frac{d^{n}k}{(2\pi)^{n}}(2-n)\frac{\gamma^{\mu}\gamma\mu\gamma^{\sigma}\frac{g\rho\sigma}{n}k^{2}}{(k^{2})^{3}} \end{aligned}$$

$$= (ig\mu^{\epsilon})(-ig^{2})(-1)^{n/2} \frac{i}{(4\pi)^{n/2}} \frac{(2-n)^{2}}{n} \Gamma(2-\frac{n}{2})\gamma\mu$$

$$\rightarrow (ig\mu^{\epsilon}) \frac{g^{2}}{(4\pi)^{2}} \frac{1}{\epsilon} \gamma\mu, \qquad (475)$$

which reproduces the expected result once the color factor of $C_F - \frac{1}{2}C_A$ is supplied. The reason we could not use this technique for $i\Pi$ is that the contributing diagrams had quadratic $\Gamma(1-\frac{n}{2})$ divergences that had to be combined to get the residual $\Gamma(2-\frac{n}{2})$ logarithmic divergence. In the present case, the superficial divergence of each diagram is from the beginning only logarithmic, $\Gamma(2-\frac{n}{2})$.

We may attack the final diagram in which the gluon attaches to the gluon in the triangle graph in the same simplified manner. The diagram notation is that the fermion flows into the diagram with momentum p, encounters vertex γ_{ν} after which it changes to momentum k, then encounters γ_{λ} and turns into p'. The three-gluon vertex has momentum p - k entering from left with associated index σ which exits with momentum p' - k with associated index ρ . Meanwhile, the external gluon entering from the top has momentum p' - pand index μ . The resulting non-color part of the calculation is:

$$\begin{split} ig\mu^{\epsilon}\Lambda\mu &= \int \frac{d^{n}k}{(2\pi)^{n}}(ig\mu^{\epsilon}\gamma_{\lambda})\frac{i}{k}(ig\mu^{\epsilon}\gamma_{\nu})\left(\frac{-ig^{\nu\sigma}}{(p-k)^{2}}\right)\left(\frac{-ig^{\rho\lambda}}{(p'-k)^{2}}\right)\times\\ &\quad (-ig\mu^{\epsilon})\left[g_{\sigma\rho}(-(p-k)-(p'-k))\mu+g_{\mu\rho}((p'-k)+(p'-p))\sigma+g_{\mu\sigma}(-(p'-p)+(p-k))\rho\right] \end{split}$$

$$\rightarrow g^{3} \int \frac{d^{n}k}{(2\pi)^{n}} \gamma_{\lambda} \gamma^{\alpha} \gamma_{\nu} \frac{k_{\alpha}}{(k^{2})^{3}} \left[g^{\nu\lambda} 2k_{\mu} - g^{\lambda}_{\mu} k^{\nu} - g^{\nu}_{\mu} k^{\lambda} \right]$$
 keeping only the most singular stuff

$$\rightarrow \frac{1}{4} g^{3} \int \frac{d^{n}k}{(2\pi)^{n}} \gamma_{\lambda} \gamma^{\alpha} \gamma_{\nu} \frac{1}{(k^{2})^{2}} \left[g^{\nu\lambda} 2g_{\mu\alpha} - g^{\lambda}_{\mu} g^{\nu}_{\alpha} - g^{\nu}_{\mu} g^{\lambda}_{\alpha} \right]$$
 using $k_{\alpha} k_{\beta} \rightarrow k^{2} g_{\alpha\beta} / n$ with $n \rightarrow 4$

$$\rightarrow \frac{1}{4} g^{3} \int \frac{d^{n}k}{(2\pi)^{n}} \frac{1}{(k^{2})^{2}} \left[-12\gamma_{\mu} \right]$$
 using γ matrix identities in 4-dimensions

$$\rightarrow -3g^{3} \gamma_{\mu} \frac{i}{(4\pi)^{2}} \Gamma(2 - \frac{n}{2})$$
 using $n = 4$ everywhere except in singular Γ function

$$\rightarrow -3g^{3} (-\frac{1}{2}C_{A}) \gamma_{\mu} \frac{i}{(4\pi)^{2}\epsilon}$$
 after bringing in the color factor we computed earlier

So, now let us assemble the $\frac{1}{\epsilon}$ pieces, neglecting all finite corrections,

$$ig\Lambda_{\mu} = ig\frac{g^{2}}{(4\pi)^{2}}\frac{1}{\epsilon}\gamma_{\mu}\left(C_{F} - \frac{1}{2}C_{A} + \frac{3}{2}C_{A}\right) = ig\frac{g^{2}}{(4\pi)^{2}}\frac{1}{\epsilon}\gamma_{\mu}(C_{F} + C_{A}),$$
(477)

to be multiplied (in the convention we are using) by the color matrix L^a . We must counter this $\frac{1}{\epsilon}$ piece by a counter term. We introduce (taking into account the basic sign of the gluon-quark-antiquark interaction, $+g\overline{\psi}/A\psi$,

(476)

$$A = \gamma^{\mu} A^{a}_{\mu} L^{a})$$

$$\Delta \mathcal{L} = Dg \overline{\psi} A \psi$$
(478)

which yields Feynman rule $iDg\gamma_{\mu}L^{a}$ from which we see that we require

$$D = -\frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} (C_F + C_A) \,. \tag{479}$$

The net form of the Lagrangian is then

 $\mathcal{L} + \Delta \mathcal{L} \ni (1+D)g\overline{\psi} \, \mathcal{A}\psi \equiv Z_1 g\overline{\psi} \, \mathcal{A}\psi$ (480)

which gives

$$Z_1 = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} (C_F + C_A).$$
(481)

We also recall our earlier results of

$$Z_2 = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} C_F$$
(482)

and

$$Z_3 = 1 + \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \left(\frac{5}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right) \,. \tag{483}$$

The final stage is to relate the bare coupling to the renormalized coupling:

$$g_{0}\overline{\psi}_{0}A_{0}\psi_{0} = g\mu^{\epsilon}Z_{1}\overline{\psi}A\psi$$

$$= g\mu^{\epsilon}\frac{Z_{1}}{Z_{2}\sqrt{Z_{3}}}\overline{\psi}_{0}A_{0}\psi_{0} \qquad (484)$$

from which we have

$$g_{0} = g\mu^{\epsilon} \frac{Z_{1}}{Z_{2}\sqrt{Z_{3}}} = g\mu^{\epsilon} \left(1 - \frac{g^{2}}{(4\pi)^{2}} \frac{1}{\epsilon} \left[(C_{F} + C_{A}) - C_{F} + \frac{1}{2} \left(\frac{5}{3}C_{A} - \frac{4}{3}(\frac{1}{2}n_{f})\right) \right] \right)$$
$$= g\mu^{\epsilon} \left(1 - \frac{1}{2} \frac{g^{2}}{(4\pi)^{2}} \frac{1}{\epsilon} \left[\frac{11}{3}C_{A} - \frac{4}{3}(\frac{1}{2}n_{f})\right] \right).$$
(485)

Taking the $\mu \frac{\partial g_0}{\partial \mu}$ derivative gives us

$$0 = \epsilon g \mu^{\epsilon} \left(1 - \frac{1}{2} \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \left[\frac{11}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right] \right) + \mu^{\epsilon} \mu \frac{\partial g}{\partial \mu} \left(1 - \frac{3}{2} \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon} \left[\frac{11}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right] \right)$$
(486)

leading to (after dividing through, expanding the denominator, and taking $\epsilon
ightarrow 0$)

$$\mu \frac{\partial g}{\partial \mu} \equiv \beta(g) = -\frac{g^3}{(4\pi)^2} \left[\frac{11}{3} C_A - \frac{4}{3} (\frac{1}{2} n_f) \right] \,. \tag{487}$$

For $C_A = 3$ we have

$$\mu \frac{\partial g}{\partial \mu} \equiv \beta(g) = -\frac{g^3}{(4\pi)^2} \left[11 - \frac{2}{3} n_f \right] \,, \tag{488}$$

which is indeed negative so long as $n_f < 33/2$. We have 2 colored fermions per family and 3 families, for a total of 6 flavors that we know of. So, we are not very close to 17. It is actually a closer call when we go to supersymmetry. But that is the subject of (hopefully) later discussion.

Some further notes

Before continuing, it is worth noting that $Z_1 \neq Z_2$ for this calculation because of the important C_A part coming into Z_1 from the graph in which the gluon attaches to the gluon propagator of the gluon loop correction to the fermion propagation. This is expected in Feynman gauge.

However, there is still an underlying Ward Identity. In fact, if one were to carry out this calculation in an appropriate axial gauge, one would find that $Z_1 = Z_2$ and that all the moving coupling constant information would be contained in Z_3 (which of course takes a (required) different value than in Feynman gauge).

We also reemphasize the fact that we could have gotten our relation

between g_0 and g using any of the other fundamental interactions that define g at tree-level. Altogether there are 4 vertices that we can use:

- 1. the $gq\overline{q}$ vertex, as employed here;
- 2. the *ggg* vertex;
- 3. the g ghost ghost vertex;
- 4. the 4g vertex.

Explicit calculation for 2, 3, or 4 would result in our obtaining exactly the same relation between g and g_0 as we have found. This is required by the gauge invariance of the theory, which is present at tree-level and would otherwise be destroyed by going to the one-loop or higher level. Gauge invariance should be preserved at all orders of perturbation theory.

Of course, Eq. (488) can be integrated. For the moment, we write

$$\mu \frac{\partial g}{\partial \mu} = -\frac{b}{16\pi^2} g^3 \quad \Rightarrow \frac{dg}{g^3} = -\frac{b}{16\pi^2} \frac{d\mu}{\mu} \Rightarrow dg^{-2} = \frac{b}{8\pi^2} d\ln\mu \qquad (489)$$

which integrates to $(g_0$ refers to an initial value at μ_0)

$$\frac{1}{g^2(\mu)} - \frac{1}{g_0^2} = \frac{b}{8\pi^2} \ln \frac{\mu}{\mu_0}$$
(490)

or, equivalently,

$$g^{2}(\mu) = \frac{g_{0}^{2}}{1 + \frac{b}{8\pi^{2}}g_{0}^{2}\ln\frac{\mu}{\mu_{0}}}$$
(491)

or, equivalently, defining $lpha_s=rac{g^2}{4\pi}$

$$\alpha_s(\mu) = \frac{\alpha_0}{1 + \frac{b}{2\pi}\alpha_0 \ln \frac{\mu}{\mu_0}}.$$
(492)

As in the QED case, we can define a Λ such that the above equation can be rewritten as

$$\alpha_s(\mu) = \frac{1}{\frac{b}{2\pi} \ln(\mu/\Lambda)} = \frac{4\pi}{(11 - 2n_f/3) \ln(\mu^2/\Lambda^2)}.$$
 (493)

For $11 - 2n_f/3 > 0$ (as we believe), α_s decreases with increasing μ^2 .

The quantity Λ is sometimes denoted by Λ_{QCD} . A fit to the α_s behavior of high momentum transfer scattering gives a determination of $\Lambda_{QCD} \sim$ 500 MeV ÷ 1 GeV. This is consistent with the fact that we know that strong interactions become truly strong as we approach the hadron mass scale/size, which we know resides at the \leq GeV scale. The connection between hadron masses and the like and large momentum transfer scales (ranging from \geq 2 GeV as probed in the early days at SLAC up to the current \geq 100 GeV scales being probed at the Tevatron as well as LEP and SLD) can be made reasonably precise using the lattice calculation techniques for the hadron mass computations and performing a careful matching. Some plots are below.



Figure 12: Experimental results for α_s from the current 2014 PDG.

Some discussion of these experimental extractions is perhaps worthwhile.

The most rigorous manner for extracting α_s is to employ the order g^2 (and higher) QCD predictions for $e^+e^- \rightarrow hadrons$ based on computations of $e^+e^- \rightarrow q\overline{q}$ with appropriate QCD corrections. The first such correction takes the form (I hope that we will have time to get to this calculation, but we might have to rush — you should try to read about it if we don't).

$$\sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left(1 + C_F \frac{\alpha_S}{2\pi} \frac{3}{2} \right)$$
(494)
$$= \sigma_0 \left(1 + \frac{\alpha_S}{\pi} \right).$$
(495)
$$\mathbf{P}(e^+e^- \to e^- e^- e^-) = \mathbf{1} + \frac{\alpha_S}{\pi}$$
(496)

$$\Rightarrow R(e^+e^- \to hadrons) = 1 + \frac{\alpha_s}{\pi}.$$
 (496)

To calculate the total cross section for $e^+e^- \rightarrow hadrons$, we must sum over all $e^+e^- \rightarrow partons$ processes. At the e^2g^2 order of perturbation theory of interest, only the $q\bar{q}g$ and $q\bar{q}$ final states are relevant.

For the $q\bar{q}g$ final state, the amplitudes are those in which the incoming

 e^+e^- annihilate to give you an outgoing $q\overline{q}$ pair, either one of which can radiate the additional g. We sum these two amplitudes and then square the net amplitude and integrate over the three-parton phase space in ndimensions. The net result is of order e^2g^2 and it has $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ singularities and various finite terms. Since this result corresponds to an absolute square, equivalently, a cross section, it is positive definite. In particular, the singular terms are positive.

For the $q\bar{q}$ final state, the diagrams of interest are the three virtual correction diagrams shown in Fig. 13. We compute these diagrams using dimensional regularization.



Figure 13: One-loop diagrams for $e^+e^- \rightarrow q\bar{q}$.

It is important to note that these diagrams are one power of $\alpha_{\rm S}$ relative to the tree-level diagram,

$$\mathcal{M}_1 \propto \alpha_{\rm S} \mathcal{M}_0. \tag{497}$$

Therefore $|\mathcal{M}_1|^2$ is two powers down and hence negligible at the order g^2 to which we are working. The order e^2g^2 terms are obtained as the interference $\operatorname{Re}\{\mathcal{M}_0^*\mathcal{M}_1\}$. This order e^2g^2 interference again has $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ singularities and finite terms. Since this result comes from an interference term, there is no necessity that it be positive definite. In particular, the singular terms are negative in sign.

The net result is obtained by combining the $q\overline{q}g$ final state result with the $q\overline{q}$ final state result. One finds that all the singular terms cancel, leaving us with the finite result stated earlier.

 $R(e^+e^- \rightarrow hadrons)$ is one of the most fundamental quantities in QCD and is certainly one of the most well-calculated and measured.

Despite the fact that the $\alpha_{\rm S}/\pi$ is a relatively small correction to the total rate, experimental and theoretical systematic errors are so small that they can almost be neglected – even with the large statistics of τ decays and Z decays at LEP, the statistical errors dominate. This means that not only does $e^+e^- \rightarrow hadrons$ provide one of the most accurate measurements

of α_s , but the accuracy is rather easy to interpret and implement in global analyses for example, unlike measurements that are dominated by systematics.

The β function has, of course, been computed now to quite high order. Writing

$$\mu^2 \frac{d\alpha_{\rm S}}{d\mu^2} = \beta(\alpha_{\rm S}) = -(b_0 \alpha_{\rm S}^2 + b_1 \alpha_{\rm S}^3 + b_2 \alpha_{\rm S}^4 + \ldots)$$
(498)

one finds ⁸

$$b_0 = (11C_A - 4n_fT_R)/(12\pi) = (33 - 2n_f)/(12\pi)$$

 $b_1 = (17C_A^2 - n_f T_R(10C_A + 6C_F))/(24\pi^2) = (153 - 19n_f)/(24\pi^2)$

$$b_2 \hspace{.1in} = \hspace{.1in} (2857 - rac{5033}{9} n_f + rac{325}{27} n_f^2)/(128 \pi^3)$$

 b_3 = complicated but computed

(499)

Here, n_f is always the number of active quark flavors, *i.e.* those having $m_q < \mu$.

 $^{{}^{8}}b_{2}$ and b_{3} are scheme-dependent and given here in the \overline{MS} scheme.

As noted above, the β -function coefficients above, the b_i , are given for the coupling of an *effective theory* in which n_f of the quark flavors are considered light $(m_q \ll \mu)$, and in which the remaining heavier quark flavors decouple from the theory. One can relate the coupling for the theory with $n_f + 1$ light flavors to that with n_f flavors through an equation of the form

$$\alpha_{\rm S}^{(n_f+1)}(\mu^2) = \alpha_{\rm S}^{(n_f)}(\mu^2) \left(1 + \sum_{n=1}^{\infty} \sum_{l=0}^{n} c_{nl} \left[\alpha_{\rm S}^{(n_f)}(\mu^2)\right]^n \ln^l \frac{\mu^2}{m_h^2}\right), \quad (500)$$

where m_h is the mass of the $(n_f + 1)^{\text{th}}$ flavor, and the first few c_{nl} coefficients are $c_{11} = \frac{1}{6\pi}$, $c_{10} = 0$, $c_{22} = c_{11}^2$, $c_{21} = \frac{19}{24\pi^2}$, and $c_{20} = -\frac{11}{72\pi^2}$ when m_h is the \overline{MS} mass at scale m_h ($c_{20} = \frac{7}{24\pi^2}$ when m_h is the pole mass.) For more information see the PDG QCD section. Terms up to c_{4l} are known. Numerically, when one chooses $\mu = m_h$, the matching is a small effect, owing to the zero value for c_{10} .

For constant n_f , a simple exact analytic solution to Eq. (498) exists only if one neglects all but the b_0 term, giving

$$\alpha_{\rm S}(\mu^2) = \frac{1}{b_0 \ln(\mu^2/\Lambda^2)},$$
(501)

where Λ is a constant of integration as discussed previously. A convenient approximate analytic solution to Eq. (498) that includes also the b_1 , b_2 and b_3 terms is given by

$$\begin{aligned} \alpha_{\rm S}(\mu^2) &\simeq \frac{1}{b_0 t} \left(1 - \frac{b_1 \ln t}{b_0^2} + \frac{b_1^2 (\ln^2 t - \ln t - 1) + b_0 b_2}{b_0^4 t^2} \right. \\ &\left. - \frac{b_1^3 (\ln^3 t - \frac{5}{2} \ln^2 t - 2 \ln t + \frac{1}{2}) + 3b_0 b_1 b_2 \ln t - \frac{1}{2} b_0^2 b_3}{b_0^6 t^3} \right), \end{aligned}$$
(502)

where $t \equiv \ln \frac{\mu^2}{\Lambda^2}$. This is only one of several possible approximate 4-loop solutions for $\alpha_{\rm S}(\mu^2)$ and Λ depends on which particular approximation is being used.

One can also solve Eq. (498) numerically. Then Λ is not defined. Rather the results must be phrased in terms of the value of $\alpha_{\rm S}$ at a given scale, the canonical scale being m_Z , as we do here.

Returning to R, Eq. (496) is now known up to order $\alpha_{\rm S}^4$.

$$\sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \Biggl\{ 1 + \frac{\alpha_{\text{S}}(\mu)}{\pi} + C_2 \Biggl(\frac{\mu^2}{s} \Biggr) \Biggl(\frac{\alpha_{\text{S}}(\mu)}{\pi} \Biggr)^2 + C_3 \Biggl(\frac{\mu^2}{s} \Biggr) \Biggl(\frac{\alpha_{\text{S}}(\mu)}{\pi} \Biggr)^3 + C_4 \Biggl(\frac{\mu^2}{s} \Biggr) \Biggl(\frac{\alpha_{\text{S}}(\mu)}{\pi} \Biggr)^4 \Biggr\} (503)$$

Numerically, the coefficients above are, using $\mu = Q$,

$$\begin{split} C_1 &= 1, \quad C_2 = 1.9857 - 0.1152 \, n_f, \\ C_3 &= -6.63694 - 1.20013 \, n_f - 0.00518 \, n_f^2 - 1.24 \, \eta \\ C_4 &= -156.61 + 18.77 \, n_f - 0.7974 \, n_f^2 + C \eta \end{split} \tag{504}$$

where $\eta = (\sum_{q} e_{q})^{2}/(3\sum_{q} e_{q}^{2})$ and *C* has yet to be determined. Note the big coefficients at higher orders; this slows down the convergence of the series.

We can display explicitly the μ dependencies:

$$C_1(\mu^2/Q^2) ~=~ 1,$$

$$C_{2}(\mu^{2}/Q^{2}) = C_{2} + \pi b_{0}C_{1}\ln(\mu^{2}/Q^{2})$$

$$C_{3}(\mu^{2}/Q^{2}) = C_{3} + (2b_{0}C_{2}\pi + b_{1}C_{1}\pi^{2})\ln(\mu^{2}/Q^{2})$$

$$+ b_{0}^{2}C_{1}\pi^{2}\ln^{2}(\mu^{2}/Q^{2})$$
(505)

Thus, renormalization introduces a renormalization scale dependence not only into α_s but also into the coefficient functions beyond the first one.

(Note: the above results are for photon exchange only; including Z exchange is necessary unless $Q \ll m_Z$.)

Note that large logs develop if μ is very different from Q. Nonetheless, after summing enough terms there is little μ dependence of the final answer.

Reducing this renormalization-scale dependence is one of the biggest reasons for going to higher orders.

A graph of the reduced μ dependence up to including C_3 terms is in Fig. 14 — the scale-dependence is indeed significantly smaller at each order, giving stability over a wider range of μ .



Figure 14: The QCD prediction for the corrections to $R_{e^+e^-}$ at $\sqrt{s} = 33$ GeV as a function of renormalization scale at leading, next-to-leading and next-to-next-to-leading order.

It can also be seen that provided μ is of order Q, the higher order corrections are relatively small. We will see shortly that simply taking the leading order result with $\mu = \sqrt{s}$ does surprisingly well and is certainly sufficient to understand the phenomenology.



Figure 15: Green dashed line is naive parton model; red solid is 3-loop pQCD prediction. 230C, U.C. Davis, 246



Figure 16: *R* in Light-Flavor, Charm, and Beauty Threshold Regions. Green/red as above. 230C, U.C. Davis, 247

As noted above, the experimental measurement of $R_{e^+e^-}$ gives one of the best measurements of α_s . In fact the LEP combined Z-pole value of R_{had} is

$$R(LEP) = 20.767 \pm 0.025, \tag{506}$$

while the tree-level prediction is

$$R_0(m_Z) = 19.984. \tag{507}$$

Combining the two, and simply using the leading order result with $\mu = m_Z$, we obtain our first measurement of α_S ,

$$\alpha_{\rm S}(m_Z) = 0.124 \pm 0.004, \tag{508}$$

surprisingly close to the current world average value of 0.1184 ± 0.0007 .

Since QCD predicts the scale dependence of $\alpha_{\rm S}$, one measurement at any scale is sufficient to give a prediction for all scales. We can therefore phrase measurements at other scales either as tests of QCD throughout the intervening energy range or, by translating them all into measurements at a single scale, as different measurements of the same quantity that can be combined to give a better overall measurement.

As an example, the average measurement of R over several energy points around 34 ${
m GeV}$ is

$$R(PETRA) = 3.88 \pm 0.03, \tag{509}$$

while the tree-level prediction is

$$R_0(34 \text{ GeV}) = 3.69. \tag{510}$$

Again using the leading order result, we obtain

$$\alpha_{\rm S}(34 \; {\rm GeV}) = 0.162 \pm 0.026.$$
 (511)

Finally, using the one-loop renormalization group equation, we can convert this into a measurement of $\alpha_{\rm S}(m_Z)$,

$$\alpha_{\rm S}(m_Z) = 0.134 \pm 0.018. \tag{512}$$

This is in good agreement with the value from LEP, although with much larger uncertainties, simply due to the fact that the statistics of the PETRA experiments were much lower.

As a final example, we consider τ decays. The QCD corrections to the hadronic decay rate actually have two effects: on the ratio of various

branching fractions, and also directly on the total decay rate of the τ . These can form the basis for two analyses in which the experimental errors are largely independent. The combined result for the two is

$$\alpha_{\rm S}(M_{ au} = 1.77 \; {\rm GeV}) = 0.33 \pm 0.03.$$
 (513)

This time, because we are translating over such a wide energy range the one-loop renormalization group equation does not do quite such a good job. From the above experimental result and using one-loop evolution one finds

$$\alpha_{\rm S}^{(\text{one-loop})}(m_Z) = 0.126 \pm 0.004,$$
 (514)

compared to the 'official' value

$$\alpha_{\rm S}(m_Z) = 0.1184 \pm 0.0007, \tag{515}$$

but it is not so far out. Including higher loops yields a value very close to the official value. In fact, one should note in this case the phenomenon of the "incredible shrinking error". Although the measurement at the τ mass scale only has a precision of 10%, after evolving it to m_Z the relative uncertainty gets scaled down by the ratio of the two α_s values, and τ decays give one of the best measurements of $\alpha_s(m_Z)$.

First, we must endure a few more technical details.

Two-component fermion results vs. Dirac four-component

The first of these is to simply note that the computation that we did of the fermion-loop contribution to $\Pi_{\mu\nu}$ implicitly employed a full Dirac fermion loop with coupling to the gauge boson current given by $ig\mu^{\epsilon}\gamma_{\mu}$ — i.e. no helicity restriction was imposed at the vertex. For what is about to come, we need to note the alteration that occurs if we deal only with a right- or left-handed fermion, with coupling $-ie\mu^{\epsilon}\gamma_{\mu}\frac{1}{2}(1\pm\gamma_5)$. Referring back to Eq. (309)

$$i\Pi_{\mu\nu} = -(-ie\mu^{\epsilon})^{2} \int \frac{d^{n}p}{(2\pi)^{n}} \operatorname{Tr}\left(\gamma_{\mu}\frac{i}{\not{p}-m}\gamma_{\nu}\frac{i}{\not{p}-\not{k}-m}\right)$$

$$= -e^{2}\mu^{2\epsilon} \int \frac{d^{n}p}{(2\pi)^{n}} \frac{\operatorname{Tr}\left[\gamma_{\mu}(\not{p}+m)\gamma_{\nu}(\not{p}-\not{k}+m)\right]}{(p^{2}-m^{2})[(p-k)^{2}-m^{2}]}$$

$$= -e^{2}\mu^{2\epsilon} \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \frac{\operatorname{Tr}\left[\gamma_{\mu}(\not{p}'+\alpha\not{k}+m)\gamma_{\nu}(\not{p}'-(1-\alpha)\not{k}+m)\right]}{[p'^{2}-m^{2}+k^{2}\alpha(1-\alpha)]^{2}}$$
(516)

but inserting the helicity projected vertices above, we would have had (remembering the minus sign for a closed fermion loop and shifting to
$$p' = p - \alpha k \text{ at a certain stage and assuming a left-handed fermion}:$$

$$i\Pi_{\mu\nu} = -(-ie\mu^{\epsilon})^{2} \int \frac{d^{n}p}{(2\pi)^{n}} \operatorname{Tr} \left(\gamma_{\mu} \frac{1}{2} (1 - \gamma_{5}) \frac{i}{\not{p} - m} \gamma_{\nu} \frac{1}{2} (1 - \gamma_{5}) \frac{i}{\not{p} - \not{k} - m} \right)$$

$$= -e^{2} \mu^{2\epsilon} \int \frac{d^{n}p}{(2\pi)^{n}} \frac{\operatorname{Tr} \left[\gamma_{\mu} \frac{1}{2} (1 - \gamma_{5}) (\not{p} + m) \gamma_{\nu} \frac{1}{2} (1 - \gamma_{5}) (\not{p} - \not{k} + m) \right]}{(p^{2} - m^{2})[(p - k)^{2} - m^{2}]}$$

$$= -e^{2} \mu^{2\epsilon} \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \frac{\operatorname{Tr} \left[\gamma_{\mu} \frac{1}{2} (1 - \gamma_{5}) (\not{p}' + \alpha \not{k} + m) \gamma_{\nu} \frac{1}{2} (1 - \gamma_{5}) (\not{p}' - (1 - \alpha) \not{k} + m) \right]}{[p'^{2} - m^{2} + k^{2} \alpha (1 - \alpha)]^{2}}$$

$$= -e^{2} \mu^{2\epsilon} \int d\alpha \int \frac{d^{n}p'}{(2\pi)^{n}} \frac{\operatorname{Tr} \left[\gamma_{\mu} \frac{1}{2} (1 - \gamma_{5}) (\not{p}' + \alpha \not{k}) \gamma_{\nu} (\not{p}' - (1 - \alpha) \not{k}) \right]}{[p'^{2} - m^{2} + k^{2} \alpha (1 - \alpha)]^{2}}$$
(517)

At this point, let us work on simplifying the numerator. We must remember that terms that are odd in p' will integrate to 0. Further, we must recall that the trace of an odd number of γ matrices is 0. Finally, the trace involving the γ_5 is 0 for the terms even in p' because of the appearance of either two p''s or two k's. Using these ingredients, we obtain:

$$N = \frac{1}{2} [p'^{\rho} p'^{\sigma} - k^{\rho} k^{\sigma} \alpha (1 - \alpha)] \operatorname{Tr}[\gamma_{\mu} \gamma_{\rho} \gamma_{\nu} \gamma_{\sigma}]$$

$$= \frac{1}{2} [p'^{\rho} p'^{\sigma} - k^{\rho} k^{\sigma} \alpha (1 - \alpha)] 2^{n/2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho})].$$
(518)

If we neglect the m^2 term in our earlier Eq. (310) (to get the m^2 terms

correct, we would have to get involved with helicity flip diagrams — this is unnecessary for the important point below) then this is the same result as in our earlier calculation, but with one difference — the extra factor of $\frac{1}{2}$.

This $\frac{1}{2}$ carries through everything and so for a fermion of given helicity we get just half of our earlier contribution to Z_3 . As a result, the β function contribution is $\frac{1}{2}$ of what we had for a 4-component Dirac particle. For each 2-component fermion of definite helicity we get (cf. Eq. (358))

$$\frac{ie^2}{24\pi^2} \frac{1}{\epsilon} \left[k_{\mu} k_{\nu} - k^2 g_{\mu\nu} \right] = \frac{ie^2}{16\pi^2} \left(\frac{2}{3} \right) \frac{1}{\epsilon} \left[k_{\mu} k_{\nu} - k^2 g_{\mu\nu} \right] \,. \tag{519}$$

A spin-zero loop — e.g. a Higgs boson loop

Imagine that we have a charged Higgs boson in QED, with charge 1, i.e. the same absolute charge as that of the electron. In order to include its effect in the running of e, we need to compute its impact on Z_3 through its impact on $\Pi_{\mu\nu}$.⁹

⁹One-loop diagrams for the fermion $-i\Sigma$ and the photon-electron-electron vertex, Λ_{μ} , in which the charged Higgs boson is emitted by the electron, leaving behind a neutrino, and then the neutrino reabsorbs the charged Higgs can be neglected in the small m limit — Higgs bosons couple proportional to mass. Thus, only $\Pi_{\mu\nu}$ needs to be computed.

So, first recall the Feynman rules for a charged spin-zero particle coupling to the photon. These are generated from the Lagrangian form

$$[(\partial^{\mu}-ieA^{\mu})\phi]^{\dagger}[(\partial_{\mu}-ieA_{\mu})\phi] \ni -ieA_{\mu}[\partial^{\mu}\phi^{\dagger}]\phi + ieA^{\mu}\phi^{\dagger}\partial_{\mu}\phi + e^{2}A^{\mu}A_{\mu}\phi^{\dagger}\phi.$$
(520)

(Note the opposite sign of the $-ieA_{\mu}$ in the covariant derivative compared to our convention for the electron. This is because I am discussing a positively charged Higgs vs. the negatively charged electron.)

These generate two Feynman rules. The first is one generated by the first two terms above and is one in which the H^+ enters from the left with momentum p, absorbs a photon with momentum q and Lorentz index μ and exits to the right with momentum p' = p + q. The Feynman rule for this vertex is

$$ie(p'+p)_{\mu}.$$
 (521)

The second diagram is generated by the last term above. It is a 4-particle interaction involving two photons with Lorentz indices μ and ν connected to an incoming H^+ and an outgoing H^+ . There are two contractions of the two external photons with the $A^{\mu}A_{\mu}$ fields yielding a factor of 2. The resulting Feynman rule is

$$2ie^2g_{\mu\nu}.$$
 (522)

With these Feynman rules, we find there are two one-charged-Higgs-loop diagrams contributing to $\Pi_{\mu\nu}$. These are: a) photon to charged Higgs pair (H^+ with k entering vertex μ and exiting with p + k), following by charged Higgs pair recombining at vertex ν ; and b) charged Higgs "tadpole" correction to the photon propagator, with tadpole loop momentum k. (For some reason, I switched p and k in this derivation.) The expressions for these are

• a)

$$(ie\mu^{\epsilon})^{2} \int \frac{d^{n}k}{(2\pi)^{n}} \frac{i}{k^{2} - m^{2}} (p + 2k)_{\mu} \frac{i}{(p+k)^{2} - m^{2}} (p + 2k)_{\nu}; \quad (523)$$

• b)

$$2ie^2 g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2}.$$
 (524)

These we evaluate using standard techniques.

$$\begin{array}{ll} a) & = & (ie\mu^{\epsilon})^2 \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} (p + 2k) \mu \frac{i}{(p + k)^2 - m^2} (p + 2k) \nu \\ \\ & = & (e\mu^{\epsilon})^2 \int d\alpha \int \frac{d^n k}{(2\pi)^n} \frac{(p + 2k) \mu (p + 2k) \nu}{(k^2 + 2\alpha k \cdot p + \alpha p^2 - m^2)^2} \end{array}$$

$$= (e\mu^{\epsilon})^{2} \int d\alpha \int \frac{d^{n}k'}{(2\pi)^{n}} \frac{((1-2\alpha)p+2k')\mu((1-2\alpha)p+2k')\nu}{(k'^{2}+\alpha(1-\alpha)p^{2}-m^{2})^{2}}$$

$$= (e\mu^{\epsilon})^{2} \int d\alpha \int \frac{d^{n}k'}{(2\pi)^{n}} \frac{(1-2\alpha)^{2}p\mu\rho\nu+4k'_{\mu}k'_{\nu}}{(k'^{2}+\alpha(1-\alpha)p^{2}-m^{2})^{2}}$$

$$\stackrel{n \to 4}{=} e^{2} \frac{i}{(4\pi)^{2}} \int d\alpha \left[(1-2\alpha)^{2}p\mu\rho\nu\Gamma(2-\frac{n}{2}) + 4(\frac{1}{2}g\mu\nu)[\alpha(1-\alpha)p^{2}-m^{2}]\Gamma(1-\frac{n}{2}) \right]$$

$$= \frac{ie^{2}}{(4\pi)^{2}} \left[\frac{1}{3}p\mu\rho\nu\Gamma(2-\frac{n}{2}) + (\frac{1}{3}p^{2}-2m^{2})g\mu\nu\Gamma(1-\frac{n}{2}) \right]$$

$$b) = 2i(e\mu^{\epsilon})^{2}g\mu\nu \int \frac{d^{n}k}{(2\pi)^{n}} \frac{i}{k^{2}-m^{2}}$$

$$= 2i(e\mu^{\epsilon})^{2}g\mu\nu \int \frac{d^{n}k}{(2\pi)^{n}} \frac{(1-\alpha)^{2}p^{2}+k'^{2}-m^{2}}{(p+k)^{2}-m^{2}}$$

$$= 2i(e\mu^{\epsilon})^{2}g\mu\nu i \int d\alpha \int \frac{d^{n}k'}{(2\pi)^{n}} \frac{(1-\alpha)^{2}p^{2}+k'^{2}-m^{2}}{(p+k)^{2}-m^{2}]^{2}}$$

$$= 2i(e\mu^{\epsilon})^{2}g\mu\nu i \int d\alpha \int \frac{d^{n}k'}{(2\pi)^{n}} \left[\frac{[(1-\alpha)^{2}-\alpha(1-\alpha)]p^{2}}{[k'^{2}+\alpha(1-\alpha)p^{2}-m^{2}]^{2}} + \frac{1}{k'^{2}+\alpha(1-\alpha)p^{2}-m^{2}} \right]$$

$$\stackrel{n \to 4}{=} -2e^{2}g\mu\nu \frac{i}{(4\pi)^{2}} \left[\frac{1}{6}p^{2}\Gamma(2-\frac{n}{2}) + \left(\frac{1}{6}p^{2}-m^{2}\right)\Gamma(1-\frac{n}{2}) \right]$$

$$+b) = \frac{ie^{2}}{(4\pi)^{2}} \left[\frac{1}{3}p\mu\rho\nu\Gamma(2-\frac{n}{2}) - \frac{1}{3}p^{2}g\mu\nu\Gamma(2-\frac{n}{2}) \right]$$

$$(525)$$

This should be compared to the old result for the 4-component Dirac electron

a)

which was

$$\frac{ie^2}{12\pi^2} \frac{1}{\epsilon} \left[p_{\mu} p_{\nu} - p^2 g_{\mu\nu} \right] \,. \tag{526}$$

The rule is: a charged spin-0 loop gives 1/4 the result of a Dirac fermion (i.e. including both electron helicities) or 1/2 the result for a fermion of definite helicity.

In the above, you will see I used the same sort of trick as previously for gauge boson loop corrections to a gauge propagator to cancel away the $\Gamma(1-\frac{n}{2})$ that cannot actually be present. (There should be no quadratically divergent contribution to $\Pi_{\mu\nu}$ because of gauge invariance requirements: the Ward identity $p^{\mu}\Pi_{\mu\nu} = 0$ should hold in an arbitrary number of dimensions.)

Putting it all together

So let us summarize where we are.

1. For SU(3) we have

$$eta_3(g_3) = -rac{g_3^3}{16\pi^2} \left[rac{11}{3} 3 - rac{2}{3} \left(rac{1}{2}
ight) n_{SU(3) \ triplets \ of \ given \ helicity}
ight] \ , \ \ (527)$$

where we have introduced the $\frac{1}{2}$ in the last term appropriate when counting fundamental representations of given helicity. We have also introduced the

notation g_3 for the SU(3) strong coupling constant which we had been calling g.

Each generation or family has u_L , d_L , u_R , d_R (L, R denote helicity), each coming with three "colors" of the fundamental SU(3) triplet. Thus, $n_{SU(3) \ triplets \ of \ given \ helicity} = 4N_g$, and we obtain

$$\beta_3(g_3) = -\frac{g_3^3}{16\pi^2} \left[11 - \frac{4}{3} N_g \right] \,. \tag{528}$$

Note that the Higgs boson does not have "color" (does not interact strongly) and so is not counted above.

2. For the weak SU(2) we have

$$\beta_{2}(g_{2}) = -\frac{g_{2}^{3}}{16\pi^{2}} \left[\frac{11}{3} 2 - \frac{2}{3} \left(\frac{1}{2} \right) n_{weak-isospin \ doublets \ of \ given \ helicity} \right],$$
(529)
What weak-isospin doublets do we have in the SM? In each family or generation we have:
$$\begin{pmatrix} u \\ d \end{pmatrix}_{L}$$
in three colors and
$$\begin{pmatrix} \nu_{e} \\ e^{-} \end{pmatrix}_{L}$$
(uncolored).

The right-handed quarks and right-handed electron are weak-isospin singlets (i.e. are not seen by the SU(2) gauge bosons) and do not contribute. Thus, we have $n_{weak-isospin\ doublets\ of\ given\ helicity} = 4N_g$ leading to

$$\beta_2(g_2) = -\frac{g_2^3}{16\pi^2} \left[\frac{22}{3} - \frac{4}{3} N_g \right]$$
(530)

from the family members. But, we are not done! In the SM, we have a (complex) Higgs doublet: $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. It counts $\frac{1}{2}$ as much as a fermion doublet of given helicity. (This is a slight generalization of the QED calculation we did under the technical notes precursor discussion.) (Note, I will do the counting ignoring electroweak symmetry breaking — the β functions obtained are those appropriate above the EWSB scale of $v \sim 246 \text{ GeV.}$)

Adding this guy in (I actually allow for an arbitrary number of such Higgs doublets) gives:

$$\beta_2(g_2) = -\frac{g_2^3}{16\pi^2} \left[\frac{22}{3} - \frac{4}{3}N_g - \frac{1}{6}N_{D_1} \right]$$
(531)

where D means doublet and the 1 subscript indicates the Y = 1hypercharge needed for the $Q = T_3 + \frac{1}{2}Y$ operator to give correct charges to the two components of the Higgs doublet.

3. Finally, we have the SM U(1) group. I will temporarily keep the notation g' for the coupling. For reasons to be explained, $g' = \sqrt{\frac{3}{5}}g_1$.

For this group, the well-known coupling is $g'\frac{Y}{2}$. This is the charge that replaces the QED charge e. We must count all particles that can give a contribution to Z_3 . The list is extensive. In the case of fermions, we count separately different helicities and remember that we have to reduce the 4-component QED result by a factor of $\frac{1}{2}$. That is we employ Eq. (519) with $e \rightarrow g'$ for each helicity state. The definite helicity states to be considered are:



(532)

This is what we get for each of the N_g generations.

For the Higgs doublet, we get

particle
$$Y \left(\frac{Y}{2}\right)^2$$

 H^+ 1 $\frac{1}{4}$
 H^0 1 $\frac{1}{4}$
Sum $= \frac{1}{2}$

$$(533)$$

Here, we have to use the spin-0 calculation which showed that a scalar particle comes in with 1/2 the weight of a fermion of given helicity.

The result is

$$\beta(g') = \frac{g'^3}{24\pi^2} \left[\frac{10}{3} N_g + \frac{1}{2} \left(\frac{1}{2} \right) N_{D_1} \right] \,. \tag{534}$$

If we now throw in the $g'=\sqrt{rac{3}{5}}g_1$ factor, the effective $eta_1(g_1)$ is

$$\beta_1(g_1) = \frac{g_1^3}{16\pi^2} \left[\frac{4}{3} N_g + \frac{1}{10} N_{D_1} \right] \,. \tag{535}$$

(Recall that $\beta(g)$ is defined by $\mu \frac{\partial g}{\partial \mu} = \beta(g)$ for a general coupling g. Thus, if we define $g = f\hat{g}$, since $\beta(g) = Kg^3$, the β function for \hat{g} only picks up a factor of f^2 : $\beta(\hat{g}) = Kf^2\hat{g}^3$. In our case $f^2 = 3/5$.)

The only question is why is it appropriate to consider g_1 (when discussing coupling constant unification) as opposed to g'? This has to do with the idea that a group such as SU(5) is the appropriate unification group.

SU(5)

The idea of grand unification is that all of our fundamental forces should actually be manifestations of a single force. In the QFT language, this means

that there should be a single group that contains subgroups corresponding to the subgroups required for the forces we observe: i.e. we need a master non-abelian group that contains $SU(3) \times SU(2) \times U(1)$. One will then write a covariant derivative in the form

$$\partial_{\mu} - ig_{GUT} A_{\mu} \tag{536}$$

where A_{μ} is a matrix of some sort defined by

$$A_{\mu} = \sum_{a} A^{a}_{\mu} L^{a} \tag{537}$$

where the *a* runs over all the generators for the group and L^a are the matrices that represent that group in some representation or other. So, it is then convenient to think of this matrix set in the fundamental representation of the group and in that fundamental representation the matrices being used must be "big" enough to incorporate as distinct matrices all of the matrices needed to include $SU(3) \times SU(2)_L \times U(1)$.

This means that the master group should be large enough to contain the 4 commuting generators of the SM — 2 from color SU(3) (denoting colors by 1, 2, 3 = r, g, b the diagonal traceless ones could be chosen as $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$

and $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}))$ and 1 for the SU(2) T^3 generator associated with the W^3 field and 1 for the U(1) generator associated with the B field. The generator for the U(1) group must be chosen so that it commutes with the SU(3) and SU(2) representation matrices.

The simplest such group is SU(5). SU(5) contains 24 generators associated with 24 fields, which we might denote by A_j^i . They belong to the adjoint representation, which is a 24 and decomposes to SU(3) and SU(2) representations as

$$24 = (8,1) + (1,3) + (1,1) + (3,2) + (\bar{3},2).$$
 (538)

We will denote the fundamental representation matrices for SU(5) by T^a (a = 1, ..., 24).

Considering the generators in a 5×5 traceless matrix (fundamental) representation, we would associate:

1. the upper 3×3 sub-block matrix with SU(3) (A_b^a , a, b = 1, 2, 3, associated with the gluon fields G_b^a – more or less, as we shall explain) — this would be the (8, 1) part of the 24;

2. the lower 2×2 sub-block with SU(2) $(A_s^r, r, s = 4, 5, which can be$

identified with the W^{\pm} , W^{3} fields of the SM) — this would be the (1,3) part of the 24;

3. the remaining orthogonal traceless composition of the diagonal matrices that commute with all upper 3×3 matrices and all lower 2×2 matrices with the U(1) generator — this would be the (1, 1) part of the 24.

This means that the U(1) generator should be a linear combination of

1	1	0	0	0	0	
	0	1	0	0	0	
	0	0	1	0	0	
	0	0	0	0	0	
	0	0	0	0	0 /	
					í.	

and

(0	0	0	0	0	
	0	0	0	0	0	
	0	0	0	0	0	•
	0	0	0	1	0	
	0	0	0	0	1 /	

(540)

(539

The appropriate matrix, T^0 , normalized so that $Tr[T^0T^0] = \frac{1}{2}$ (our convention) is (up to a sign, which is chosen by convention)

$$T^{0} = \frac{1}{2}\lambda^{0} = \frac{1}{2\sqrt{15}} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{15}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{20}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{20}} \end{pmatrix} .$$
 (541)

The corresponding T^3 generator associated with the W^3 is

The *B* field of the U(1) will then multiply T^0 .

An overall matrix representation of the full set of A_i^j fields would take the

form:

$$A \equiv \begin{pmatrix} A_{1}^{1} & A_{1}^{2} & A_{1}^{3} & A_{1}^{4} & A_{1}^{5} \\ A_{2}^{1} & A_{2}^{2} & A_{2}^{3} & A_{2}^{4} & A_{2}^{5} \\ A_{3}^{1} & A_{3}^{2} & A_{3}^{3} & A_{4}^{4} & A_{5}^{5} \\ A_{4}^{1} & A_{4}^{2} & A_{3}^{3} & A_{4}^{4} & A_{5}^{5} \\ A_{5}^{1} & A_{5}^{2} & A_{5}^{3} & A_{5}^{4} & A_{5}^{5} \end{pmatrix} .$$
(543)

In terms of the SU(3) gluonic G fields, the SU(2) weak boson W^{\pm}, W^{3} fields, and the U(1) field B, we then have

$$A = \sum_{a=0}^{a=23} A^{a} \frac{\lambda^{a}}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} G^{\alpha}_{\beta} - \frac{2}{\sqrt{30}} \delta^{\alpha}_{\beta} B \end{pmatrix} & X_{2} & Y_{2} \\ X_{1} & X_{2} & Y_{3} \\ X^{1} & X^{2} & X^{3} & \frac{1}{\sqrt{2}} W^{3} + \frac{3}{\sqrt{30}} B & W^{+} \\ Y^{1} & Y^{2} & Y^{3} & W^{-} & -\frac{1}{\sqrt{2}} W^{3} + \frac{3}{\sqrt{30}} B \end{pmatrix}.$$
(544)

This means that, in terms of the A_j^i fields, the *B* field takes the form:

$$B = 2\left(-\frac{1}{\sqrt{15}}A^{\alpha}_{\alpha} + \sqrt{\frac{3}{20}}A^{r}_{r}\right), \qquad (545)$$

As a check of Eq. (545), note that

$$-\frac{1}{\sqrt{15}} \left(\frac{1}{\sqrt{2}} \frac{-2}{\sqrt{30}}\right) \times 3B + \sqrt{\frac{3}{20}} \left(\frac{1}{\sqrt{2}} \frac{3}{\sqrt{30}}\right) \times 2B = \frac{1}{2}B.$$
 (546)

Here, all normalizations are fixed by the requirement that the kinetic energy terms for the individual fields have proper canonical normalization. Recall that for a single real gauge field, the free-particle normalization should be $-\frac{1}{4}F^{a}_{\mu\nu}F^{a\ \mu\nu}$. In the non-abelian structure, recall that we write our kinetic energy as $-\frac{1}{2}\text{Tr}[F_{\mu\nu}F^{\mu\nu}]$, where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig[A_{\mu}, A_{\nu}]$ and A represents the full $A^{a}L^{a}$ summation, where L^{a} is the representation matrix, that we are here calling T^{a} for the fundamental 5×5 representation.

One can, in particular, check that this works for the *B* field. It is not necessary to keep the derivatives for this check, just imagine they are there. Then, if we take $-\frac{1}{2}\text{Tr}[AA]$ we get



which is the desired result. Of course, the kinetic energies for the other fields also work out correctly. As one more example, we can check the W^3 field. There,

$$-\frac{1}{2} \operatorname{Tr}[AA] \ni -\frac{1}{2} \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right)^2 2(W^3)^2 = -\frac{1}{4}(W^3)^2.$$
 (548)

In the above, the extra gauge bosons denoted by X_i and Y_i , associated with the (3, 2) part of the 24 (and their conjugates X^i and Y^i , associated with the $(\bar{3}, 2)$ part of the 24) must be very heavy since they would yield transitions that would lead to proton decay, something that can only occur at a very suppressed level. Presumably, their masses are of order the M_U scale at which this master SU(5) group is broken down to $SU(3) \times SU(2) \times U(1)$. We do not have time to go into details about how this breaking occurs. Usually, extra Higgs fields (in particular a 24 Higgs field) are introduced to accomplish this breaking in a manner analogous to that you have studied for breaking $SU(2) \times U(1)$ down to $U(1)_{EM}$.

So, now the key point related to the matrix T^0 and the relation between g' and g_1 is to note that the covariant derivative for the SU(5) group will take the form (prior to breaking down to $SU(3) \times SU(2) \times U(1)$)

$$\partial_{\mu} - ig_5 A_{\mu} \tag{549}$$

where

$$A_{\mu} = \sum_{a=0}^{a=23} A_{\mu}^{a} \frac{\lambda^{a}}{2} = \sum_{a=0}^{a=23} A_{\mu}^{a} T^{a}$$
(550)

with the T^a being the 5 \times 5 matrices normalized according to our convention

 $(\operatorname{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab})$ when we consider this covariant derivative operating on a fundamental 5-dimensional representation of SU(5). Thus, if the groups all unify at M_U to a common group, we will have $g_1 = g_2 = g_3 = g_5$ at M_U . Below M_U , the coupling constants will diverge according to the individual $\beta_i(g_i)$, but we must use unification of, in particular, g_1 (and not g') with g_2 and g_3 as our starting point.

Now, we must fit the fermions into representations of SU(5). This is done by putting part of them into a 5 (or, equivalently, $\overline{5}$) and part into a 10 (antisymmetric) representation. For a single family, we have to accommodate

$$(\nu_{e}, e^{-})_{L}: (1, 2)$$

$$e_{R}^{-} = e_{L}^{+}: (1, 1)$$

$$(u_{a}, d_{a})_{L}: (3, 2)$$

$$(u_{a})_{R} = \bar{u}_{L}^{a}: (\bar{3}, 1)$$

$$(d_{a})_{R} = \bar{d}_{L}^{a}: (\bar{3}, 1)$$

$$(551)$$

where I have used the equivalence of a f_R state with the \overline{f}_L state (under charge conjugation). Placement of these 1st family members is accomplished

in the form

$$\bar{\mathbf{5}} = (\bar{d}^1, \bar{d}^2, \bar{d}^3, e^-, -\nu_e)_L, \qquad (552)$$

which could also be written as

$$5 = (d_1, d_2, d_3, e^+, -\bar{\nu}_e)_R, \qquad (553)$$

and

$$10 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \bar{u}^3 & -\bar{u}^2 & u_1 & d_1 \\ -\bar{u}^3 & 0 & \bar{u}^1 & u_2 & d_2 \\ \bar{u}^2 & -\bar{u}^1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L$$
(554)

In the above, the 1, 2, 3 indices are the three SU(3) indices in the fundamental representation, which we have earlier called r, g, b.

I have no time to go into details about the 10 representation. What we want to focus on is the fundamental 5 representation. This is the key to establishing the relation between the Y operator and the T^0 generator of the U(1) subgroup of SU(5). We know from our SM work that the hypercharge assignments of the 5 members imply (here [5] is the column matrix of 5 above)

$$Y[5] = diag[-2/3, -2/3, -2/3, 1, 1][5].$$
(555)

Meanwhile,

$$T^{0}[5] = \frac{1}{2\sqrt{15}} diag[-2, -2, -2, 3, 3][5].$$
 (556)

Thus, if we wish to make the identification

$$g_1 T^0 B = g' \frac{1}{2} Y B$$
, (557)

where g_1 will unify with g_2 and g_3 at scale M_U so that all take the value g_5 of the universal SU(5) coupling constant at that scale, we must have

$$g_1 \frac{1}{2\sqrt{15}} \times (-2) = g' \frac{1}{2} \times \left(-\frac{2}{3}\right)$$
 (558)

or

$$g' = \sqrt{\frac{3}{5}}g_1, \qquad (559)$$

as employed earlier in getting the form of β_1 .

There are many other fascinating aspects to SU(5) (and its generalizations), not the least of which is that it leads to the observed relation between quark and lepton charges. Since the charge operator Q must commute with all the

color operators, it can only be some linear combination of T^3 and T^0 . In fact, from the discussion just above, we see that

$$Q = T^3 + \frac{Y}{2} = T^3 + \sqrt{\frac{5}{3}}T^0$$
 (560)

so that for the fundamental 5 representation we have

$$Q[5] = diag[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1, 0][5].$$
(561)

We could have multiplied by an arbitrary overall factor, but, once we have defined the e^+ charge to be +1 in charge units, the quarks must have the phenomenologically required fractional values!

Coupling Constant Unification

To review, we have:

$$\beta_1(g_1) = \frac{g_1^3}{16\pi^2} \left[\frac{4}{3} N_g + \frac{1}{10} N_{D_1} \right] ; \qquad (562)$$

$$\begin{aligned} \beta_2(g_2) &= -\frac{g_2^3}{16\pi^2} \left[\frac{22}{3} - \frac{4}{3} N_g - \frac{1}{6} N_{D_1} \right] ; \end{aligned} \tag{563} \\ \beta_3(g_3) &= -\frac{g_3^3}{16\pi^2} \left[11 - \frac{4}{3} N_g \right] . \end{aligned} \tag{564}$$

The coupling evolution equations are then summarized in the form

$$\frac{\partial g_n}{\partial \ln \mu} = -\frac{b_n}{16\pi^2} g_n^3 \tag{565}$$

with

$$-b_{1} = \frac{4}{3}N_{g} + \frac{1}{5}(N_{S_{2}} + 4N_{S_{4}}) + \frac{1}{10}(N_{D_{1}} + 9N_{D_{3}}) + \frac{3}{5}N_{T_{2}} + \frac{28}{5}N_{34}$$

$$-b_{2} = \frac{4}{3}N_{g} + \frac{1}{6}(N_{D_{1}} + N_{D_{3}}) + \frac{2}{3}(N_{T_{0}} + N_{T_{2}}) + \frac{28}{27}N_{34} - \frac{22}{3}$$

$$-b_{3} = \frac{4}{3}N_{g} - 11.$$
(566)

where I have introduced a bunch of other possible Higgs representations that might be present in a more general model: I denote the number of |Y| = 1doublets by N_{D_1} , the number of |Y| = 2 triplets by N_{T_2} , and so forth; N_{34} denotes the number of (I = 3, |Y| = 4) representations. I do not consider $|Y| \ge 6$ singlets, $|Y| \ge 5$ doublets, or $|Y| \ge 4$ triplets. Here, by doublets and triplets and so forth, I refer to the weak isospin representation.

In the Minimal Supersymmetric Model, one adds only the superpartners of the SM particles plus a 2nd Higgs doublet (as required for anomaly cancellation and for giving masses to both up and down type quarks). The resulting b_i are derived in the Appendix following this section. One finds:

$$-b_{1} = 2N_{g} + \frac{3}{5}(N_{S_{2}} + 4N_{S_{4}}) + \frac{3}{10}(N_{D_{1}} + 9N_{D_{3}}) + \frac{9}{5}N_{T_{2}} + \frac{84}{5}N_{34}$$
$$-b_{2} = 2N_{g} + \frac{1}{2}(N_{D_{1}} + N_{D_{3}}) + 2(N_{T_{0}} + N_{T_{2}}) + \frac{28}{9}N_{34} - 6$$
$$-b_{3} = 2N_{g} - 9$$
(567)

Note that in the above equations there is no influence from Y = 0 singlets, but $Y \neq 0$ singlets affect b_1 . Note also that the N_g terms cancel in the difference of any two b_i 's in both the SM and the MSSM. Finally, in the MSSM we note the upper limit of $N_g = 4$ in order that QCD be asymptotically free. (Phenomenological constraints on an extra generation are quite severe if one assumes coupling unification — basically, its members cannot be much heavier than the top quark of the 3rd generation.) Now, the evolution equations are easily integrated to give

$$\frac{1}{g_n^2(\mu)} - \frac{1}{g_n^2(\mu_0)} = \frac{b_n}{8\pi^2} \ln\left(\frac{\mu}{\mu_0}\right)$$
(568)

or

$$\frac{1}{\alpha_n(\mu)} - \frac{1}{\alpha_n(\mu_0)} = \frac{b_n}{2\pi} \ln\left(\frac{\mu}{\mu_0}\right) \,. \tag{569}$$

We identify μ_0 with the unification scale M_U at which all the α_i are supposed to unify to a single value that we denote by $\alpha_5(M_U)$. The lower scale μ will be chosen to be of order the EWSB scale of order m_Z or $v \sim 246$ GeV. By subtracting equations for two of the α_i assuming that all the α_i unify to α_5 at M_U we obtain two equations. A useful pair is (for simplicity, I write $\alpha_i \equiv \alpha_i(\mu)$)

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_3} = \frac{b_2 - b_3}{2\pi} L$$

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_1} = \frac{b_2 - b_1}{2\pi} L$$
 (570)

where $L = \ln(\mu/M_U)$. We can solve the first equation for $L/2\pi$ and

substitute, obtaining

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_1} = \frac{b_2 - b_1}{b_2 - b_3} \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_3} \right) \,. \tag{571}$$

We write this in the form:

$$\frac{1}{\alpha_2} \left(1 - \frac{b_2 - b_1}{b_2 - b_3} \right) = \frac{1}{\alpha_1} - \frac{1}{\alpha_3} \frac{b_2 - b_1}{b_2 - b_3}$$
(572)

or

$$\frac{1}{\alpha_2}(b_1 - b_3) - \frac{1}{\alpha_1}(b_2 - b_3) = \frac{1}{\alpha_3}(b_1 - b_2).$$
 (573)

At the EWSB scale, we have the phenomenological results that

$$egin{array}{rcl} lpha_1(\mu)&=&rac{5\,{g'}^2}{3\,4\pi}=rac{5\,lpha_{QED}}{3\,\cos^2 heta_W}\ lpha_2(\mu)&=&rac{g^2}{4\pi}=rac{lpha_{QED}}{\sin^2 heta_W}\ lpha_3(\mu)&=&lpha_s(\mu)\,. \end{array}$$

(574)

We substitute these into the above equation. The requirement of unification then takes the form (with $\mu \sim m_Z$)

$$\alpha_s(m_Z) = \alpha_{QED}(m_Z) \frac{5(b_1 - b_2)}{\sin^2 \theta_W(5b_1 + 3b_2 - 8b_3) - 3(b_2 - b_3)}$$
(575)

We emphasize that the above equation assumes "standard" SU(5) normalization of the U(1) coupling constant and a desert between m_Z and M_U . Once we have a solution to Eq. (575), we can solve for L and thence M_U using either of the two equations (570). We can then solve for $\alpha_5(M_U)$ using one of the equations (569).

Using $\alpha_s(m_Z) = 0.118$ and $\sin^2 \theta_W = .2315$, and the results of Eqs. (566) and (567), the constraint of Eq. (575) reduces to

SM:
$$1 \simeq -0.09N_{S_2} - 0.36N_{S_4} + 0.13N_{D_1} - 0.22N_{D_3}$$

+ $0.71N_{T_0} + 0.44N_{T_2} - 1.39N_{34}$ (576)
SUSY: $1 \simeq -0.33N_{S_2} - 1.31N_{S_4} + 0.49N_{D_1} - 0.82N_{D_3}$
+ $2.61N_{T_0} + 1.63N_{T_2} - 5.11N_{34}$ (577)

in the non-SUSY and MSSM cases, respectively. The exact coefficients in

the equations above are sensitive to the precise α_s and $\sin^2 \theta_W$ choices as well as to whether all the Higgs bosons (and SUSY sparticles) have mass near m_Z , as assumed, or nearer to 1 TeV. Two-loop corrections also lead to small changes in the unification conditions. Thus, the following discussion of 'solutions' should be regarded as being a somewhat rough, but indicative, guide to the possibilities.

First, we note that the simple SM with $N_{D_1} = 1$ and all others zero does not work, whereas the MSSM with (as required for anomaly cancellation) $N_{D_1} = 2$ provides an almost exact solution. If you plot graphically, the SM failure looks less bad since the $\frac{1}{\alpha_i}$'s come close to crossing. Nonetheless, they do not. The figure below shows this lack of crossing. (In a change of notation, I write $N_{T,Y} =$ for the number of Higgs representations of a given type in the figure and in the subsequent discussion.)



In the SM case, we can only fix things up by bringing in the other Higgs representations (or other particles that we do not consider). We find that coupling unification can be achieved without SUSY by introducing additional Higgs representations in the standard model.

Some simple choices are:

Table 1: Higgs models that yield coupling unification in the SM context with α_s not far from 0.118

$N_{1/2,1}$	$N_{1/2,3}$	$N_{0,2}$	$N_{0,4}$	$N_{1,0}$	$N_{1,2}$	$lpha_{s}$	M_U (GeV)
1	0	0	2	0	0	0.106	$4 imes 10^{12}$
1	0	4	0	0	1	0.112	$7.7 imes10^{12}$
1	0	0	0	0	2	0.120	$1.6 imes10^{13}$
2	0	0	0	1	0	0.116	$1.7 imes10^{14}$
2	0	2	0	0	2	0.116	$4.9 imes10^{12}$
2	1	0	0	0	2	0.112	$1.7 imes10^{12}$
3	0	0	0	0	1	0.105	$1.2 imes10^{13}$

Remarks:

• Find lower M_U than comfortable for proton decay.

Can fix by not having true group unification, as in some string models.

• My personal favorite: $N_{rac{1}{2},1}=2, N_{1,0}=1 \Rightarrow lpha_s(m_Z)=0.116$, $M_U=1.7 imes10^{14}~{
m GeV}$

The unification of the couplings is displayed in the figure below.



This model has no phenomenological problems (other than needing the above fix for proton decay) so long as the neutral member of the T = 1, Y = 0 triplet has zero vacuum expectation value. In fact, this neutral triplet member is even a viable dark matter candidate.

Of course, without supersymmetry there is no cancellation of the quadratic divergences, implying that a light mass for the Higgs boson (as required by LEP data in the SM context) has no natural explanation. Something like supersymmetry is required to make a light Higgs boson natural.

Appendix: derivation of the MSSM b_i

β_3

It is simplest to begin with the b_3 additions due to the spin-0 partners of the spin-1/2 (two-component) quarks. In the first family, we have the partners \tilde{u}_L , \tilde{u}_R , \tilde{d}_L , and \tilde{d}_R . In total, there are four spin-0 fields per family, a total of $4N_g$ spin-0 fields. Since they are spin-0, they enter with only 1/2 the weight of the two-component spin-1/2 fermions. Referring back to Eq (527), we get a correction to the bracket which then takes the form

$$[\ldots] \rightarrow \left[\ldots - \frac{2}{3} \left(\frac{1}{2} \right) \left(n_{fermionic \ triplets \ of \ given \ helicity} + \frac{1}{2} n_{bosonic \ triplets} \right) \right].$$
(578)

We have

 $n_{fermionic\ triplets\ of\ given\ helicity} = n_{bosonic\ triplets} = 4N_g$, (579)

so that the bracket becomes

$$\left[\ldots -\frac{4}{3}N_g\right] \to \left[\ldots - 2N_g\right]. \tag{580}$$

But, we are not done. We also have fermionic partners to the gluons, the so-called gluinos. These are fermion partners of the gluons. There is one (effectively with just one helicity since we are talking about a Majorana fermion in this case) for each gluon. Let us recall that the contribution of a fermion to the standard $[\ldots]$ is

$$-\frac{2}{3} \times color \ factor \,, \tag{581}$$

where the color factor $=\frac{1}{2}$ for a fundamental triplet insertion into a gluon line is now replaced by the usual adjoint representation C_A which is 3 for SU(3). The result in the case of β_3 is the replacement

$$\left[\frac{11}{3}3 - \ldots\right] \rightarrow \left[\left(\frac{11}{3} - \frac{2}{3}\right)3 - \ldots\right], \qquad (582)$$

implying that the previous $11 \rightarrow 9$ inside the [...].

 eta_2

Here, we get a modification of Eq. (529) in which

 $^{n}weak-isospin \ doublets \ of \ given \ helicity \
ightarrow$

 $^{n}weak-isospin$ doublets of given helicity

$$+\frac{1}{2}n_{weak-isospin\ bosonic\ doublets}$$
 (583)

Since there is a squark or slepton for each L or R helicity quark or lepton, we have

 $n_{weak-isospin \ bosonic \ doublets} = n_{weak-isospin \ doublets \ of \ given \ helicity} = 4N_g$ (584)

and the [...] in the form of $\beta_2(g_2)$ obtained before including the Higgs field gets modified in a manner very analogous to the color case:

$$\left[\dots -\frac{4}{3}N_g\right]] \rightarrow \left[\dots -\frac{4}{3}N_g\frac{3}{2}\right] = \left[\dots -2N_g\right].$$
 (585)

Note that once again we are getting the same (now $-2N_g$) term in both β_3 and β_2 .

Similar to β_3 , we must include the effect on β_2 of the fermionic partners of the weak gauge bosons, the so-called "winos". By analogy, it should be clear that in the [...], we have

$$\left[\frac{11}{3}2-\ldots\right] \rightarrow \left[\left(\frac{11}{3}-\frac{2}{3}\right)2-\ldots\right] = \left[6-\ldots\right], \quad (586)$$

where the 2 outside the parenthesis is just C_A for the SU(2) group.

The Higgs fields require special discussion. First, there are now two Higgs complex bosonic doublet fields. Second, each of these fields comes with a fermionic partner which contributes twice as much. As a result, the $-\frac{1}{6}N_{D_1}$ component of β_2 is replaced by

$$-\frac{1}{6}N_{D_1} \to -\frac{1}{6}N_{D_1} \times 3 = -\frac{1}{2}N_{D_1}, \qquad (587)$$

where, in addition, $N_{D_1} = 2, 4, 6, ...$ is required for anomaly cancellation.

β_1

This requires going through the listing of the earlier tables and multiplying each fermion contribution listed by $\frac{3}{2}$ and each boson contribution listed by 3. The result is

$$\left[\frac{4}{3}N_g + \frac{1}{10}N_{D_1}\right] \to \left[2N_g + \frac{3}{10}N_{D_1}\right].$$
 (588)

There is, of course, also a bino partner to the B, but since the group is U(1) there is no B-bino-bino vertex and so no effect on b_1 from this particle.

You can now check that these modifications give the b_i values quoted for
the MSSM.

Unification plots

Of course, it is interesting to plot the coupling unification for the MSSM to compare to the graphs obtained earlier for the SM.



Note the much higher scale of $M_U \sim 1.9 \times 10^{16}$ GeV for the unification point. This means that the MSSM is much safer (but still not entirely safe) with regard to proton decay if there is true group unification — the required X and Y bosons are much heavier than in the ad hoc SM case I gave as an example earlier.

Also note the large value of α_5 at the unification point.

Computation of $\Pi_{\mu\nu}$ to order $g^2 e^2$

This I will do from my notes (Renormalization Notebook). The idea is to derive the first important correction appearing in the formula

$$R = \frac{\sum_{q} \sigma(e^+e^- \to q\overline{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{q} e_q^2 \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right)$$
(589)

using the $\Pi_{\mu\nu}$ computation. (As we have noted, there are alternative approaches, but this approach teaches us some interesting things along the way.) Incidentally, I will not keep track of up vs. down Lorentz indices. They always enter in the right way in the end.

The Optical Theorem

The first basic idea is that $\sigma(e^+e^- \rightarrow q\overline{q})$ should be proportional to the imaginary part of the amputated photon propagator function we called $\Pi(q^2)$. We actually don't care about the constant of proportionality since this will cancel out in the ratio used to define R. This "unitarity"

relationship is not something we have gone into in detail, but it is an intuitively obvious generalization of the non-relativisitic QM result.

Let us give a general and then a specific derivation.

General Derivation

Let us write the S matrix in the form S = 1 + iT, in which case unitarity $S^{\dagger}S = 1$ takes the form

$$-i(T-T^{\dagger}) = T^{\dagger}T.$$
(590)

In the notation that we have been using for most of these lectures, which derived from Mandl and Shaw, we wrote

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 \left(\sum_f p'_f - \sum_i p_i\right) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE'_f}\right)^{1/2} \mathcal{M}.$$
(591)

However, it has actually become more conventional to use

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 (\sum_f p'_f - \sum_i p_i) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE'_f}\right)^{1/2} i\mathcal{M},$$
(592)

i.e. differing by a factor of *i*. It is better to use this latter definition when discussing unitarity in particular (this was also done in QFT-III notes on unitarity, which we had skipped over — see about page 270 — there I called it \widetilde{M} , but here I will for convenience drop the tilde.). So, in this discussion, I will use the latter definition and one needs to keep in mind that all our feynman rules are then for $i\mathcal{M}$ in our new definition.

So, we now take the matrix element of Eq. (590) between two-particle states, $\langle p_1 p_2 |$ and $|k_1 k_2 \rangle$, and use the definition of \mathcal{M} that has an extra *i*, as discussed above. For example,

$$\langle p_1 p_2 | iT | k_1 k_2 \rangle = \frac{1}{\sqrt{2VE_{\vec{p}_1}}} \frac{1}{\sqrt{2VE_{\vec{p}_2}}} \frac{1}{\sqrt{2VE_{\vec{k}_1}}} \frac{1}{\sqrt{2VE_{\vec{k}_2}}} (2\pi)^4 \delta^4 (p_1 + p_2 - k_1 - k_2) i\mathcal{M}(k_1 k_2 \to p_1 p_2) .$$
(593)

The left hand side of Eq. (590) becomes

$$\frac{1}{\sqrt{2VE_{\vec{p}_1}}} \frac{1}{\sqrt{2VE_{\vec{p}_2}}} \frac{1}{\sqrt{2VE_{\vec{k}_1}}} \frac{1}{\sqrt{2VE_{\vec{k}_2}}} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \\ \times \left[-i\mathcal{M}(k_1k_2 \to p_1p_2) + i\mathcal{M}^*(p_1p_2 \to k_1k_2) \right] .$$
(594)

Meanwhile, on the right hand side, we insert a complete set of states labeled by momentum states $\{q_i\}$ and find (remembering the box to continuum equivalence $\sum_{\vec{q}} = \int V \frac{d^3q}{(2\pi)^3}$)

$$\langle p_{1}p_{2}|T^{\dagger}T|k_{1}k_{2}\rangle = \sum_{n} \left(\prod_{i=1}^{n} \int V \frac{d^{3}q_{i}}{(2\pi)^{3}} \right) \langle p_{1}p_{2}|T^{\dagger}|\{q_{i}\}\rangle \langle \{q_{i}\}|T|k_{1}k_{2}\rangle$$

$$= (2\pi)^{4} \delta^{4}(p_{1}+p_{2}-k_{1}-k_{2}) \sum_{n} \left(\prod_{i=1}^{n} \int V \frac{d^{3}q_{i}}{(2\pi)^{3}} \right)$$

$$\times \frac{1}{\sqrt{2VE_{\vec{p}1}}} \frac{1}{\sqrt{2VE_{\vec{p}2}}} \frac{1}{\sqrt{2VE_{\vec{k}1}}} \frac{1}{\sqrt{2VE_{\vec{k}2}}} \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2VE_{\vec{q}i}}} \right)^{2}$$

$$\times \mathcal{M}^{*}(p_{1}p_{2} \to \{q_{i}\}) \mathcal{M}(k_{1}k_{2} \to \{q_{i}\}) (2\pi)^{4} \delta^{4}(k_{1}+k_{2}-\sum_{i} q_{i})$$
(595)

where we used the relationship between T and \mathcal{M} . Equating the two sides of our starting equation and removing the overall $(2\pi)^4 \delta^4 (p_1 + p_2 - k_1 - k_2)$

gives

$$= \sum_{n} \left(\prod_{i=1}^{n} \int \frac{d^{3}q_{i}}{(2\pi)^{3}} \frac{1}{2E_{i}} \right) \times \mathcal{M}^{*}(p_{1}p_{2} \to \{q_{i}\}) \mathcal{M}(k_{1}k_{2} \to \{q_{i}\})(2\pi)^{4}\delta^{4}(k_{1} + k_{2} - \sum_{i} q_{i}).$$
(596)

Let us now take the forward case of $p_1 = k_1$ and $p_2 = k_2$. In this case, the above relation can be abbreviated as

$$2 \text{Im} \mathcal{M}(k_1 k_2 \to k_1 k_2) = \sum_{f} \int d\Pi_f \mathcal{M}^*(k_1 k_2 \to f) \mathcal{M}(k_1 k_2 \to f)$$

=
$$4 E_{cm} p_{cm} \sigma_{\text{tot}}(k_1 k_2 \to \text{anything})$$

small masses
$$2 s \sigma_{\text{tot}}(k_1 k_2 \to \text{anything}).$$
(597)

Feynman Diagram Derivation

The remarkable result is that this unitarity relationship applies at the

Feynman diagram level in the sense of what are called the Cutkosky cutting rules.

Let us derive it in the simplest case of a fermion loop in QED. We compute the one loop insertion correction to the $e^+e^- \rightarrow \gamma^* \rightarrow \text{loop} \rightarrow \gamma^* \rightarrow e^+e^-$ forward amplitude, *i.e.* with the outgoing final e^+ momentum p equal to the incoming initial e^+ momentum, and final e^- momentum p' equal to the incoming initial e^- momentum. The calculation for incoming $p, p' \rightarrow \gamma^*(\mu, \mu') \rightarrow \text{loop} \rightarrow \gamma^*(\nu, \nu') \rightarrow p, p'$ takes the form (we will neglect e^+, e^- masses)

$$i\mathcal{M} = \overline{v}(p)(-ie\gamma_{\mu})u(p')\frac{-ig_{\mu\mu'}}{k^{2}}i\Pi_{\mu'\nu'}(k)\frac{-ig_{\nu'\nu}}{k^{2}}\overline{u}(p')(-ie\gamma_{\nu})v(p)$$
$$= e^{2}\overline{v}(p)\gamma_{\mu}u(p')\frac{1}{k^{2}}i\Pi_{\mu\nu}\frac{1}{k^{2}}\overline{u}(p')\gamma_{\nu}v(p), \qquad (598)$$

where, by convention, the first index (μ) of $i\Pi_{\mu\nu}$ goes with the incoming γ^* and the ν goes with the outoing γ^* . So, now let us consider the one-loop $\mu^+\mu^-$ type of contribution to $\Pi_{\mu\nu}(k)$. We know the form that it takes very well from our earlier work.

$$\begin{split} i\Pi_{\mu\nu}(k) &= (-)(-ie)^2 \int \frac{d^4l}{(2\pi)^4} \mathrm{Tr} \left[\gamma_{\mu} \frac{i}{\not{l}-m+i\epsilon} \gamma_{\nu} \frac{i}{\not{l}+\not{k}-m} \right] \\ &= (-)e^2 \int \frac{d^4l}{(2\pi)^4} \mathrm{Tr} \left[\gamma_{\mu}(\not{l}+m) \gamma_{\nu}(\not{l}+\not{k}+m) \right] \frac{1}{l^2-m^2+i\epsilon} \quad \frac{1}{(l+k)^2-m^2+i\epsilon} \end{split}$$

$$= e^{2} \int \frac{d^{4}q}{(2\pi)^{4}} \operatorname{Tr} \left[\gamma_{\mu} \left(\left(\frac{1}{2} \not{k} - q \right) - m \right) \gamma_{\nu} \left(\left(\frac{1}{2} \not{k} + q \right) + m \right) \right] \\ \times \frac{1}{\left(\frac{1}{2} k - q \right)^{2} - m^{2} + i\epsilon} \frac{1}{\left(\frac{1}{2} k + q \right)^{2} - m^{2} + i\epsilon},$$
(599)

where the explicit (-) is for the fermion loop and in the last step I switched to $q = l + \frac{1}{2}k$. We must now discuss how to take the imaginary part of \mathcal{M} . It is easy to see from Eq. (598) that $\mathrm{Im}\mathcal{M} \propto \mathrm{Im}\Pi_{\mu\nu}$ and so we turn now to the latter calculation. The key is understanding how to do the q^0 integral given the four poles implicit in the two denominators shown. At a certain point it will be important to recall the following standard identity:

$$\frac{1}{x+i\epsilon} = P\left(\frac{1}{x}\right) - i\pi\delta(x).$$
(600)

What we will show is that $-2 \times \text{Im}\Pi_{\mu\nu}$ is obtained by replacing

$$= \begin{bmatrix} \frac{1}{(\frac{1}{2}k-q)^{2}-m^{2}+i\epsilon} \end{bmatrix} \begin{bmatrix} \frac{1}{(\frac{1}{2}k+q)^{2}-m^{2}+i\epsilon} \end{bmatrix} \\ \rightarrow \begin{bmatrix} -2\pi i\delta((\frac{1}{2}k-q)^{2}-m^{2}) \end{bmatrix} \theta(\frac{1}{2}k^{0}-q^{0}) \\ \times \left[-2\pi i\delta((\frac{1}{2}k+q)^{2}-m^{2})) \right] \theta(\frac{1}{2}k^{0}+q^{0}) .$$
 (601)

in the right-hand side of Eq. (599). Assuming this result, defining $r = \frac{1}{2}k - q$ and $r' = \frac{1}{2}k + q$, and taking the - sign to the other side, we obtain

$$2 \text{Im} \Pi_{\mu\nu} = -(-2\pi i)^{2} e^{2} \int \frac{d^{4}r}{(2\pi)^{4}} \frac{d^{4}r'}{(2\pi)^{4}} (2\pi)^{4} \delta^{4}(r+r'-k) \text{Tr}[\gamma_{\mu}(\eta'-m)\gamma_{\nu}(\eta'+m)] \\ \times \delta(r^{2}-m^{2})\theta(r^{0})\delta(r'^{2}-m^{2})\theta(r'^{0}) \\ = e^{2} \int \frac{d^{3}r}{(2\pi)^{3}2E_{r}} \frac{d^{3}r'}{(2\pi)^{3}2E_{r'}} (2\pi)^{4} \delta^{4}(r+r'-k) \text{Tr}[\gamma_{\mu}(\eta'-m)\gamma_{\nu}(\eta'+m)] \\ = e^{2} \int \frac{d^{3}r}{(2\pi)^{3}2E_{r}} \frac{d^{3}r'}{(2\pi)^{3}2E_{r'}} (2\pi)^{4} \delta^{4}(r+r'-k) \sum_{s_{r},s_{r'}} [\overline{u}(r')\gamma_{\mu}v(r)\overline{v}(r)\gamma_{\nu}u(r')] \quad (602)$$

In order to write the last line, we had to know that the r and r' were on-shell as assumed when writing a u or v spinor.

If we now include the e^+e^- and γ^* propagator stuff appearing in $i\mathcal{M}$,

$$e^{2}\overline{v}(p)\gamma_{\mu}u(p')\overline{u}(p')\gamma_{\nu}v(p)\left(\frac{1}{k^{2}}\right)^{2}$$
(603)

and use $(1/k^2)^2 = 1/s^2$ in the ${\cal M}$ expression, we obtain

$$2 \operatorname{Im} \mathcal{M} = e^{2} \overline{u}(p') \gamma_{\nu} v(p) \overline{v}(p) \overline{v}(p) \gamma_{\mu} u(p') \left(\frac{1}{s}\right)^{2} e^{2} \int \frac{d^{3}r}{(2\pi)^{3} 2E_{r}} \frac{d^{3}r'}{(2\pi)^{3} 2E_{r'}} \times (2\pi)^{4} \delta^{4}(r+r'-k) \sum_{s_{r},s_{r'}} [\overline{u}(r') \gamma_{\mu} v(r) \overline{v}(r) \gamma_{\nu} u(r')]$$
(604)

We now wish to compare this expression to the total cross section for $e^+e^- \rightarrow \mu^+\mu^-$. For this latter, we would write down the amplitude, take the absolute square, and then integrate over final state phase space, and finally divide by the appropriate flux factor. The amplitude for $e^+(p)e^-(p') \rightarrow \mu^+(r)\mu^-(r')$ (where I am using momenta that match our previous calculation conventions) is given by

$$i\mathcal{M}(e^+e^- \to \mu^+\mu^-) = \overline{u}(r')(-ie\gamma_{\mu})v(r)\frac{-ig_{\mu\mu'}}{s}\overline{v}(p)(-ie\gamma_{\mu'})u(p')$$
$$= ie^2\overline{u}(r')\gamma_{\mu}v(r)\frac{1}{s}\overline{v}(p)\gamma_{\mu}u(p').$$
(605)

Next we take the absolute square:

$$|\mathcal{M}|^{2} = e^{4}\overline{u}(r')\gamma_{\mu}v(r)\overline{v}(r)\gamma_{\nu}u(r')\frac{1}{s^{2}}\overline{v}(p)\gamma_{\mu}u(p')\overline{u}(p')\gamma_{\nu}v(p).$$
(606)

Finally, we multiply by the flux factor, simply 1/(2s) in this case, and integrate over phase space which has the form

$$\int \frac{d^3r}{(2\pi)^3 2E_r} \frac{d^3r'}{(2\pi)^3 2E_{r'}} (2\pi)^4 \delta^4(r+r'-k), \qquad (607)$$

and sum over final spins to obtain

$$\sigma(e^{+}e^{-} \to \mu^{+}\mu^{-}) = \frac{1}{2} \frac{e^{4}}{s^{3}} \overline{v}(p) \gamma_{\mu} u(p') \overline{u}(p') \gamma_{\nu} v(p) \int \frac{d^{3}r}{(2\pi)^{3} 2E_{r}} \frac{d^{3}r'}{(2\pi)^{3} 2E_{r'}} \times (2\pi)^{4} \delta^{4}(r+r'-k) \sum_{s_{r},s_{r'}} [\overline{u}(r') \gamma_{\mu} v(r) \overline{v}(r) \gamma_{\nu} u(r')]$$
(608)

By comparing Eq. (608) to Eq. (604) you will see that

$$s\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \text{Im}\mathcal{M}(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^- \rightarrow \gamma^* \rightarrow e^+e^-).$$
(609)
This result agrees with the general optical theorem of Eq. (597) since
 $2E_{cm}p_{cm} = s$ in the limit where the electron mass is neglected.

Proof of the replacement Eq. (601)

We wish to show, dropping the trace algebra stuff, which has no imaginary

2

part, and defining¹⁰

$$i\mathcal{I} \equiv \int \frac{d^4q}{(2\pi)^4} \left[\frac{1}{(\frac{1}{2}k+q)^2 - m^2 + i\epsilon} \right] \left[\frac{1}{(\frac{1}{2}k-q)^2 - m^2 + i\epsilon} \right] , \quad (610)$$

that (c.f. Eq. (602))

$$2 \text{Im} \mathcal{I} = -\int \frac{d^4 q}{(2\pi)^4} \left[-2\pi i \delta \left(\left(\frac{k}{2} + q\right)^2 - m^2 \right) \right] \theta \left(\frac{k^0}{2} + q^0\right) \\ \times \left[-2\pi i \delta \left(\left(\frac{k}{2} - q\right)^2 - m^2 \right) \right] \theta \left(\frac{k^0}{2} - q^0\right) . \quad (611)$$

By way of partial motivation for the rule, we note that \mathcal{I} is real unless one of the virtual propagator denominators vanishes, so that the $i\epsilon$ prescription for treating the poles becomes relevant. This is equivalent to saying that the propagating particles can go on-shell as part of the q integration (which

¹⁰ The explicit i in front of \mathcal{I} below is there because we want the imaginary part of \mathcal{M} in the unitarity theorem, whereas our Feynman rules apply to $i\mathcal{M}$.

we note can be performed by Wick rotating the dq^0 , thereby generating an i to cancel the explicit i in the definition of \mathcal{I}). The θ functions are simply the usual positive energy requirement for an on-shell particle.

Let us begin by simply figuring out what the result is from applying the Cutkosky rule. For our discussion, it is most convenient to employ the center of mass frame where $k = (k^0, 0, 0, 0)$. The product of the two δ functions then takes the form:

$$\delta\left((\frac{1}{2}k+q)^{2}-m^{2}+i\epsilon\right)\delta\left((\frac{1}{2}k-q)^{2}-m^{2}+i\epsilon\right) \\ \rightarrow \quad \delta\left(\left[(q^{0}+\frac{1}{2}k^{0})-(E_{\vec{q}}-i\epsilon)\right]\left[(q^{0}+\frac{1}{2}k^{0})+(E_{\vec{q}}-i\epsilon)\right]\right) \\ \times \delta\left(\left[(q^{0}-\frac{1}{2}k^{0})-(E_{\vec{q}}-i\epsilon)\right]\left[(q^{0}-\frac{1}{2}k^{0})+(E_{\vec{q}}-i\epsilon)\right]\right) \\ \equiv \delta\left([1][2]\right)\delta\left([3][4]\right)$$
(612)

Let us use $\delta([1])$ (i.e. $q^0 = -\frac{1}{2}k^0 + E_{\vec{q}}$) to do the q^0 integration (we cannot use $\delta([2])$ since it does not obey the θ function requirement of $q^0 + \frac{1}{2}k^0 > 0$), from which we obtain (assuming the Cutkosky rule formula)

$$2 \text{Im} \mathcal{I} = -(-2\pi i)^{2} \int \frac{d^{3}q}{(2\pi)^{4}} \frac{1}{2E_{\vec{q}}} \delta([-k^{0}][-k^{0}+2E_{\vec{q}}])\theta(k^{0}-E_{\vec{q}})$$

$$= -(-2\pi i)^{2} \int \frac{4\pi dE_{\vec{q}}E_{\vec{q}}|\vec{q}|}{(2\pi)^{4}} \frac{1}{2E_{\vec{q}}} \frac{1}{k^{0}} \delta([-k^{0}+2E_{\vec{q}}])$$

$$= 4\pi^{2} \frac{4\pi E_{\vec{q}}\vec{q}}{(2\pi)^{4}} \frac{1}{2E_{\vec{q}}} \frac{1}{2E_{\vec{q}}} \frac{1}{2}$$

$$= +\frac{1}{8\pi} \frac{|\vec{q}|}{E_{\vec{q}}}$$
(613)

where in going from the first to the second line we used the fact that the residual $\theta(k^0 - E_{\vec{q}})$ implies that $k^0 > 0$ and so we cannot employ the $\delta([-k^0])$ part of the δ function of the 1st line. The final $\frac{1}{2}$ in the 3rd line is from the Jacobian rule for the δ function.

To prove the Cutkosky rule formula, we must directly obtain $\text{Im}\mathcal{I}$. First, we note that \mathcal{I} above defines an analytic function of $s = k^2$. We next note that the appearance of an imaginary part of $\mathcal{I}(s) \propto \mathcal{M}(s)$ always requires a branch cut singularity. By the unitarity argument, \mathcal{M} only acquires an imaginary part for $s > s_0$, where s_0 is the threshold for production of the

lightest multiparticle state, here $s_0 = 4m^2$. For real $s < s_0$ the intermediate state (here consisting of the two particles with mass m) cannot go on-shell, and so $\mathcal{I}(s)$ is real and

$$\mathcal{I}(s) = [\mathcal{I}(s^*)]^* \,. \tag{614}$$

In fact, each side of the above equation is an analytic function (a standard fact of complex analysis that you should know) and so this equation can be continued to the entire complex s plane. Once $s > s_0$, \mathcal{I} will develop an imaginary part (which we shall explicitly compute) and from Eq. (614) we have

$$\operatorname{Re}\mathcal{I}(s+i\epsilon) = \operatorname{Re}\mathcal{I}(s-i\epsilon), \quad \operatorname{Im}\mathcal{I}(s+i\epsilon) = -\operatorname{Im}\mathcal{I}(s-i\epsilon).$$
 (615)

The - sign in the 2nd relation implies the presence of a branch cut across the real axis for $s \ge s_0$. The discontinuity across this branch cut is obviously

$$Disc \mathcal{I}(s) = 2i Im \mathcal{I}(s + i\epsilon).$$
(616)

In fact, it is easier to compute this discontinuity than it is to compute the imaginary part directly. The $i\epsilon$ prescription in the Feynman propagator indicates that physical scattering amplitudes should be evaluated above the cut, at $s + i\epsilon$.

To evaluate $i\text{Disc}\mathcal{I}$ we again go to the center of mass frame where $k = (k^0, 0, 0, 0)$. The product of the two denominators then takes the form:

$$[(\frac{1}{2}k+q)^{2}-m^{2}+i\epsilon][(\frac{1}{2}k-q)^{2}-m^{2}+i\epsilon] \rightarrow [(q^{0}+\frac{1}{2}k^{0})-(E_{\vec{q}}-i\epsilon)][(q^{0}+\frac{1}{2}k^{0})+(E_{\vec{q}}-i\epsilon)] \times [(q^{0}-\frac{1}{2}k^{0})-(E_{\vec{q}}-i\epsilon)][(q^{0}-\frac{1}{2}k^{0})+(E_{\vec{q}}-i\epsilon)] \equiv [1][2][3][4]$$

$$(617)$$

From this we see that the integrand in the expression for $i\mathcal{I}$ has four poles in the integration variable q^0 located at

$$q^{0} = \frac{1}{2}k^{0} \pm (E_{\vec{q}} - i\epsilon) \text{ from [3] and [4]}$$

$$q^{0} = -\frac{1}{2}k^{0} \pm (E_{\vec{q}} - i\epsilon) \text{ from [1] and [2]}.$$
(618)

Two of these poles lie above the real axis([2] and [4]) at $-\frac{1}{2}k^0 - E_{\vec{q}}$ and $\frac{1}{2}k^0 - E_{\vec{q}}$ and two lie below ([1] and [3]) at $-\frac{1}{2}k^0 + E_{\vec{q}}$ and $\frac{1}{2}k^0 + E_{\vec{q}}$.

We can perform the dq^0 integration by closing the contour either in the upward direction or the downward direction. We choose to close downward and pick up the [1] and [3] poles in the lower half plane. (These correspond to the actual on-shell positive energy poles, but we would get the same answer by closing up.) At the locations of these two poles, the product of the rest of the brackets in Eq. (617)) reduces to

$$[2][3][4] \to [2E_{\vec{q}}][-k^0][-k^0 + 2E_{\vec{q}}] \text{ at pole of [1]} [1][2][4] \to [k^0][k^0 + 2E_{\vec{q}}][2E_{\vec{q}}] \text{ at pole of [3]}$$
(619)

The residues at these two pole locations are, respectively, then the inverse

of these expressions:

$$[1]: \quad -\frac{1}{2}k^{0} + E_{\vec{q}} \Rightarrow \frac{1}{-k^{0}} \frac{1}{-k^{0} + 2E_{\vec{q}}} \frac{1}{2E_{\vec{q}}}$$
(620)
$$[3]: \quad \frac{1}{2}k^{0} + E_{\vec{q}} \Rightarrow \frac{1}{k^{0}} \frac{1}{2E_{\vec{q}}} \frac{1}{k^{0} + 2E_{\vec{q}}}.$$
(621)

Only the pole residue of the [1] pole at $q^0 = -\frac{1}{2}k^0 + E_{\vec{q}}$ can contribute to the discontinuity since the residue at the other [3] pole has positive definite stuff in the denominators (recall that k^0 is the *positive* energy of the incoming photon). We are left with (since the contour is clockwise, there is an extra minus sign in the residue theorem)

$$i\mathcal{I} = -2\pi i \int rac{d^3 ec q}{(2\pi)^4} rac{1}{2E_{ec q}} rac{1}{k^0(k^0 - 2E_{ec q})}$$
 (622)

This result from picking up the residue of this pole is equivalent to replacing

$$rac{1}{(rac{1}{2}k+q)^2-m^2+i\epsilon} ~
ightarrow -2\pi i\delta\left(\left(rac{k}{2}+q
ight)^2-m^2
ight) heta(rac{1}{2}k^0+q^0)$$

$$= -2\pi i \delta \left((q^{0} + \frac{1}{2}k^{0} - E_{\vec{q}})(q^{0} + \frac{1}{2}k^{0} + E_{\vec{q}}) \right) \theta(\dots)$$

$$\sim -2\pi i \frac{1}{2E_{\vec{q}}} \delta \left(q^{0} + \frac{1}{2}k^{0} - E_{\vec{q}} \right), \qquad (623)$$

using the standard delta function Jacobian factor. The denominators containing k^0 in Eq. (620) come from the other propagator. In any case, we are left with

$$i\mathcal{I} = -2\pi i \int \frac{d^{3}\vec{q}}{(2\pi)^{4}} \frac{1}{2E_{\vec{q}}} \frac{1}{k^{0}(k^{0}-2E_{\vec{q}})} = -2\pi i \frac{4\pi}{(2\pi)^{4}} \int_{m}^{\infty} dE_{\vec{q}} E_{\vec{q}} |\vec{q}| \frac{1}{2E_{\vec{q}}} \frac{1}{k^{0}(k^{0}-2E_{\vec{q}})}$$
(624)

The integral has a pole at $E_{\vec{q}} = \frac{1}{2}k^0$, which means that if $k^0 < 2m$ the pole is not on the integration contour (since $E_{\vec{q}} \ge m$) and \mathcal{I} is manifestly real. When $k^0 > 2m$, the pole lies just above or below the contour of integration, depending upon whether k^0 has a small positive or small negative imaginary part. Thus, the integral will acquire a discontinuity which is revealed by

$$\frac{1}{k^0 - 2E_{\vec{q}} \pm i\epsilon} = P \frac{1}{k^0 - 2E_{\vec{q}}} \mp i\pi\delta(k^0 - 2E_{\vec{q}})$$
(625)

yielding discontinuity of $-2\pi i \delta(k^0 - 2E_{\vec{q}})$. Substituting into Eq. (624) we find for $i\text{Disc}\mathcal{I} = i(2i\text{Im}\mathcal{I})$ (see Eq. (616)) the *real* result

$$i(2i \text{Im}\mathcal{I}) = (-2\pi i)^{2} \frac{4\pi}{(2\pi)^{4}} \int_{m}^{\infty} dE_{\vec{q}} E_{\vec{q}} |\vec{q}| \frac{1}{2E_{\vec{q}}} \frac{1}{k^{0}} \delta(k^{0} - 2E_{\vec{q}})$$

$$= (-2\pi i)^{2} \frac{4\pi}{(2\pi)^{4}} E_{\vec{q}} |\vec{q}| \frac{1}{2E_{\vec{q}}} \frac{1}{2E_{\vec{q}}} \frac{1}{2}$$

$$= -\frac{1}{8\pi} \frac{|\vec{q}|}{E_{\vec{q}}}, \qquad (626)$$

or $2\mathrm{Im}\mathcal{I} = +\frac{1}{8\pi} \frac{|\vec{q}|}{E_{\vec{q}}}$ (627)

which is the same as the Cutkosky rule result of Eq. (613).

The result we have found is the first example of the Cutkosky cutting rules for obtaining the discontinuity of a Feynman diagram across its branch cut.

The general result is that the discontinuity of any Feynman diagram contribution to $i\mathcal{M}$ (which gives $-2\mathrm{Im}\mathcal{M}$)) is given by the following rules:

- 1. Cut through the diagram in all possible ways consistent with all the cut propagators simultaneously vanishing (i.e. all propagators on-shell).
- 2. For each cut propagator replace $\frac{i}{q^2-m^2+i\epsilon} \rightarrow 2\pi\delta(q^2-m^2)\theta(q^0)$ (note how *i* in numerator of propagator cancels -i in $-2\pi i$ so that phase is not changing as you add to number of cut propagators).
- 3. Sum the contributions from all such cuts.
- 4. Perform integrations over the internal " q^0 " 's using the above δ functions leaving you with just the on-shell phase-space part: $\int d^4q \delta(q^2 - m^2) \theta(q^0) \rightarrow \int \frac{d^3\vec{q}}{2E_{\vec{r}}}.$

Using these cutting rules, the optical theorem can be proven to arbitrary order in perturbation theory.

General proof of the Cutkosky prescription

The general Cutkosky prescription states that to obtain the discontinuity of the matrix element associated with cutting a particular set of propagators, which cut divides the diagram into two disconnected pieces, we replace $\frac{1}{q^2-m^2} \rightarrow -2\pi i \delta(q^2 - m^2)$ for each cut propagator. To obtain the full discontinuity, one sums over the results from all such cuts. Of course, only those cuts for which the energy or *s* being considered is sufficient to allow all cut particles to be on-shell will contribute to the discontinuity.

A full proof of the Cutkosky prescription requires a fairly sophisticated approach.

From the point of view of coordinate space, the proof can be found in "Diagrammar" by 't Hooft and Veltman.

From the point of view of the momentum space representation of Feynman diagrams an excellent reference is Eden, Landshoff, Olive and Polkinghorne "The Analytic S-Matrix".

I will try to sketch the latter, leaving out a lot of details, but giving enough so that you get a first glimpse of this whole analytic structure approach that was so important in the pre-1970 period and that is now coming back into use as a means of computing loop integrals by getting their imaginary parts and then integrating in the context of dispersion relations, etc.

We begin with some simple remarks based on complex analysis. Let us

consider the integral

$$f(s) = \int_{a}^{b} ds' \frac{1}{s' - s} = \log\left(\frac{s - b}{s - a}\right) \,. \tag{628}$$

The function f(s) develops a singularity at s = a and s = b since the integration contour cannot be deformed at these end points so as to avoid the points s' = a and s' = b. This kind of singularity is thus called an "end-point" singularity.

For *s* real and in between *a* and *b*, we can analytically continue the function by deforming the contour away from the real axis. Thus, there is no actual singularity in this region, but there is a branch-cut connecting the points *a* and *b* and there will be a discontinuity across this branch cut. Related to the presence of this branch cut, one can think of the above integral as a spectral representation of the function f(s) of the type

$$f(s) = \frac{1}{2\pi i} \int_{a}^{b} ds' \operatorname{Disc}[f(s')] \frac{1}{s' - s}$$
(629)

where by direct computation the discontinuity of f(s) defined by f(s + s)

 $i\epsilon$) – $f(s - i\epsilon) \equiv \text{Disc} f(s)$ is simply $2\pi i$ along the $s \in [a, b]$ interval. This, we can verify in two ways.

1.

$$f(s+i\epsilon) - f(s-i\epsilon) = \int_{a}^{b} ds' \left(\frac{1}{s'-s-i\epsilon} - \frac{1}{s'-s+i\epsilon} \right)$$
$$= \int_{a}^{b} ds' 2\pi i \delta(s'-s)$$
$$= 2\pi i \quad \text{for } s \in [a,b]$$
(630)

and 0 elsewhere.

2. Alternatively, let us write $s = a + re^{i\theta}$, where r < b - a and $\theta = +\epsilon'$ defines a location corresponding to $s = s + i\epsilon = a + r + i\epsilon$ and $\theta = 2\pi - \epsilon'$ defines a location corresonding to $s - i\epsilon = a + r - i\epsilon$. By direct computation, we see that (the $\log(s - b)$ is the same at these two locations)

$$\log\left(\frac{1}{s+i\epsilon-a}\right) - \log\left(\frac{1}{s-i\epsilon-a}\right) = -\log(re^{i(\epsilon')}) + \log(re^{i(2\pi-\epsilon')})$$
$$\stackrel{\epsilon' \to 0}{=} 2\pi i \qquad (631)$$

as consistent with the claimed discontinuity.

There are also "pinch" singularities that arise when two (or more) singularities approach the contour of integration defining some function from opposite sides of the contour *and* coincide. In such a case, there is no way to deform the contour so as to avoid these singularities.

An example of this type is provided by (think of a as the m^2 of some Feynman propagator and the $i\epsilon$ is the Feynman prescription – meanwhile $s + i\epsilon$ is always the physical location at which we evaluate some amplitude)

$$f(s) = \int_{0}^{1} \frac{ds'}{(s'-s-i\epsilon)(s'-a+i\epsilon)}$$

= $\frac{1}{s-a} \left[\log \frac{1-s-i\epsilon}{-s-i\epsilon} - \log \frac{1-a+i\epsilon}{-a+i\epsilon} \right]$ (632)

For a > 1 the $i\epsilon$'s don't matter and we simply get

$$f(s) = \log\left(\frac{a(1-s)}{(1-a)s}\right) \quad (a > 1),$$
 (633)

which does have logarithmic singularities at s = 1, 0 of the end-point type, but there is no singularity at s = a when a > 1 since the log vanishes there. This corresponds to the fact that the contour is not pinched when a > 1 since the integral only runs over s' between 0 and 1. However, if a < 1 one must be careful with the $i\epsilon$'s and the fact that the log function has a cut when its argument is negative. The $i\epsilon$'s are not important for the $\log(1 - s - i\epsilon)$ and $\log(1 - a + i\epsilon)$ and these cancel as $s \rightarrow a$. But for the other two logs we must write

$$\log(-a+i\epsilon) = \log(|a|e^{i\pi}) = \log|a| + i\pi,$$

$$\log(-s-i\epsilon) \rightarrow \log(|a|e^{-i\pi}) = \log|a| - i\pi$$
(634)

implying

$$f(s) \stackrel{s \to a}{\to} \frac{1}{s-a} (-2\pi i) \quad 0 < a < 1.$$
(635)

What is happening is that if you try to distort the contour so as, let us say, to avoid the $s' = s + i\epsilon$, you must push the contour past the $s' = a - i\epsilon$ singularity which leaves behind a (clockwise) circular contour that circles the pole at $s' = a - i\epsilon$ and gives you $-1 \times$ the residue at that pole. But this is only necessary if $s \sim a \in [0, 1]$.

With this warm up we can now turn to a function of the type we are interested in:

$$\mathcal{I}(s) = \int \frac{d^4 k_1 d^4 k_2 \dots d^4 k_l}{\prod_{i=1}^N (q_i^2 - m_i^2)}$$
(636)

where of the N propagators we are going to imagine that we can cut through (with a simple line) the Feynman diagram by cutting through the first r of these propagators. The 4l-dimensional k_l integrations can be transformed to new integration variables $q_1^2, q_2^2, \ldots, q_r^2$ and 4l - r other variables ξ_s . We thus write

$$\mathcal{I}(s) = \int_{a_1}^{b_1} dq_1^2 \int_{a_2}^{b_2} dq_2^2 \dots \int_{a_r}^{b_r} dq_r^2 \int \prod d\xi_s \frac{1}{J \prod_{1}^{r} (q_i^2 - m_i^2) \prod_{r+1}^{N} (q_k^2 - m_k^2)} \frac{1}{(637)}$$
where J is the Jacobian of the variable transform. The limits (a_j, b_j) for
the q_j^2 integration are the extrema of q_j^2 with respect to all values of the k_l

when q_i^2 for i < j are held fixed. Suppose we now write

$$\mathcal{I}(s) = \int_{a_1}^{b_1} dq_1^2 \frac{\mathcal{I}_1(q_1^2, s)}{q_1^2 - m_1^2},$$
(638)

where m_1^2 is really $m_1^2 - i\epsilon$ by the Feynman prescription.

Now, just as in the simple example we just did, $\mathcal{I}(s)$ will have a branch cut singularity structure and we are wanting to know the discontinuity. To do so we must understand its singularities.

$$\begin{array}{c} & & & & & \\ & & & & \\ a_1 & & & & \\ & & & m_1^2 \end{array} \xrightarrow{b_1} & & a_1 & & \\ & & & & & \\ & & & & & \\ \end{array}$$

Figure 17: Example of contour pinching/trapping for $\int dq_1^2$.

First note that \mathcal{I} depends on the value of m_1^2 but neither $\mathcal{I}_1(q_1^2, s)$ nor a_1, b_1 explicitly contains m_1^2 . A singularity of \mathcal{I} at $s = s_0$ will develop as a result of the q_1^2 contour integration being trapped between the fixed m_1^2 singularity and a singularity \tilde{q}_1^2 of \mathcal{I}_1 that approaches m_1^2 when $s \to s_0$. Let us take this trapping explicitly into account by passing the contour in q_1^2 which runs along the real axis past $m_1^2 - i\epsilon$ by moving the contour down and keeping the extra clockwise (yielding the - sign below) circle around $m_1^2 - i\epsilon$ needed to compensate. The result of this explicit circle is simply to pick up $-2\pi i$ times the residue

$$-2\pi i \mathcal{I}_1(m_1^2,s)$$
. (639)

The contribution from the shifted down contour will not have any singularity.

However, \mathcal{I}_1 may.

But, now we can write

$$\mathcal{I}_1(m_1^2, s) = \int_{a_2}^{b_2} dq_2^2 \frac{\mathcal{I}_2(q_2^2, s)}{q_2^2 - m_2^2} \tag{640}$$

and repeat the above arguments, picking up all the poles until we arrive at

$$\int_{a_r}^{b_r} dq_r^2 \frac{\mathcal{I}_r(m_1^2, \dots, m_{r-1}^2, q_r^2, s)}{q_r^2 - m_r^2} \,. \tag{641}$$

We still do not know for sure whether this expression actually has a singularity in s — so far we have managed to shift integration contours, but picking of the residual written above that might or might not have a singularity at s^0 . For this final residual to have a singularity, it has to be that as $s \rightarrow s_0$ one of the end points of integration (a_r and b_r depend on s) moves toward $q_r^2 = m_r^2$, let us say it is $a_r(s)$ that does so. So, picture the interval $[a_r, b_r]$ and we are integrating along this interval and $m_r^2 - i\epsilon$ lies a bit below the interval near a_r .



Figure 18: The discontinuity associated with the end-point singularity coming from $\int dq_r^2$.

Write

$$a_r(s) = a_r(s_0) + \delta \frac{da_r}{ds}(s=s_0) = a_r(s_0) + |\delta|e^{i\theta} \frac{da_r}{ds}(s=s_0). \quad (642)$$

Now rotate s (assumed $> s_0$) from $s + i\epsilon$ to $s - i\epsilon$ by writing $s - s_0 = |\delta|e^{i\theta}$ with θ going from 0 to $+2\pi$. Then, $a_r(s)$ will describe a circle that wraps in a counterclockwise direction (for $da_r/ds > 0$ — the final result is the same if $da_r/ds < 0$ put the picture is different) around m_r^2 as θ goes from 0 to 2π . The result for the $s + i\epsilon$ result minus the $s - i\epsilon$ result is $-2\pi i$ times the residue of the pole at $q_r^2 = m_r^2$. Plugging this into the interatively developed expression for $\mathcal{I}(s)$, we have

$$\operatorname{Disc}\mathcal{I}(s) = (-2\pi i)^r \int \prod d^4k_l \frac{\delta(q_1^2 - m_1^2) \dots \delta(q_r^2 - m_r^2)}{(q_{r+1}^2 - m_{r+1}^2) \dots (q_N^2 - m_N^2)}.$$
 (643)

You will recognize this as being the generalization of what we found in the two propagator case. If this integral is non-zero (as it will be if s is large enough to put all the q_r^2 's on-shell) then we have a discontinuity of $\mathcal{I}(s)$ and a corresponding contribution to the total cross section.

Of course, I have glossed over lots of subtleties in the above. A fully correct discussion requires lots more effort and time. You will need to look at *Diagrammar* or *The Analytic S Matrix* for a more thorough treatment.

Relating to Π

We now return to the problem at hand by first relating $\Pi_{\mu\nu}$ to $\Pi(k^2)$ using

$$i\Pi_{\mu\nu}(k) = i\Pi(k^2)[k^2g_{\mu\nu} - k_{\mu}k_{\nu}].$$
(644)

To repeat, the generic one loop insertion correction to $e^+e^- \rightarrow \gamma^* \rightarrow \log \rho \rightarrow \gamma^* \rightarrow e^+e^-$ takes the form (again using p for the e^+ and p' for the

 e^- and neglecting their masses):

$$i\mathcal{M} = \overline{u}(p')(-ie\gamma_{\mu})v(p)\frac{-ig_{\mu\mu'}}{k^{2}}i\Pi(k^{2})[k^{2}g_{\mu'\nu'} - k_{\mu'}k_{\nu'}]\frac{-ig_{\nu'\nu}}{k^{2}}\overline{v}(p)(-ie\gamma_{\nu})u(p')$$

$$= e^{2}\overline{u}(p')\gamma_{\mu}v(p)\frac{1}{k^{2}}i\Pi(k^{2})[k^{2}g_{\mu\nu} - k_{\mu}k_{\nu}]\frac{1}{k^{2}}\overline{v}(p)\gamma_{\nu}u(p').$$
(645)

Note that even though we used the Feynman style propagator for the virtual photons, any other form of the propagator (e.g. axial gauge) would contain terms proportional to k^{α} and/or k'^{α} with $\alpha = \mu, \mu', \nu$ or ν' , any of which would vanish either because they multiply the $k^2 g_{\mu\nu} - k_{\mu} k_{\nu}$ structure or are sandwiched between spinors in such a way that you get, for example, $\overline{u}(p')(-ie \not k)v(p) = -ie\overline{u}(p')(\not p + \not p')v(p)$ which is zero after using the Dirac equations for the spinors.

We will now average over the e^- and e^+ spins, which we take to be the same in the initial and final state in order to connect to the $e^+e^$ annihilation cross section. We convert the spin average into a trace in the usual way to obtain:

$$egin{aligned} &i\mathcal{M} &= rac{1}{4}e^2 ext{Tr}[\, p' \gamma_\mu \, p \gamma_
u] rac{1}{k^2} i\Pi(k^2) [k^2 g_{\mu
u} - k_\mu k_
u] rac{1}{k^2} \ &= e^2 [p'_\mu p_
u + p'_
u p_\mu - g_{\mu
u} p \cdot p'] rac{1}{k^2} i\Pi(k^2) [k^2 g_{\mu
u} - k_\mu k_
u] rac{1}{k^2} \ &= e^2 [2k^2 p \cdot p' - 2p \cdot k p' \cdot k - 4k^2 p \cdot p' + k^2 p \cdot p'] rac{1}{k^2} i\Pi(k^2) rac{1}{k^2} \end{aligned}$$

$$= -k^{4}e^{2}\frac{1}{k^{2}}i\Pi(k^{2})\frac{1}{k^{2}} \text{ using } p \cdot p' = \frac{k^{2}}{2} \text{ and } p \cdot k = p' \cdot k = \frac{k^{2}}{2}$$

= $-ie^{2}\Pi(k^{2}).$ (646)

So, after averaging over e^+e^- spins,

$$s\sigma(e^+e^- \to f\overline{f}) = \mathrm{Im}\mathcal{M} = -e^2\mathrm{Im}\Pi(s).$$
 (647)

where $\Pi(s)$ is to be computed for the fermionic loop for fermion type f.

Getting the imaginary part

So, the first order of business is to understand how to get the required imaginary part in a manner that will be easily generalized to some arbitrary number of loops. This is best done using the RGE equations.

We ignore QCD for the moment, since we know that only enters Π at the two-loop level as a correction to the basic fermion=quark loops that contribute to the photon propagator Π . We use the fact that Π will obey

an RGE equation:

$$\left[\frac{\partial}{\partial t} - \beta(e)\frac{\partial}{\partial e} + (1 - \gamma_m)m\frac{\partial}{\partial m} - d_N + N\gamma_d\right]\Pi(e^t q_0, m, e, \mu) = 0,$$
(648)

where $e^t = s$ was our momentum scaling factor and

$$m\gamma_m = \mu \frac{\partial m}{\partial \mu}, \quad \gamma_d = \frac{1}{2} \mu \frac{\partial}{\partial \mu} (\ln Z_3)$$
 (649)

where for the latter we keep in mind that the only external fields are the two photons and therefore only Z_3 renormalization enters. Also, since we have N = 2 external photon fields, we have an engineering dimension of $d_N = 4 - N = 2$. We will neglect the *m* dependence. The solution of the equation is then

$$\Pi(e^{t}q_{0}, e, \mu) = e^{td_{N}} \exp\left[-N \int_{0}^{t} \gamma_{d}(\overline{e}(t', e))dt'\right] \Pi(q_{0}, \overline{e}(t, e), \mu).$$
(650)
We will actually neglect the moving \overline{e} and focus only on γ_{d} . In QED we

found

$$Z_3 = 1 - \frac{e^2}{12\pi^2} \frac{1}{\epsilon}$$
 (651)

which implies that

$$\gamma_d Z_3 = \frac{1}{2} \mu \frac{\partial Z_3}{\partial \mu} = -\frac{1}{2} \mu \frac{\partial e}{\partial \mu} \frac{e}{6\pi^2} \frac{1}{\epsilon}.$$
 (652)

We then have to remember that

$$\mu \frac{\partial e}{\partial \mu} = d_0 + d_1 \epsilon \,, \tag{653}$$

with $d_0 = \frac{e^3}{12\pi^2}$ giving us the $\beta(e)$ function, and, the crucial thing for this calculation, $d_1 = -e$. So, we now write our defining equation for γ_d in the form:

$$\gamma_d \left(1 - \frac{e^2}{12\pi^2} \frac{1}{\epsilon} + \dots \right) = -\frac{1}{2} (d_0 + d_1 \epsilon) \frac{e}{6\pi^2} \frac{1}{\epsilon}.$$
 (654)

To determine the $\epsilon
ightarrow 0$ result for γ_d , we match powers of ϵ^0 to obtain

$$\gamma_d = -\frac{1}{2}d_1 \frac{e}{6\pi^2} = \frac{e^2}{12\pi^2}.$$
(655)

Alternatively, we can go back to the more naive style of derivation and write

$$e_0 = \mu^{\epsilon} \left[e + \frac{e^3}{24\pi^2} \frac{1}{\epsilon} \right] \,. \tag{656}$$

We then take $\mu \frac{\partial}{\partial \mu}$ on this to get

$$0 = \epsilon \mu^{\epsilon} \left[e + \frac{e^3}{24\pi^2} \frac{1}{\epsilon} \right] + \mu^{\epsilon} \left[\mu \frac{\partial e}{\partial \mu} \right] \left[1 + \frac{e^2}{8\pi^2} \frac{1}{\epsilon} \right] , \qquad (657)$$

which gives

$$\mu \frac{\partial e}{\partial \mu} = -\epsilon \left[e + \frac{e^3}{24\pi^2 \epsilon} \frac{1}{\epsilon} \right] \left[1 + \frac{e^2}{8\pi^2 \epsilon} \frac{1}{\epsilon} \right]^{-1} = -\epsilon e \left[1 - \frac{e^2}{12\pi^2 \epsilon} \frac{1}{\epsilon} + \dots \right]$$
$$= -\epsilon e + \frac{e^3}{12\pi^2}.$$
(658)

Substituting into the γ_d defining equation, Eq. (652), we have

$$\gamma_d \left(1 - \frac{e^2}{12\pi^2} \frac{1}{\epsilon} + \ldots \right) = -\frac{1}{2} \left[-\epsilon e + \frac{e^3}{12\pi^2} \right] \frac{e}{6\pi^2} \frac{1}{\epsilon}.$$
 (659)
At order ϵ^0 , this gives

$$\gamma_d = \frac{e^2}{12\pi^2}.$$
 (660)

which also leads to consistency at order $\frac{1}{\epsilon}$.

The above calculation was for QED. For the SM, we actually have to sum over all fermion loop contributions to γ_d . In this way, we get [defining $C = 1/(12\pi^2)$]

$$\gamma_d = Ce^2 \sum_{k=\text{all relevant fermions}} e_k^2 \tag{661}$$

where e_k is the fractional charge, relative to e, of a given fermion. By "all relevant fermions", what I mean to imply is all the fermions that are contributing to the particular total cross section of interest. If I am interested in the total cross section for $e^+e^- \rightarrow q\overline{q}$, then I will be summing over all quarks. We can now plug this into our RGE solution form (with

N = 2 and $d_N = 4 - N = 2$):

$$\Pi(e^t q_0, e, \mu) = e^{2t} \exp\left[-2 \int_0^t C \sum_k e_k^2 \overline{e}^2(t', e) dt'\right] \Pi(q_0, \overline{e}(t, e), \mu).$$
(662)

For our purposes, we can neglect the movement of \overline{e} (which would give a small correction to this and later results). Let us also take $\mu = q_0$. Then

$$\Pi(q_0, \overline{e}(t, \mu), \mu) = -q_0^2 D, \qquad (663)$$

where q_0^2 will be negative and D is just some numerical multiplier. The factor of $-q_0^2$ is required by the dimensionality of Π and the fact that only q_0 is available to carry the dimensions (neglecting the electron mass). One can explicitly compute Π from the one loop diagram (see Peskin), but it is not necessary for our purposes. Inserting Eq. (663) into Eq. (662), we obtain

$$\Pi(q = e^t q_0, e, \mu) = -q^2 D \exp[-2te^2 C \sum_k e_k^2], \qquad (664)$$

with $q^2 < 0$. Expanding to order e^2 we get (using $e^{2t}q_0^2 = q^2$ in the form

 $t=rac{1}{2}\ln q^2/q_0^2$)

$$\Pi(q, e, \mu) = -q^2 D \left[1 - 2te^2 C \sum e_k^2 + \dots \right]$$

= $-q^2 D \left[1 - e^2 C \sum e_k^2 \ln(q^2/q_0^2) + \dots \right].$ (665)

Now, I did not stress yet that all these computations are absolutely valid in the deep Euclidean region in which $q^2 < 0$ and large, and we presume we started the evolution at a $q_0^2 < 0$ point.

So, where is the imaginary part that we want to identify with the cross section?

We must continue (with care!) to the physical region where $q^2 > 0$. We write:

$$q^2 = Q^2 e^{i\theta}$$
, and $q_0^2 = Q_0^2 e^{i\theta_0}$, (666)

where $Q^2, Q_0^2 > 0$, and $\theta = \theta_0 = \pi$ to begin with. We now move to $q^2 > 0$ and just above the physical cut along the positive q^2 axis by taking $\theta \to \epsilon$, leaving $\theta_0 = \pi$. Then

$$\ln\frac{q^2}{q_0^2} \to \ln\left(\frac{Q^2}{Q_0^2 e^{i\pi}}\right) = \ln\frac{Q^2}{Q_0^2} - \ln e^{i\pi}, \Rightarrow \operatorname{Im}\ln\frac{q^2}{q_0^2} = -\pi.$$
(667)

Eq. (665) then gives

$$\mathrm{Im}\Pi = -q^2 D[-e^2 C \sum_k e_k^2(-\pi)] = -q^2 C D e^2 \sum_k e_k^2 \pi \,. \tag{668}$$

Substituting into Eq. (647) then gives

$$s\sigma(e^+e^- \to q\overline{q}) = -e^2 \text{Im}\Pi = Q^2 D e^4 C \sum e_k^2 \pi$$
, (669)

where we used the fact that $q^2 = Q^2 > 0$ after this continuation in the $-q^2$ premultiplier appearing in Eq. (665).

Taking the ratio of the $q\overline{q}$ to the $\mu^+\mu^-$ imaginary parts gives us the ratio of the cross sections and we obtain the usual result:

$$R = \frac{\sigma(e^+e^- \to q\overline{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_{quarks} e_k^2}{e_{k=\mu}^2} = \sum_{quarks} e_k^2.$$
(670)

You may wonder why I bothered with all this formalism just to get the result you know from simple calculation. The reason will become clear when we go to the next order in QCD.

In any case, let us review how well this tree-level result does in comparison to the data.

The following plots show that the tree-level result gets the gross features of R as a function of \sqrt{s} correct, but that there is clearly something missing in that the data is clearly above the tree-level result at lower \sqrt{s} values where $\alpha_s(s=Q^2)$ is largest.



Figure 40.6: World data on the total cross section of $e^+e^- \rightarrow hadrons$ and the ratio $R(s) = \sigma(e^+e^- \rightarrow hadrons, s)/\sigma(e^+e^- \rightarrow \mu^+\mu^-, s)$. $\sigma(e^+e^- \rightarrow hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \rightarrow \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.12) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ , $\psi(2S)$, and $\Upsilon(nS)$, n = 1, 2, 3, 4 are also shown. The full list of references to the original data and the details of the *R* ratio extraction from them can be found in [arXiv:hep-ph/0312114]. Corresponding computer-readable data files are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, August 2007. Corrections by P. Janot (CERN) and M. Schmitt (Northwestern U.)) See full-color version on color pages at end of book.

Figure 19: The ratio R as a function of \sqrt{s} . The green dashed line is the tree-level result. The red line is the 3-loop QCD corrected result.



Figure 40.7: R in the light-flavor, charm, and beauty threshold regions. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are the same as in Fig. 40.6. Note: CLEO data above $\Upsilon(4S)$ were not fully corrected for radiative effects, and we retain them on the plot only for illustrative purposes with a normalization factor of 0.8. The full list of references to the original data and the details of the R ratio extraction from them can be found in [arXiv:hep-ph/0312114]. The computer-readable data are available at http://pdg.lbl.gov/current/xsect/. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, August 2007) See full-color version on color pages at end of book.

Figure 20: The ratio R at low \sqrt{s} : note threshold jumps. The green dashed line is the tree-level result. The red line is the 3-loop QCD corrected result.

The one-loop QCD correction

We are finally in a position to use this technology to compute the QCD correction which comes from the two-loop diagrams for the photon propagator in which there is a fermion loop and a gluon exchange somewhere. We wish to sum four graphs contributing to Π :

- 1. the quark loop graph without QCD correction;
- 2. the quark loop graph with *g* exchange between upper and lower quark lines;
- 3. the quark loop graph with upper quark line corrected by g emission and reabsorption.
- 4. the quark loop graph with lower quark line corrected by g emission and reabsorption.

The result found (see later for a summary of the calculation) is

$$Z_3 = 1 + \mathcal{O}(e^2) + \mathcal{O}(e^2 g^2), \qquad (671)$$

where the $\mathcal{O}(e^2)$ is that we have already discussed. After carrying out the

calculation and then using Z_3 to determine γ_d for the gluon, we find

$$\gamma_d = Ce^2 \sum e_k^2 \left[1 + \frac{3}{16\pi^2} C_F g^2 + \dots \right] \,. \tag{672}$$

Our renormalization group result for Π is then (neglecting the very small effect from the movement of \overline{e} with t')

$$\Pi(q^{2}) = -q^{2}D \exp\left[-2\int_{0}^{t} \gamma_{d}(\overline{e}(t',e),\overline{g}(t',g)dt')\right]$$

$$= -q^{2}D\left\{1 - 2Ce^{2}\sum_{k}e_{k}^{2}\int_{0}^{t}dt'\left[1 + \frac{3C_{F}}{16\pi^{2}}\overline{g}^{2}(t',g) + \ldots\right](\phi^{7})\right\}$$

Into this we insert (using short-hand $b \equiv 11 - (2/3)n_f$)

$$\overline{g}^{2}(t') = \frac{g^{2}(0)}{1 + \frac{b}{8\pi^{2}}g^{2}(0)t'}$$
(674)

for which

$$\int_{0}^{t} dt' \overline{g}^{2}(t') \;\; = \;\; rac{8\pi^{2}}{b} {
m ln} \left[1 + rac{b}{8\pi^{2}} g_{0}^{2} t'
ight]_{0}^{t}$$

$$= \frac{8\pi^2}{b} \ln \left[1 + \frac{b}{8\pi^2} g_0^2 t \right] \,. \tag{675}$$

Thus, we have

$$\Pi(q^2) = -q^2 D\left(1 - 2Ce^2 \sum e_k^2 \left[t + \frac{3C_F}{16\pi^2} \frac{8\pi^2}{b} \ln\left(1 + \frac{b}{16\pi^2} g_0^2 2t\right)\right]\right)$$
(676)

where after continuation to positive $q^2 = Q^2$ we have $2t = \ln rac{Q^2}{Q_0^2} - i\pi$. We need

$$\ln\left(1 + \frac{b}{16\pi^2}g_0^2\left[\ln\frac{Q^2}{Q_0^2} - i\pi\right]\right)$$

$$= \ln\left(1 + \frac{b}{16\pi^2}g_0^2\ln\frac{Q^2}{Q_0^2}\right) + \ln\left(1 - \frac{\frac{b}{16\pi^2}g_0^2i\pi}{1 + \frac{b}{16\pi^2}g_0^2\ln\frac{Q^2}{Q_0^2}}\right)$$

$$\sim \ln\left(1 + \frac{b}{16\pi^2}g_0^2\ln\frac{Q^2}{Q_0^2}\right) - \frac{\frac{b}{16\pi^2}g_0^2i\pi}{1 + \frac{b}{16\pi^2}g_0^2\ln\frac{Q^2}{Q_0^2}}$$
(677)

with imaginary part given by

$$\operatorname{Im}\ln\left(\ldots\right) = -\pi \frac{\frac{b}{16\pi^2} g_0^2}{1 + \frac{b}{16\pi^2} g_0^2 \ln \frac{Q^2}{Q_0^2}} = -\pi \frac{b}{16\pi^2} \overline{g}^2(Q^2) = -\pi \frac{b}{4\pi} \alpha_s(Q^2) \,.$$
(678)

Our net result for ${\rm Im}\Pi$ is then

$$Im\Pi = Q^{2}DCe^{2}\sum e_{k}^{2}\left(-\pi + 2\frac{3C_{F}}{16\pi^{2}}\frac{8\pi^{2}}{b}(-\pi)\frac{b}{4\pi}\alpha_{s}(Q^{2})\right)$$
$$= -\pi Q^{2}DCe^{2}\sum e_{k}^{2}\left(1 + \frac{3C_{F}}{4\pi}\alpha_{s}(Q^{2})\right)$$
$$= -\pi Q^{2}DCe^{2}\sum e_{k}^{2}\left(1 + \frac{1}{\pi}\alpha_{s}(Q^{2})\right)$$
(679)

using $C_F = 4/3$. Using Eq. (647) this gives

$$s\sigma(e^+e^- \to q\overline{q}) = \pi Q^2 D C e^4 \sum e_k^2 \left(1 + \frac{1}{\pi} \alpha_s(Q^2)\right)$$
 (680)

In taking the ratio of the cross section for $e^+e^-
ightarrow q\overline{q}$ proportional to

the above imaginary part to that for $e^+e^- \rightarrow \mu^+\mu^-$ (which has no QCD correction and no $\sum e_k^2$ factor), the $\pi Q^2 D C e^4$ stuff cancels and we are left with

$$\frac{\sum_{q} \sigma(e^+e^- \to q\overline{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{k=quarks} e_k^2 \left(1 + \frac{\alpha_s(Q^2)}{\pi}\right) \,. \tag{681}$$

We still have to compute the QCD loop diagrams to get the form of Z_3 given earlier, but it is clear that this procedure generalizes nicely to higher orders. The advantage of this procedure over that in which the cross section is computed directly is that you do not have to consider the infrared singularities arising from real gluon emission and how they cancel with the infrared singularities present in the virtual gluon vertex correction diagrams sketched diagramatically earlier.

In any case, we see from the previous two figures that the QCD corrections are needed in order to get precision agreement with the measurements of R.

Summary of Two-Loop Calculation

We now consider the two-loop QCD corrections to $\Pi_{\mu\nu}$. It is possible to reference the Itzykson-Zuber two-loop calculation for QED and generalize

to QCD in a fairly obvious manner. Here, I will sketch the results so obtained.

The two-loop diagrams we must compute are given in the figure. Note the counter-term diagrams that we must include at this order.



Figure 21: Two-loop corrections to the photon propagator at order e^2g^2 . The diagrams are called: 1st row: a_1 , a_2 , b; 2nd row: a'_1 , a'_2 , b'_1 3rd row: b'_2 .

I now outline the calculation results. I use ρ and σ for the external photon indices and define $\hat{\gamma} \equiv \gamma_E - \ln 4\pi$ (where γ_E is what I called γ earlier). (Details of this calculation for the 2nd diagram in the first row of the above figure, called a_2 , are given later.) The result is (we drop C_F for the

moment and will restore this factor at the end and, also, we are doing the calculation for fixed quark color — the sum over colors is done as part of \sum_k):

$$\Pi_{\rho\sigma}^{a} = \frac{\alpha\alpha_{\rm S}}{12\pi^{2}} \left\{ k_{\rho}k_{\sigma} \left[-\frac{1}{\epsilon^{2}} + \frac{1}{\epsilon} \left(-\frac{7}{2} + 2\widehat{\gamma} + 2\ln\left(\frac{-k^{2}}{\mu^{2}}\right) \right) + \left((7 - 4\widehat{\gamma})\ln\left(\frac{-k^{2}}{\mu^{2}}\right) - 2\ln^{2}\left(\frac{-k^{2}}{\mu^{2}}\right) \right) \right] - k^{2}g_{\rho\sigma} \left[-\frac{1}{\epsilon^{2}} + \frac{1}{\epsilon} \left(2\widehat{\gamma} - 4 + 2\ln\left(\frac{-k^{2}}{\mu^{2}}\right) \right) + \left((8 - 4\widehat{\gamma})\ln\left(\frac{-k^{2}}{\mu^{2}}\right) - 2\ln^{2}\left(\frac{-k^{2}}{\mu^{2}}\right) \right) \right] \right\}$$
(682)

Note that this result is not yet entirely "transverse", i.e. in the gauge-invariant form. But, we have lots left to do. Next, we have the counter term diagrams

$$\Pi_{\rho\sigma}^{a'} = \Pi_{\rho\sigma}^{a'_{1}} + \Pi_{\rho\sigma}^{a'_{2}}$$

$$= -\frac{\alpha\alpha_{\rm S}}{12\pi^{2}} (k^{2}g_{\rho\sigma} - k_{\rho}k_{\sigma}) \left\{ \frac{2}{\epsilon^{2}} + \frac{1}{\epsilon} \left(-2\ln\left(\frac{-k^{2}}{\mu^{2}}\right) + \frac{10}{3} - 2\widehat{\gamma} \right) + \ln^{2}\left(\frac{-k^{2}}{\mu^{2}}\right) + \left(2\widehat{\gamma} - \frac{10}{3}\right)\ln\left(\frac{-k^{2}}{\mu^{2}}\right) \right\}$$
(683)

Note that $\Pi_{\rho\sigma}^{a} + \Pi_{\rho\sigma}^{a'}$ still has $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ terms. The $\frac{1}{\epsilon^2}$ terms will cancel after including all diagrams, but the $\frac{1}{\epsilon}$ terms will not. This is to be expected: the

diagram as a whole will require a $(Z_3 - 1)^{e^2g^2}$ counter term to be rendered finite. We now turn to the vertex diagram and its two counterterms. We have

$$\Pi_{\rho\sigma}^{b} = \frac{\alpha\alpha_{\rm S}}{12\pi^{2}} \left\{ k\rho k\sigma \left[\frac{1}{\epsilon^{2}} + \frac{1}{\epsilon} \left(5 - 2\widehat{\gamma} - 2\ln\left(\frac{-k^{2}}{\mu^{2}}\right) \right) + 2\ln^{2}\left(\frac{-k^{2}}{\mu^{2}}\right) + \ln\left(\frac{-k^{2}}{\mu^{2}}\right) (4\widehat{\gamma} - 10) + \dots \right] - g\rho\sigma k^{2} \left[\frac{1}{\epsilon^{2}} + \frac{1}{\epsilon} \left(\frac{11}{2} - 2\widehat{\gamma} - 2\ln\left(\frac{-k^{2}}{\mu^{2}}\right) \right) + 2\ln^{2}\left(\frac{-k^{2}}{\mu^{2}}\right) + \ln\left(\frac{-k^{2}}{\mu^{2}}\right) (4\widehat{\gamma} - 11) + \dots \right] \right\}$$

$$(584)$$

Now, because of the Ward identity in the form $Z_1 = Z_2$, the $\Pi_{\rho\sigma}^{b'} = \Pi_{\rho\sigma}^{b'_1} + \Pi_{\rho\sigma}^{b'_2}$ counter term diagrams exactly cancel the $a'_1 + a'_2$ counter terms. Indeed, the latter are proportional to $-(Z_2 - 1)^{g^2}$ while the former are proportional to $(Z_1 - 1)^{g^2}$, where these Z_1 and Z_2 at order g^2 are the QCD corrections to the photon vertex and quark propagator respectively. Since the photon does not couple to the exchange gluon, we do not have the extra C_A piece that upsets $Z_1 = Z_2$ that was present when doing gluon propagator corrections.

We now combine (including the C_F color factor now)

$$i\Pi_{\rho\sigma}^{e^{2}g^{2}} = i\left[\Pi_{\rho\sigma}^{a} + \Pi_{\rho\sigma}^{b}\right]$$

$$= (g_{\rho\sigma}k^{2} - k_{\rho}k_{\sigma})i\Pi^{e^{2}g^{2}}(k^{2}) \quad \text{with}$$

$$i\Pi^{e^{2}g^{2}}(k^{2}) = i\frac{\alpha\alpha_{S}}{\pi^{2}}C_{F}\left(-\frac{1}{8\epsilon} + \frac{1}{4}\ln\frac{k^{2}}{\mu^{2}} + \dots\right). \quad (685)$$

Note that the $\frac{1}{\epsilon^2}$, $\ln^2 \frac{k^2}{\mu^2}$ and $\frac{1}{\epsilon} \ln \frac{k^2}{\mu^2}$ terms have all canceled and that the result is now completely transverse (at least for the terms we have kept, but in fact for all terms if one is careful).

The cancellation of the $\frac{1}{\epsilon} \ln \frac{k^2}{\mu^2}$ terms is quite crucial as there would be no way to remove such a momentum dependent singularity using a counter term approach.

The cancellation of the $\frac{1}{\epsilon^2}$ terms and the lock-step $\ln^2 \frac{k^2}{\mu^2}$ terms is a general property of the vacuum polarization, valid to all orders when we restrict ourselves to diagrams with a single fermion loop. Indeed, it is possible to fix order by order the gauge parameter so as to make $Z_1 = Z_2$ equal to one. Consequently, the selected subset of diagrams does not exhibit

any internal divergence and its contribution to the gauge-invariant quantity $\Pi(k^2)$ behaves as a single power of $\frac{1}{\epsilon}$ (which would be $\ln \frac{\Lambda^2}{\mu^2}$ in the cut-off regularization schemes).

The $\frac{1}{\epsilon}$ stuff does not, of course, cancel and you will note how it goes along with a certain $\ln \frac{k^2}{\mu^2}$ term with the same coefficient. This matching is encoded in our RGE equations by which we claim we can basically resum the ln stuff by simply using the RGE equations based upon the counter term information from the $\frac{1}{\epsilon}$ singularity structure.

The $-1/(8\epsilon)$ should be $-1/(4\epsilon)$, but I cannot yet find my mistake. I will use the $-1/(4\epsilon)$ result in the following. (Actually, my result agrees with IZ once you take their $\epsilon_{IZ} = 2\epsilon$, so they are wrong too, or else their Γ^b , which I have not checked, but simply used with my ϵ definition, is wrong in a compensating way.)

It is the $\frac{1}{\epsilon}$ in the final expression that must be counter-termed in the usual sort of way. Recalling first the $\mathcal{O}(e^2)$ result, Eq. (358), we had

$$i\Pi_{\rho\sigma}^{e^2} = -\frac{1}{\epsilon} \frac{ie^2}{12\pi^2} (g_{\rho\sigma}k^2 - k_\rho k_\sigma) = -i\frac{1}{\epsilon} \frac{\alpha}{3\pi} (g_{\rho\sigma}k^2 - k_\rho k_\sigma)$$
(686)

leading to, Eq. (359), $C^{e^2} = -\frac{\alpha}{3\pi\epsilon}$ which in turn gave

$$Z_3^{e^2} = 1 + C^{e^2} = 1 - \frac{\alpha}{3\pi\epsilon}.$$
 (687)

We now simply add in the order e^2g^2 result, yielding

$$i\Pi_{\rho\sigma}^{e^2+e^2g^2} = -i\frac{1}{\epsilon} \left[\frac{\alpha}{3\pi} + \frac{\alpha\alpha_{\rm S}}{4\pi^2}C_F\right] (g_{\rho\sigma}k^2 - k_{\rho}k_{\sigma})$$
(688)

leading to $C^{e^2+e^2g^2} = -\frac{lpha}{3\pi\epsilon} - \frac{lpha lpha_{
m S}}{4\pi^2\epsilon} C_F$ which in turn gives

$$Z_{3}^{e^{2}+e^{2}g^{2}} = 1 + C^{e^{2}+e^{2}g^{2}}$$

$$= 1 - \frac{\alpha}{3\pi\epsilon} - \frac{\alpha\alpha_{S}}{4\pi^{2}\epsilon}C_{F}$$

$$= 1 - \frac{\alpha}{3\pi\epsilon} \left(1 + \frac{3}{4}\frac{\alpha_{S}}{\pi}C_{F}\right)$$

$$C_{F} = 4/3 = 1 - \frac{\alpha}{3\pi\epsilon} \left(1 + \frac{\alpha_{S}}{\pi}\right). \quad (689)$$

The computations of γ_d and then of the $e^+e^- \rightarrow q\overline{q}$ process proceed by just including this extra parenthetical correction factor, as done earlier.

Derivation of $\Pi^a_{\rho\sigma}$

We will now descend to one more level of detail so that you can glimpse how the two-loop calculations are performed in an explicit fashion. I focus on the *a* diagrams a_1 and a_2 . In fact, these two contributions are equal and so I will simply compute a_2 and multiply by 2 at the end.

Assigning loop momentum p to the lower line of diagram a_2 as it enters into the photon vertex, so that the upper line will have p + k, we find (neglecting masses and defining $-i\Sigma$ as the gluon correction to the fermion propagator and including an overall minus sign for the fermion loop)

$$i\Pi_{\rho\sigma}^{a_{2}}(k) = -\int \frac{d^{n}p}{(2\pi)^{n}} \operatorname{Tr}\left[\frac{i}{\not p + \not k}(-ie\mu^{\epsilon})\gamma_{\rho}\frac{i}{\not p}[-i\Sigma(p)]\frac{i}{\not p}(-ie\mu^{\epsilon})\gamma_{\sigma}\right]$$
$$= -e^{2}\mu^{2\epsilon}\int \frac{d^{n}p}{(2\pi)^{n}} \operatorname{Tr}\left[\frac{1}{\not p + \not k}\gamma_{\rho}\frac{1}{\not p}\Sigma(p)\frac{1}{\not p}\gamma_{\sigma}\right], \qquad (690)$$

where $\Sigma(p)$ can be taken from the QCD analogue of the 2nd line of Eq. (307) with m = 0 and $e \to -g$ (and we will wait to bring in the C_F factor which is present in the QCD analogue):

$$\Sigma(p) \;\;=\;\; g^2 \mu^{2\epsilon} rac{\Gamma\left(2-rac{n}{2}
ight)}{(4\pi)^{n/2}} \int dlpha [(1-lpha)\, p\!\!\!\!/(2-n)] [-lpha(1-lpha)p^2]^{n/2-2}$$

$$= g^{2} \mu^{2\epsilon} \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}} (-p^{2})^{n/2-2} (2-n) \not p \int d\alpha \alpha^{n/2-2} (1-\alpha)^{n/2-1}$$

$$= g^{2} \mu^{2\epsilon} \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}} (-p^{2})^{n/2-2} (2-n) \not p \frac{\Gamma\left(\frac{n}{2}-1\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma(n-1)}$$

$$= \not p g^{2} \mu^{2\epsilon} \frac{(-1)^{n/2-1} \Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}} \frac{\Gamma\left(\frac{n}{2}-1\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma(n-2)(p^{2})^{2-n/2}}.$$
(691)

Inserting this expression, there is a cancellation of a p in the numerator with one in the denominator, leaving you with the sub-calculation (for the 2nd line we employ the generalized Feynman trick as derived in the Appendix; for the 3rd line we shift to p' = p + wk):

$$\int \frac{d^{n}p}{(2\pi)^{2}} \frac{\operatorname{Tr}\left[(\not\!p+k\!\!)\gamma\rho\not\!p\gamma\sigma\right]}{(p+k)^{2}(p^{2})^{3-n/2}}$$

$$= \int dw(1-w)^{2-n/2} \int \frac{d^{n}p}{(2\pi)^{2}} \frac{\operatorname{Tr}\left[(\not\!p+k\!\!)\gamma\rho\not\!p\gamma\sigma\right]}{\left[(1-w)p^{2}+w(p+k)^{2}\right]^{4-n/2}} \frac{\Gamma\left(4-\frac{n}{2}\right)}{\Gamma(1)\Gamma\left(3-\frac{n}{2}\right)}$$

$$= \int dw(1-w)^{2-n/2} \int \frac{d^{n}p'}{(2\pi)^{2}} \frac{\operatorname{Tr}\left[(\not\!p'+(1-w)k\!\!)\gamma\rho(\not\!p'-wk\!\!)\gamma\sigma\right]}{\left[p'^{2}+w(1-w)k^{2}\right]^{4-n/2}} (3-n/2).$$
(692)

At this point, we can drop the terms that are odd in p' and employ the q
ightarrow 0

limits of Eqs. (311) and (313), which I repeat here:

$$\int \frac{d^{n}p'}{(2\pi)^{n}} \frac{1}{[p'^{2} + M^{2}]^{A}} = \frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(A)} \frac{\Gamma(A - \frac{n}{2})}{(M^{2})^{A - n/2}}$$
(693)
$$\int \frac{d^{n}p'}{(2\pi)^{n}} \frac{p'_{\mu}p'_{\nu}}{[p'^{2} + M^{2}]^{A}} = \frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(A)} \frac{1}{2}g_{\mu\nu} \frac{\Gamma(A - 1 - \frac{n}{2})}{(M^{2})^{A - 1 - n/2}}.$$
(694)

These will be employed with A = 4 - n/2 and $M^2 = w(1 - w)k^2$. The two structures in the trace then yield (for consistency with Itzykson and Zuber, I adopt their convention of Tr[I] = 4 instead of my $2^{n/2}$ — however, they also use $n = 4 - \epsilon$; I will stick with $n = 4 - 2\epsilon$, so be careful in comparing my results with theirs):

$$\operatorname{Tr}[\gamma_{\mu}\gamma_{\rho}\gamma_{\nu}\gamma_{\sigma}] \int p'_{\mu}p'_{\nu} \rightarrow \operatorname{Tr}[\gamma_{\mu}\gamma_{\rho}\gamma_{\nu}\gamma_{\sigma}] \frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(4-n/2)} \frac{1}{2} g_{\mu\nu} \frac{\Gamma(3-n)}{[w(1-w)k^{2}]^{3-n}} = (1-n/2)\operatorname{Tr}[\gamma_{\rho}\gamma_{\sigma}] \frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(4-n/2)} \frac{\Gamma(3-n)}{[w(1-w)k^{2}]^{3-n}}$$

$$= (1 - n/2) 4g_{\rho\sigma} \frac{i(-1)^{n/2}}{(4\pi)^{n/2} \Gamma(4 - n/2)} \frac{\Gamma(3 - n)}{[w(1 - w)k^2]^{3 - n}}$$
(695)

and

$$-w(1-w)\operatorname{Tr}[k\gamma_{\rho}k\gamma_{\sigma}] \int \rightarrow -w(1-w)4(2k_{\rho}k_{\sigma}-g_{\rho\sigma}k^{2})\frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(4-n/2)}\frac{\Gamma(4-n)}{[w(1-w)k^{2}]^{4-n}} = -4(2k_{\rho}k_{\sigma}-g_{\rho\sigma}k^{2})\frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(4-n/2)}\frac{\Gamma(4-n)}{k^{2}[w(1-w)k^{2}]^{3-n}}$$
(696)

There are then two identical $\int dw$'s that must be performed:

$$\int dw (1-w)^{2-n/2} [w(1-w)]^{n-3} = \frac{\Gamma(n-2)\Gamma(n/2)}{\Gamma(3n/2-2)}$$
(697)

The net result is then, at first writing separately all the various factors,

$$-(e^2\mu^{2\epsilon})(g^2\mu^{2\epsilon})\frac{\Gamma(2-n/2)}{(4\pi)^{n/2}}(-1)^{n/2-1}\frac{\Gamma(n/2-1)\Gamma(n/2)}{\Gamma(n-2)}(3-n/2)\frac{i(-1)^{n/2}}{(4\pi)^{n/2}\Gamma(4-n/2)}\times$$

$$4\frac{\Gamma(n-2)\Gamma(n/2)}{\Gamma(3n/2-2)} \left[(1-n/2)g_{\rho\sigma}\Gamma(3-n)(k^{2})^{n-3} - (2k\rho k_{\sigma} - g_{\rho\sigma}k^{2})\Gamma(4-n)(k^{2})^{n-4} \right]$$

$$= -(e^{2}\mu^{2\epsilon})(g^{2}\mu^{2\epsilon})\frac{i(-1)^{n-1}}{(4\pi)^{n}\Gamma(3-n/2)}\Gamma(2-n/2)\frac{\Gamma(n/2-1)\Gamma^{2}(n/2)}{\Gamma(3n/2-2)}\times$$

$$4\left[(1-n/2)g_{\rho\sigma}\Gamma(3-n)(k^{2})^{n-3} - (2k\rho k_{\sigma} - g_{\rho\sigma}k^{2})\Gamma(4-n)(k^{2})^{n-4} \right]$$

$$= (e^{2}\mu^{2\epsilon})(g^{2}\mu^{2\epsilon})\frac{i(-1)^{n}}{(4\pi)^{n}(2-n/2)}\frac{\Gamma(n/2-1)\Gamma^{2}(n/2)}{\Gamma(3n/2-2)}\times$$

$$4\left[(1-n/2)g_{\rho\sigma}\Gamma(3-n)(k^{2})^{n-3} - (2k\rho k_{\sigma} - g_{\rho\sigma}k^{2})\Gamma(4-n)(k^{2})^{n-4} \right]$$

$$= i\frac{e^{2}g^{2}}{64\pi^{4}} \left(\frac{4\pi\mu^{2}}{-k^{2}}\right)^{2\epsilon} \frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)\Gamma^{2}(2-\epsilon)}{\Gamma(4-3\epsilon)}\left[(-1+\epsilon)g_{\rho\sigma}k^{2}\Gamma(-1+2\epsilon) - (2k\rho k_{\sigma} - g_{\rho\sigma}k^{2})\Gamma(2\epsilon) \right]$$
(698)

This expression is evaluated for the $\frac{1}{\epsilon^2}$, $\frac{1}{\epsilon}$ and constant terms using the Mathematica file that appears in a short section following this main section of the lecture. I summarize the output below, keeping (as in Itzykson and Zuber) only the k^2 dependent parts of the constant piece and defining $\hat{\gamma} = \gamma_E - \ln 4\pi$. Keep in mind that I use $n = 4 - 2\epsilon$ while IZ use $d = 4 - \epsilon$. So, if you look at their book, you must change their ϵ to 2ϵ in making the comparison. If you do, we seem to get the same result. The result is:

$$i\Pi_{\rho\sigma}^{a2} = i\frac{\alpha\alpha_{\rm S}}{24\pi^2} \left\{ k\rho k\sigma \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{7}{2} + 2\widehat{\gamma} + 2\ln\left(\frac{-k^2}{\mu^2}\right) \right) + \left((7 - 4\widehat{\gamma})\ln\left(\frac{-k^2}{\mu^2}\right) - 2\ln^2\left(\frac{-k^2}{\mu^2}\right) \right) \right] - k^2g\rho\sigma \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(2\widehat{\gamma} - 4 + 2\ln\left(\frac{-k^2}{\mu^2}\right) \right) + \left((8 - 4\widehat{\gamma})\ln\left(\frac{-k^2}{\mu^2}\right) - 2\ln^2\left(\frac{-k^2}{\mu^2}\right) \right) \right] \right\}$$
(699)

This result for diagram a^2 must be multiplied by $\times 2$ in order to get $a^1 + a^2$.

The general Feynman trick for combining denominators

Consider the integral

$$\int dw \frac{w^{\alpha - 1} (1 - w)^{\beta - 1}}{\left[wA + (1 - w)B \right]^{\alpha + \beta}}.$$
(700)

Change variables to

$$m{z} \equiv rac{m{w}A}{m{w}A + (1-m{w})B}, \quad (1-m{z}) = rac{(1-m{w})B}{m{w}A + (1-m{w})B}, \quad m{dz} = rac{ABdw}{[m{w}A + (1-m{w})B]^2},$$
(701)

which gives

$$\int dw \frac{w^{\alpha-1}(1-w)^{\beta-1}}{[wA+(1-w)B]^{\alpha+\beta}} = \frac{1}{A^{\alpha}B^{\beta}} \int_{0}^{1} dz z^{\alpha-1}(1-z)^{\beta-1}$$
$$= \frac{1}{A^{\alpha}B^{\beta}} B(\alpha,\beta), \qquad (702)$$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$
(703)

This gives us a way of writing $\frac{1}{A^{\alpha}B^{\beta}}$ for arbitrary (including non-integer) α and β :

$$\frac{1}{A^{\alpha}B^{\beta}} = \int dw \frac{w^{\alpha-1}(1-w)^{\beta-1}}{\left[wA + (1-w)B\right]^{\alpha+\beta}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$
 (704)

Mathematica File for e^2g^2 two-loop calculation

This file takes off from Eq. (698). The result from the Mathematica file must be divided by $384\pi^4\epsilon^2$, the factor that I remove in order to make the output simple and to make the expansion simple. Note that I have defined a quantity $game = \gamma_E - \ln 4\pi$. In the equations following Eq. (698), I have simply denoted this quantity by $\hat{\gamma}$. In comparing to Itzykson-Zuber, one must keep in mind that his calculation is in the Euclidean convention, whereas mine is in the standard metric approach. Thus, his k^2 in the ln forms is actually $k_E^2 = -k^2$. So, in the file, $absk2 = k_E^2 = -k^2$.

The two terms in the mathematica output are:

- finkk, which is the $k_{\rho}k_{\sigma}$ coefficient;
- fingksq, which is the $k^2g_{\rho\sigma}$ coefficient.

Now, there are a bunch of constants in the expression multiplying ϵ^2 , which become simply constants after dividing by ϵ^2 . These are neglected in Itzykson and Zuber and in my summary.

In[21]:= finkk2 = Simplify[finkk];

In[23]:= finkk4 = Simplify[finkk3 /. EulerGamma → game + Log[4] + Log[Pi]]

$$Out[23] = \left\{ -1 + \left(-\frac{7}{2} + 2 \text{ game} - 2 \text{ Log} \left[\frac{\text{mu}^2}{\text{absk}2} \right] \right) \text{ eps } + \left(-\frac{45}{4} + 7 \text{ game} - 2 \text{ game}^2 + \frac{\pi^2}{6} - 8 \text{ Log} [2]^2 - 2 \text{ Log} [4]^2 + \text{ Log} [4] \text{ Log} [256] + (-7 + 4 \text{ game}) \text{ Log} \left[\frac{\text{mu}^2}{\text{absk}2} \right] - 2 \text{ Log} \left[\frac{\text{mu}^2}{\text{absk}2} \right]^2 \right) \text{ eps}^2 + 0 [\text{eps}]^3 \right\}$$

In[29]:= fingksq2 = Simplify[fingksq];

In[30]:= fingksq3 = Simplify[384 * Pi^4 * Series[fingksq2, {eps, 0, 2}]];

In[31]:= fingksq4 = Simplify[fingksq3 /. EulerGamma → game + Log[4] + Log[Pi]]

$$Out[31] = \left\{1 + \left(4 - 2 \operatorname{game} + 2 \operatorname{Log}\left[\frac{\operatorname{mu}^{2}}{\operatorname{absk2}}\right]\right) \operatorname{eps} + \left(14 - 8 \operatorname{game} + 2 \operatorname{game}^{2} - \frac{\pi^{2}}{6} - 4 \left(-2 + \operatorname{game}\right) \operatorname{Log}\left[\frac{\operatorname{mu}^{2}}{\operatorname{absk2}}\right] + 2 \operatorname{Log}\left[\frac{\operatorname{mu}^{2}}{\operatorname{absk2}}\right]^{2}\right) \operatorname{eps}^{2} + 0 [\operatorname{eps}]^{3}\right\}$$

1

Unitarity plays a crucial role in our understanding of why we need a Higgs boson. Let us first discuss the general partial wave formulation of unitarity. For $2 \rightarrow 2$ scattering we have the standard formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2 \,. \tag{705}$$

If we do the partial wave decomposition (the form employed assumes I am dealing with spin-0 particles — effectively the longitudinally polarized W's that I will be discussing can be thought of as spin-0 objects)

$$\mathcal{M} = 16\pi \sum_{J=0}^{\infty} (2J+1) P_J(\cos\theta) a_J \tag{706}$$

and recall the standard orthogonality relation

$$\int_{-1}^{1} dx P_J(x) P_{J'}(x) = \frac{2}{2J+1} \delta_{J,J'}$$
(707)

$$\sigma = \frac{8\pi}{s} \sum_{J=0}^{\infty} (2J+1) \sum_{J'=0}^{\infty} (2J'+1) a_J a_{J'}^* \int_{-1}^{1} d\cos\theta P_J(\cos\theta) P_{J'}(\cos\theta)$$
$$= \frac{16\pi}{s} \sum_{J=0}^{\infty} (2J+1) |a_J|^2.$$
(708)

Meanwhile, the optical theorem says that

$$s\sigma = \operatorname{Im}\mathcal{M}(\theta = 0) = 16\pi \sum_{J=0}^{\infty} (2J+1)\operatorname{Im}a_J.$$
 (709)

If a single partial wave is dominant, we end up with the requirement that

$$\operatorname{Im} a_J = |a_J|^2 \,. \tag{710}$$

In fact, this equation applies for every value of J even if more than one partial wave is present (requires more work to show explicitly — see short appendix — but is basically a statement of angular momentum conservation for the

elastic process). However, the above equality requires that the only process contributing to the total cross section be the $2 \rightarrow 2$ process. Normally, there are many inelastic channels such as $2 \rightarrow 2'$, $2 \rightarrow 3, 2 \rightarrow 4, \ldots$. The $\text{Im}a_J$ must take into account all these extra channels. In short, what we really have is

$$\text{Im}a_J \ge |a_J|^2 = (\text{Im}a_J)^2 + (\text{Re}a_J)^2,$$
 (711)

When the equality holds, we can write

$$a_J = e^{i\delta_J} \sin \delta_J \,, \tag{712}$$

which form automatically satisfies $\text{Im}a_J = |a_J|^2$ and makes it clear that $|a_J|^2 = \sin^2 \delta_J \leq 1$ with maximum value of 1 when a_J is purely imaginary, $\delta_J = \pi/2$.

It is useful to have a graphical picture. On the unitarity circle, we can rewrite $\text{Im}a_J = |a_J|^2$ as

$$\frac{1}{4} = (\text{Im}a_J - \frac{1}{2})^2 + (\text{Re}a_J)^2.$$
 (713)

so that in the complex plane a_J will lie on a "unitarity" circle of radius 1/2.



Figure 22: The partial wave unitarity circle.

Allowing for the presence of inelasticity, we can write

$$\frac{1}{4} \ge (\mathrm{Im}a_J - \frac{1}{2})^2 + (\mathrm{Re}a_J)^2, \qquad (714)$$

implying that a_J will lie inside the unitarity circle. In this case, we can write

$$a_J = \frac{\eta_J e^{2i\delta_J} - 1}{2i},$$
 (715)

since $\operatorname{Im} a_J - \frac{1}{2} = -\frac{1}{2} \eta_J \cos 2\delta_J$ and $\operatorname{Re} a_J = \frac{1}{2} \eta_J \sin 2\delta_J$ so that

$$(\mathrm{Im}a_J - \frac{1}{2})^2 + (\mathrm{Re}a_J)^2 = \frac{1}{4}\eta_J^2$$
 (716)

which will be $\leq \frac{1}{4}$ if $\eta_J \leq 1$. The quantity η_J is called the inelasticity. To repeat, $\eta_J \leq 1$ is required in order for a_J to be within the unitarity circle.

As a point of reference, you are presumably familiar with the standard spin-J resonance form of

$$a_J = a = \frac{-m\Gamma_{el}}{s - m^2 + im\Gamma_{tot}}$$
(717)

which saturates the unitarity circle when $\Gamma_{el} = \Gamma_{tot}$. You will notice that when the latter is true then at $s = m^2$ one finds $a_J = i$ so that one is at the top of the circle. As one starts from low $s \ll m^2$ with $\text{Re}a_J > 0$, passes through $s = m^2$ with $\text{Re}a_J = 0$ and on to $s \gg m^2$ with $Rea_J < 0$ one is rotating in a counter-clockwise sense about the unitarity circle.

The important final result is that the largest value of $|\text{Re}a_J|$ that is possible if on the "unitarity" circle is $|\text{Re}a_J| = \frac{1}{2}$. And, if there is inelasticity then we have our final constraint of

$$|\operatorname{Re} a_J| \le \frac{1}{2}.\tag{718}$$

Usually it is the J = 0 constraint that is the strongest for a typical process of interest to us.

We now wish to apply this to $WW \rightarrow WW$ scattering where the W's all have longitudinal polarization.

In the SM, the partial wave amplitudes take the asymptotic form

$$a_J = A_J \left(\frac{s}{m_W^2}\right)^2 + B_J \left(\frac{s}{m_W^2}\right) + C_J, \qquad (719)$$

where s is the center-of-mass energy squared. Contributions that are divergent in the limit $s \to \infty$ appear only for J = 0, 1, 2. The A-terms vanish by virtue of gauge invariance, while, and here enters the Higgs boson, the B-term for J = 1 and 0 ($B_2 = 0$) arising from gauge interaction diagrams is canceled by Higgs-boson exchange diagrams. In the high-energy limit, the result is that a_J asymptotes to an m_h -dependent constant. Imposing the unitarity limit of $|\text{Re}a_J| < 1/2$ implies the Lee-Quigg-Thacker bound for the Higgs boson mass: $m_h \lesssim 870$ GeV.

What happens if m_h is increased beyond 870 GeV is that the perturbatively calculated a_J violates unitarity at lower and lower s values. This is not to say that the theory actually violates unitarity. When a_J approaches the unitarity bound, the theory becomes strongly interacting and we are no longer able to calculate the consequences of the theory perturbatively. At the moment we do not have any means of computing what actually happens. Nonetheless, the theory cannot actually violate unitarity since unitarity is guaranteed simply by the hermiticity of the Hamiltonian that we began with. In other words, $m_h = 870 \text{ GeV}$ is the largest value for which a perturbative approach to computing in the theory is possible.

Alternatively, if a_J reaches the unitarity limit at some value of s, we can hope that some beyond-the-SM physics enters that will be such that we can compute perturbatively in the extended model. The value of s at which the SM violates unitarity then sets the upper bound on the scale at which such new BSM physics must enter.

Anyway, let us give some more details on the perturbative SM calculation. In detail, we have the following. (In my normalization, v = 246 GeV and $m_W = \frac{gv}{2}$. And, $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}$.) The various contributions to the amplitude are given in Table 2. From the table, we see that the gauge boson contributions and Higgs exchange contributions cancel at $\mathcal{O}(s^2)$ and $\mathcal{O}(s^1)$.

diagram	$\mathcal{O}(rac{s^2}{v^4})$	$\mathcal{O}(rac{s^1}{v^2})$
γ, Z s-channel	$-rac{s^2}{g^2v^4}4\cos heta$	$-rac{s}{v^2}\cos heta$
γ, Z t-channel	$-rac{s^2}{g^2v^4}(-3+2\cos heta+\cos^2 heta)$	$-rac{s}{v^2}rac{3}{2}(1-5\cos heta)$
WWWW contact	$-rac{s^2}{g^2v^4}(3-6\cos heta-\cos^2 heta)$	$-rac{s}{v^2}2(-1+3\cos heta)$
h s-channel	0	$-\frac{s}{v^2}$
h t-channel	0	$-rac{s}{v^2} rac{-1+\cos heta}{2}$
Sum	0	0

Table 2: The leading contributions to $\mathcal{M}(W_L^+W_L^- \to W_L^+W_L^-)$ amplitude — where \mathcal{M} is defined in the convention $S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}$.

The cancellation of the $\mathcal{O}(s^2)$ contributions in Table 2 between the contact term and s- and t-channel gauge-boson exchange diagrams is guaranteed by gauge invariance. The Higgs or something like it is required for cancelling the s^1 terms.

The easiest of the amplitudes above to derive is that from the WWWW

contact interaction. First, we need to write down the polarization vector for a massive, longitudinally-polarized W of momentum p. If we start with a massive vector boson with momentum $k^{\mu} = (m, 0, 0, 0)$, then the 3 orthgonal polarizations are

 $\epsilon^{\mu} = (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1).$ (720)

If we now boost along the z direction, the first two (the transverse polarizations) remain unchanged. The third (the longitudinal polarization) boosts to $(k \equiv k_z)$

$$\epsilon_L^{\mu}(k) = \left(\frac{k}{m}, 0, 0, \frac{E_k}{m}\right) \tag{721}$$

which generalizes for arbitrary direction to

$$\epsilon_L^{\mu}(\vec{k}) = \left(\frac{|\vec{k}|}{m}, \frac{\vec{k}}{|\vec{k}|} \frac{E_{\vec{k}}}{m}\right) \sim \frac{k^{\mu}}{m} + \mathcal{O}\left(\frac{m}{E_{\vec{k}}}\right)$$
(722)

(Note that the first, exact form on the rhs of the equation is explicitly orthogonal to $k^{\mu} = (E_{\vec{k}}, \vec{k})$.) It is the proportionality of the longitudinal polarization to the momentum which makes longitudinally polarized W's so "dangerous". Of course, for our case of W's, $m \to m_W$.
Meanwhile, the *WWWW* contact interaction takes the familiar form in the all outgoing convention (and $S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 (p_f - p_i) i \mathcal{M}_{fi}$ convention):

$$i\mathcal{M}^{abcd}_{\alpha\beta\gamma\delta}(p,q,r,s) = -ig^{2} \left\{ c^{eab}c^{ecd} \left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma} \right) + c^{eac}c^{edb} \left(g^{\alpha\delta}g^{\gamma\beta} - g^{\alpha\beta}g^{\gamma\delta} \right) + c^{ead}c^{ebc} \left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} \right) \right\}, \qquad (723)$$

which, for the SU(2) group and a, b, c, d index choices corresponding to $(a = W^+)(d = W^-) \rightarrow (b = W^+)(c = W^-)$ scattering reduces to¹¹ (after noting that the incoming W^+W^- should be converted to outgoing $(a = W^-)(d = W^+)$ to apply the "all-outgoing" vertex)

$$-ig^{2}(g^{\alpha\delta}g^{\beta\gamma} + g^{\gamma\delta}g^{\alpha\beta} - 2g^{\beta\delta}g^{\alpha\gamma})$$
(724)

where the first two terms are s and t channel-like (outgoing W^-W^+) and

¹¹A convenient reference for electroweak theory Feynman rules is the book "Gauge Theories of the Strong, Weak and Electromagnetic Interactions" by Chris Quigg, pages 113-16, or Appendix B of Cheng and Li.

the last is u channel-like (outgoing W^-W^-) and thus comes with a larger coefficient and opposite sign.

To see how this works in the case of the WWWW vertex starting with the general result of Eq. (723), we first recall that $W^{\pm} = \frac{W^1 \mp i W^2}{\sqrt{2}}$. So, what we are interested in is (using a shorthand)

$$a = \frac{1+i2}{\sqrt{2}}, \quad b = \frac{1-i2}{\sqrt{2}}, \quad c = \frac{1+i2}{\sqrt{2}}, \quad d = \frac{1-i2}{\sqrt{2}}, \quad (725)$$

where we used the fact that, for example,

$$W^{a}a \ni W^{1}1 + W^{2}2 = \frac{W^{1} - iW^{2}}{\sqrt{2}} \frac{1 + i2}{\sqrt{2}} + \frac{W^{1} + iW^{2}}{\sqrt{2}} \frac{1 - i2}{\sqrt{2}}$$
$$= W^{+} \frac{1 + i2}{\sqrt{2}} + W^{-} \frac{1 - i2}{\sqrt{2}}, \qquad (726)$$

where the W^- field operator either annihilates a W^- or creates a W^+ . So, if we want to create an outgoing $b = W^+$ we connect to $b = \frac{1-i2}{\sqrt{2}}$ and if we want to connect to an outgoing $a = W^-$ we should use $a = \frac{1+i2}{\sqrt{2}}$. Thus, the general vertex of Eq. (723) reduces for the case of interest to

$$i\mathcal{M}^{abcd}_{\alpha\beta\gamma\delta}(p,q,r,s) = -ig^{2} \left\{ c^{e\frac{1+i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}} c^{e\frac{1+i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}} \left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma} \right) \right. \\ \left. + c^{e\frac{1+i2}{\sqrt{2}}\frac{1+i2}{\sqrt{2}}} c^{e\frac{1-i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}} \left(g^{\alpha\delta}g^{\gamma\beta} - g^{\alpha\beta}g^{\gamma\delta} \right) \right. \\ \left. + c^{e\frac{1+i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}} c^{e\frac{1-i2}{\sqrt{2}}\frac{1+i2}{\sqrt{2}}} \left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} \right) \right\}, \quad (727)$$

Now, for SU(2) we have $c^{abc} = \epsilon^{abc}$ so that for the 1st term we have the form

$$c^{e\frac{1+i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}}c^{e\frac{1+i2}{\sqrt{2}}\frac{1-i2}{\sqrt{2}}} = \left(\frac{-2i}{(\sqrt{2})^2}\epsilon^{e12}\right)\left(\frac{-2i}{(\sqrt{2})^2}\epsilon^{e12}\right) = -1.$$
(728)

The 2nd term is 0 because of the antisymmetry of the ϵ 's and the 3rd term has the order switched on the last c and thus gives +1. The net result for the vertex is then:

$$i\mathcal{M} = -ig^{2} \left[-\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) + \left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \right] = -ig^{2} \left(g^{\alpha\delta}g^{\beta\gamma} + g^{\gamma\delta}g^{\alpha\beta} - 2g^{\beta\delta}g^{\alpha\gamma}\right)$$
(729)

Now we can return to using this form for the scattering process of interest. We simply multiply \mathcal{M} by the polarization vectors which in leading order means we multiply by

$$\frac{p^{\alpha}}{m_{W}}\frac{q^{\beta}}{m_{W}}\frac{r^{\gamma}}{m_{W}}\frac{s^{\delta}}{m_{W}},\tag{730}$$

yielding

$$\mathcal{M}(W_L^+ W_L^- \to W_L^+ W_L^-) \sim -\frac{g^2}{m_W^4} (p \cdot sq \cdot r + r \cdot sp \cdot q - 2p \cdot rq \cdot s) .$$
(731)

Now, one goes to the center of mass frame where (all outgoing convention)

$$p = -\frac{\sqrt{s}}{2}(1,0,0,\beta_W)$$

$$s = -\frac{\sqrt{s}}{2}(1,0,0,-\beta_W)$$

$$q = \frac{\sqrt{s}}{2}(1,0,\beta_W\sin\theta,\beta_W\cos\theta)$$

$$r = \frac{\sqrt{s}}{2}(1,0,-\beta_W\sin\theta,-\beta_W\cos\theta)$$
(732)

with $\beta_W = \sqrt{1 - 4m_W^2/s}$. In this frame, one easily finds

$$2p \cdot s = 2q \cdot r = s - 2m_W^2,$$

$$2p \cdot q = 2r \cdot s = t - 2m_W^2 = -\frac{s}{2}(1 - \cos\theta) - 2m_W^2 \cos\theta,$$

$$2p \cdot r = 2q \cdot s = u - 2m_W^2 = -\frac{s}{2}(1 + \cos\theta) + 2m_W^2 \cos\theta \quad (733)$$

Substituting and collecting the leading terms of order s^2 one obtains

$$\mathcal{M}(W_L^+ W_L^- \to W_L^+ W_L^-) \ni -\frac{g^2}{m_W^4} \frac{s^2}{16} (3 - 6\cos\theta - \cos^2\theta)$$
 (734)

which agrees with the $\mathcal{O}(s^2)$ result in the table for the contact interaction term after substituting $m_W = gv/2$.

To get the $\mathcal{O}(s)$ term of the table, the exact forms of the polarization vectors must be used.

The Higgs exchange diagrams are, of course, easier. For example for the *s*-channel Higgs exchange diagram, remembering that the Feynman rule for

the $hW_{\mu}W_{\nu}$ vertex is $igm_Wg_{\mu\nu}$, we find

$$i\mathcal{M} = \frac{i}{s - m_h^2} (igm_W)^2 \epsilon_p \cdot \epsilon_s \epsilon_q \cdot \epsilon_r$$

$$\simeq -ig^2 m_W^2 \frac{1}{s - m_h^2} \left(\frac{p \cdot s \, q \cdot r}{m_W^4}\right)$$

$$\simeq \frac{-i4m_W^4}{v^2} \frac{1}{s - m_h^2} \left(\frac{\frac{1}{4}s^2}{m_W^4}\right)$$

$$= -i\frac{1}{v^2} \frac{s^2}{s - m_h^2}$$

$$= -i\frac{1}{v^2} \left(s + \frac{s \, m_h^2}{s - m_h^2}\right).$$
(735)

The first term is listed as the $\mathcal{O}(s)$ term for this diagram and the 2nd term will appear below

In any case, after including the Higgs diagrams and summing everything together, we are left with a constant behavior for the amplitude. In the limit where $s, m_h^2 \gg m_W^2, m_Z^2$ we find:

$$\mathcal{M}(W_L^+ W_L^- \to W_L^+ W_L^-) = -\frac{m_h^2}{v^2} \left[\frac{s}{s - m_h^2} + \frac{t}{t - m_h^2} \right].$$
(736)

This is too naive if s is near m_h^2 ; one must include the Higgs width appropriately. For now, we only consider $s \ll m_h^2$ or $s \gg m_h^2$.

The above result can also be derived using the Goldstone Equivalence theorem. It states that in the limit of $m_h^2 \gg m_W^2$ interactions of enhanced strength, $\mathcal{O}(G_F m_h^2)$, arise only from diagrams in which the *internal* particles are also Goldstone bosons or the Higgs boson. The relevant interactions are summarized by the Lagrangian:

$$\mathcal{L} = -\lambda \left(w^{+}w^{-} + \frac{1}{2}z^{2} + \frac{1}{2}h^{2} + vh + \frac{1}{2}v^{2} - \frac{\mu^{2}}{2\lambda} \right)^{2}$$
(737)

where h is the Higgs field, the w's, z are the charged and neutral Goldstone bosons, respectively, v is the usual vev related to m_W as above and λ is the bare coupling of the $\lambda \phi^4$ theory.

Crudely, one can arrive at this result as follows. First, we note that using $m_h^2 = 2\lambda v^2$ and $m_W = gv/2$ one finds $\lambda = g^2 \frac{m_h^2}{8m_W^2}$. When $m_h^2 \gg m_W^2$, λ is large and the Higgs self coupling term in $V(\phi) = \lambda(\phi^{\dagger}\phi)^2 - \mu^2(\phi^{\dagger}\phi)$ is the source of the strongest interactions. Writing $\phi = (h^+, \frac{1}{\sqrt{2}}(v + h + ia))$ the two terms of $V(\phi)$ are

$$\lambda(\phi^{\dagger}\phi)^2 \hspace{0.4cm} = \hspace{0.4cm} \lambda\left((h^-, rac{1}{\sqrt{2}}(v+h-ia)), (h^+, rac{1}{\sqrt{2}}(v+h+ia))
ight)$$

$$= \lambda \left((h^{+}h^{-}) + \frac{1}{2}[(v+h)^{2} + a^{2}] \right)^{2} -\mu^{2}(\phi^{\dagger}\phi) = -\mu^{2} \left((h^{+}h^{-}) + \frac{1}{2}[(v+h)^{2} + a^{2}] \right).$$
(738)

It is the h^{\pm} and a fields that are eaten and become the longitudinal modes of the W^{\pm} and Z. Thus, we denote them by w^{\pm} and z. In this notation, $\mathcal{L} \ni -V = \mu^2 (\phi^{\dagger} \phi) - \lambda (\phi^{\dagger} \phi)^2$ takes the form of Eq. (737) after including a conventional constant such that V = 0 at the tree-level minimum where $\frac{\partial V}{\partial h}|_{h=0} = 0$ requiring $\lambda v^2 = \mu^2$.

Although the last two terms in Eq. (737) cancel at tree level, they more generally yield a tadpole counterterm which is fixed at each order in perturbation theory in such a way that the physical Higgs field has zero vev. This extra counter term can be ignored for our present purposes.

The Lagrangian of Eq. (737) generates the Feynman rules of Fig. 23. For example, the upper left Feynman rule comes from the $-2\lambda vw^+w^-h$ cross term coming from the square using

$$-2\lambda v = -2\frac{g^2 m_h^2}{8m_W^2} \frac{2m_W}{g} = -\frac{gm_h^2}{2m_W}$$
(739)

and supplying the usual *i* from expanding $\exp[i\mathcal{L}]$.



FIG. 1. Feynman rules for the Higgs-Goldstone scalar theory. Closed loops containing identical particles must be multiplied by $\frac{1}{2}$.

Figure 23: Feynman rules from Eq. (737).

The Feynman rule just below comes from the $-\lambda w^+ w^- w^+ w^-$ term of \mathcal{L} after substituting $\lambda = \frac{g^2 m_h^2}{8 m_W^2}$, taking account of the fact that there are two possible ways to contract the $w^+ w^+$ with two such external states and similarly for $w^- w^-$ — thereby leading to a net contraction counting factor of 4, and supplying the standard *i*.

In these Feynman rules, we have effectively chosen to work in the Landau gauge where the w^{\pm} and z propagators have zero mass and the W^{\pm} and Z propagators (not given explicitly) are proportional to $g^{\mu\nu} - k^{\mu}k^{\nu}/k^2$. In this way, gauge-boson-scalar mixing is avoided, since any such interaction is proportional to the gauge-boson four momentum k^{μ} . Furthermore, we can neglect diagrams with internal W^{\pm} and Z propagators since they are suppressed by m_W^2/m_h^2 in this gauge. Thus, the Landau gauge is the simplest and most natural gauge in which to employ the Goldstone-boson equivalence theorem.

For $ww \rightarrow ww$ scattering there are diagrams with *s*-channel *h* exchange, *t*-channel *h* exchange and the wwww contact interaction diagram. The result for the amplitude is (in agreement with the earlier-stated result)

$$i\mathcal{M}(ww
ightarrow ww) ~=~ \left(-igrac{m_h^2}{2m_W}
ight)^2rac{i}{s-m_h^2}+\left(-igrac{m_h^2}{2m_W}
ight)^2rac{i}{t-m_h^2}-ig^2rac{m_h^2}{2m_W^2}$$

$$= -i \frac{g^2 m_h^2}{4m_W^2} \left(\frac{m_h^2}{s - m_h^2} + \frac{m_h^2}{t - m_h^2} + 2 \right) = -i \frac{m_h^2}{v^2} \left(\frac{s}{s - m_h^2} + \frac{t}{t - m_h^2} \right).$$
(740)

Let us now take the partial wave projection of $\mathcal{M}(ww
ightarrow ww)$,

$$a_J = \frac{1}{32\pi(2J+1)} \int d\cos\theta P_J(\cos\theta)\mathcal{M}$$
(741)

to obtain (using $t = -\frac{s}{2}(1 - \cos \theta)$ implying $\frac{2}{s}dt = d\cos \theta$ and the next to last form for the parenthesis in \mathcal{M} above):

$$a_{0} = \frac{1}{16\pi s} \int_{-s}^{0} dt \mathcal{M}(W_{L}^{+}W_{L}^{-} \to W_{L}^{+}W_{L}^{-})$$

$$= -\frac{1}{16\pi} \frac{m_{h}^{2}}{v^{2}} \left[2 + \frac{m_{h}^{2}}{s - m_{h}^{2}} - \frac{m_{h}^{2}}{s} \log\left(1 + \frac{s}{m_{h}^{2}}\right) \right]. \quad (742)$$

Now let us consider some interesting limits:

1.
$$s \gg m_h^2 \gg m_W^2, m_Z^2$$
:

We get

$$a_0 \to -\frac{m_h^2}{8\pi v^2}$$
. (743)

Requiring $|\text{Re}a_0| \leq \frac{1}{2}$ this gives

$$m_h^2 \le 4\pi v^2 = (872 \text{ GeV})^2,.$$
 (744)

This is the absolute upper limit on m_h in order that unitarity hold for all s. If we consider a full coupled channel analysis (which includes $Z_L Z_L \rightarrow W_L^+ W_L^-$, $hh \rightarrow hh$, $hh \rightarrow W_L^+ W_L^-$, ...) then this limit gets reduced to

$$m_h^2 \lesssim (700 \text{ GeV})^2$$
. (745)

2. $m_W^2, m_Z^2 \ll s \ll m_h^2$:

By carefully expanding the \log to 2nd order in the (small) ratio s/m_h^2 we find

$$a_0 o rac{s}{32\pi v^2}$$
. (746)

Requiring $|\text{Re}a_0| \leq \frac{1}{2}$ this gives

$$s \le 16\pi v^2 \,. \tag{747}$$

An even better bound emerges by considering the (properly normalized) isospin zero channel $\sqrt{1/6}(2W_L^+W_L^-+Z_LZ_L)$ and is

$$s \le 8\pi v^2 = (1233 \text{ GeV})^2$$
. (748)

The interpretation of this limit is that if the Higgs is very heavy, then the SM can only be valid (in the sense of satisfying unitarity perturbatively) if $\sqrt{s} < 1.23$ TeV. After that energy, new physics *must* enter or the $W_L^+W_L^- \rightarrow W_L^+W_L^-$, ... sector must become strongly interacting.

If one does a full treatment, then the kind of plot for $\operatorname{Re}a_0(W_L^+W_L^- \to W_L^+W_L^-)$ as a function of \sqrt{s} that emerges is that below (only look at the SM curves which are for $m_h = 870$ GeV and 1000 GeV). Note how near the Higgs resonance $\operatorname{Re}a_0$ goes positive, but not so positive as to violate the unitarity limit. This is the result obtained after including the Higgs width in the formula for a_0 ; basically $m_h^2 \to m_h^2 - im_h\Gamma_h$. Near the resonance, one is inside the Argand circle that typifies a resonance.

 $\eta_0 < 1$ in the case of $W_L^+ W_L^- \to W_L^+ W_L^-$ because Γ_h includes "inelastic" channels such as $h \to Z_L Z_L$ as well as $h \to W_L^+ W_L^-$.



However, for $s \gg m_h^2$, $\operatorname{Re} a_0$ asymptotes towards $-\frac{1}{2}$ in the $m_h = 870 \text{ GeV}$ case and falls much below $-\frac{1}{2}$ for $m_h = 1000 \text{ GeV}$. As m_h is increased further, the \sqrt{s} value at which a_0 falls below -1/2 decreases slowly, ultimately reaching the value of $\sqrt{s} = \sqrt{2} \times 1.233 \text{ TeV}$ (the $\sqrt{2}$ because this figure does not include the full coupled channel analysis). If m_h is decreased below $m_h = 870 \text{ GeV}$, the SM a_0 curve for $W_L^+ W_L^- \to W_L^+ W_L^$ asymptotes to a less negative value for $\sqrt{s} \to \infty$ following Eq. (743).

Well, this is only a brief introduction to all this. There are endless ramifications of unitarity in the context of every new physics model. It might turn out that the $W_L W_L$, hh, $Z_L Z_L$ sector simply becomes strongly interacting and the LHC job will be to sort out exactly what kind of theory is describing these strong interactions.

This is closely analogous to the old situation in which we had strong interactions in the $\pi\pi$ scattering and related channels and only after much misery figured out that the π 's were bound states of quarks.

In the strongly interacting *WW*-scattering scenario, the LHC will have a very tough time in sorting things out. The energy and luminosity are not quite up to the job. The old SSC was designed at higher energy precisely to cover adequately this unpleasant scenario. The problem is that there is a

huge background coming from the scattering of transversely polarized W's, such as $W_T W_T \to W_T W_T$, $W_L W_L \to W_T W_T$, $W_T W_T \to W_L W_L$, ... from which it is very difficult to extract the $W_L^+ W_L^- \to W_L^+ W_L^-$ scattering of interest.

As a final note, you will notice that the final J = 0 amplitude for $W_L W_L \rightarrow W_L W_L$ is real. And, recall that our unitarity statement is that $|\text{Re}a_0| \leq 1/2$. You might then wonder how it works that $\text{Im}a_0 \geq |\text{Re}a_0|^2$? The point is the $\text{Im}a_0$ is higher order in g^2 , so that to this order in perturbation theory in g^2 there is no contradiction — $\text{Im}a_0$ must be generated by diagrams in which one real amplitude is connected to a 2nd real amplitude by two propagators, yielding something of order g^4 !!

To prove that everything works, you would need to evaluate the imaginary part of the repeated real amplitude diagram by using the Cutkowsky cutting rules for the two propagators. Starting from the most general unitarity statement, Eq. (596) repeated below,

$$= \sum_{n} \left(\prod_{i=1}^{n} \int \frac{d^{3}q_{i}}{(2\pi)^{3}} \frac{1}{2E_{i}} \right) \times \mathcal{M}^{*}(p_{1}p_{2} \to \{q_{i}\}) \mathcal{M}(k_{1}k_{2} \to \{q_{i}\}) (2\pi)^{4} \delta^{4}(k_{1} + k_{2} - \sum_{i} q_{i})$$
(749)

we consider only the same two-body final state as the initial state (i.e. n = 2). We imagine that $\vec{k}_1 \ (= -\vec{k}_2)$ in the com is along the z axis and that \vec{p}_1 is in the x - z plane at location $\theta, \phi = 0$. The two particle intermediate state is defined to have $\vec{q}_1 \ (= -\vec{q}_2)$ defined by location θ', ϕ' . The angle between \vec{p}_1 and \vec{q}_1 we define to be γ . Two-body phase space in the massless limit reduces to

$$\int \frac{d^3 q_1}{(2\pi)^3 2E_1} \frac{d^3 q_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(k_1 + k_2 - q_1 - q_2) = \frac{1}{32\pi^2} \int d\cos\theta' d\phi' \,.$$
(750)

So, now let us do partial wave expansions of all the \mathcal{M} 's (implicitly I am assuming spin-0 equivalent as appropriate for W_L scattering)

$$\mathcal{M}(k_{1}k_{2} \to p_{1}p_{2}) = 16\pi \sum_{J} (2J+1)P_{J}(\cos\theta)a_{J}$$
$$\mathcal{M}^{*}(p_{1}p_{2} \to k_{1}k_{2}) = 16\pi \sum_{J} (2J+1)P_{J}(\cos\theta)a_{J}^{*}$$
$$\mathcal{M}(k_{1}k_{2} \to q_{1}q_{2}) = 16\pi \sum_{J} (2J+1)P_{J}(\cos\theta')a_{J}$$
$$\mathcal{M}^{*}(p_{1}p_{2} \to q_{1}q_{2}) = 16\pi \sum_{J} (2J+1)P_{J}(\cos\gamma)a_{J}^{*}$$
(751)

Inserting into Eq. (749), we get

$$32\pi \sum_{J} (2J+1)P_{J}(\cos\theta) \operatorname{Im} a_{J} = \frac{1}{32\pi^{2}} \int d\cos\theta' d\phi'$$

 $\times (256\pi^{2}) \sum_{J'} (2J'+1)P_{J'}(\cos\theta')a_{J'} \sum_{J''} (2J''+1)P_{J''}(\cos\gamma)a_{J''}^{*}$ (752)

So, now let us write (the standard addition theorem from Jackson)

$$P_{J''}(\cos\gamma) = \frac{4\pi}{2J''+1} \sum_{m} Y^*_{J''m}(\theta',\phi') Y_{J''m}(\theta,\phi=0)$$
(753)

The $\int d\phi'$ simply gives

$$\int d\phi' P_{J''}(\cos \gamma) = (2\pi) \frac{4\pi}{2J''+1} Y_{J''0}^{*}(\theta') Y_{J''0}(\theta)$$

= $(2\pi) \frac{4\pi}{2J''+1} \left[\sqrt{\frac{2J''+1}{4\pi}} \right]^2 P_{J''}(\cos \theta') P_{J''}(\cos \theta)$
= $(2\pi) P_{J''}(\cos \theta') P_{J''}(\cos \theta)$. (754)

Eq. (752) then reduces to

$$\sum_{J} (2J+1) P_{J}(\cos\theta) \operatorname{Im} a_{J}$$

$$= \frac{1}{2} \int d\cos\theta' \sum_{J'} (2J'+1) P_{J'}(\cos\theta') a_{J'} \sum_{J''} (2J''+1) P_{J''}(\cos\theta') P_{J''}(\cos\theta) a_{J''}^{*}$$

$$= \frac{1}{2} \sum_{J'} (2J'+1) a_{J'} \sum_{J''} (2J''+1) \frac{2}{2J'+1} \delta_{J',J''} P_{J''}(\cos\theta) a_{J''}^{*}$$

$$= \sum_{J'} (2J'+1) P_{J'}(\cos\theta) a_{J'} a_{J'}^{*}$$
(755)

from which we immediately conclude that

$$\mathrm{Im}a_J = a_J a_J^* \tag{756}$$

if only the same two-body intermediate state is present as in the initial and final states (equivalent to there being no inelastic scattering).